Rational bubbles and portfolio constraints

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May 11, 2012

Arbitrage

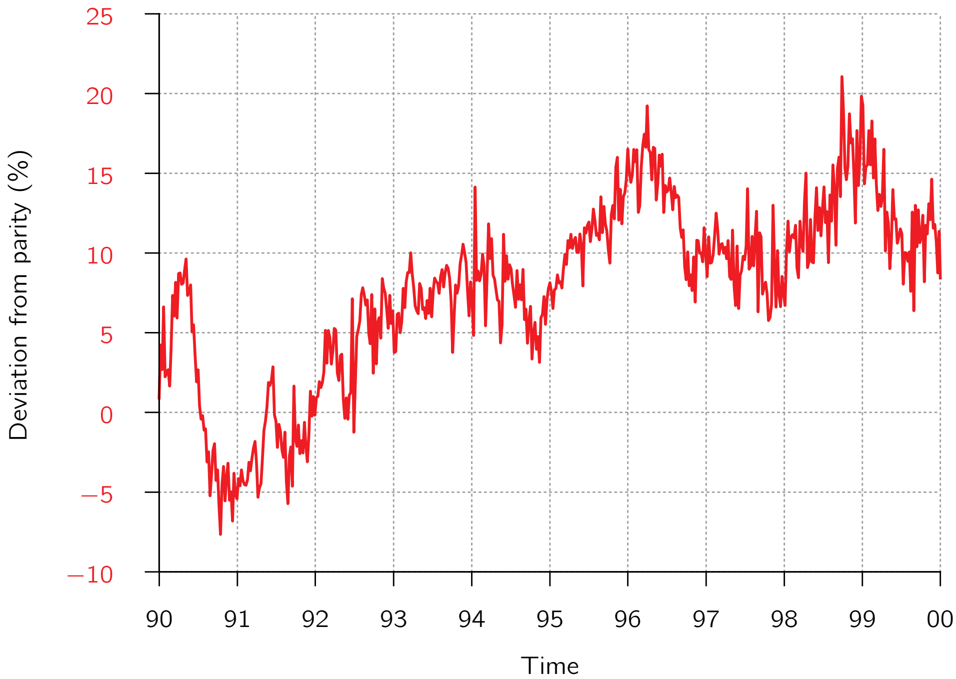
- **The absence of arbitrage**, defined as the possibility of simultaneously buying and selling the same security at different prices, is the most fundamental concept of finance.
- To make a parrot into a trained financial economist it suffices to teach him a single word: **arbitrage**.

S. Ross (1987)

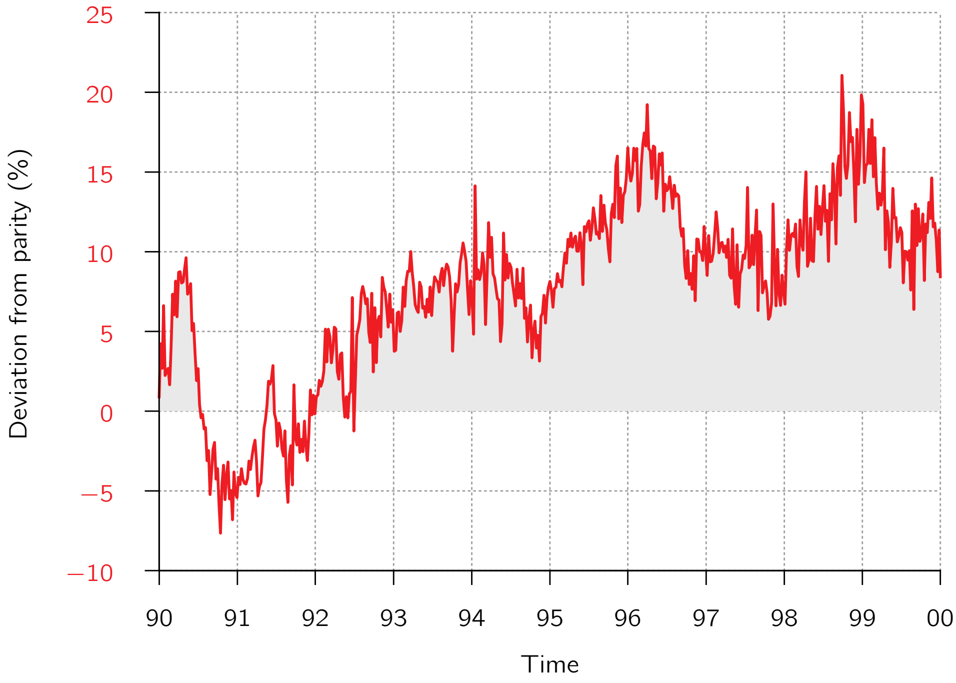
Anomalies

- **But significant violations** of this basic paradigm **are often observed** in real world markets.
- A famous example is the simultaneous trading of **Royal Dutch and Shell** in Amsterdam and London:
  - The two companies merged in 1907 on a 60/40 basis
  - Cash flows are attributed to the stocks in these proportions
  - Despite this RD traded at a significant premium relative to Shell throughout most of the 1990's.
- Other examples: Molex, Unilever NV/PLC, 3Com/Palm...

Royal Dutch/Shell



Royal Dutch/Shell



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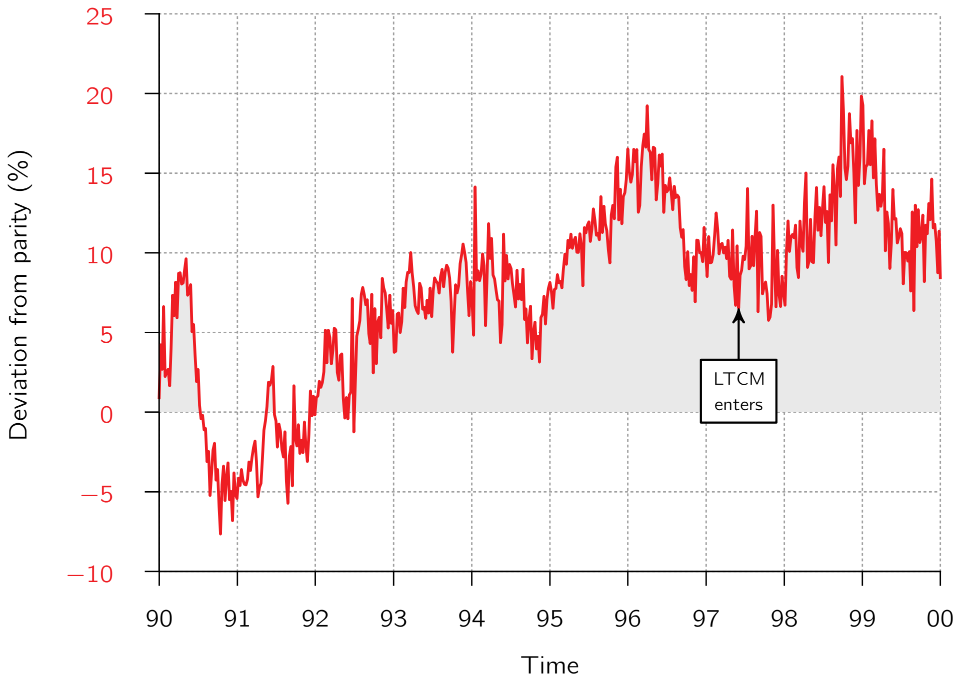

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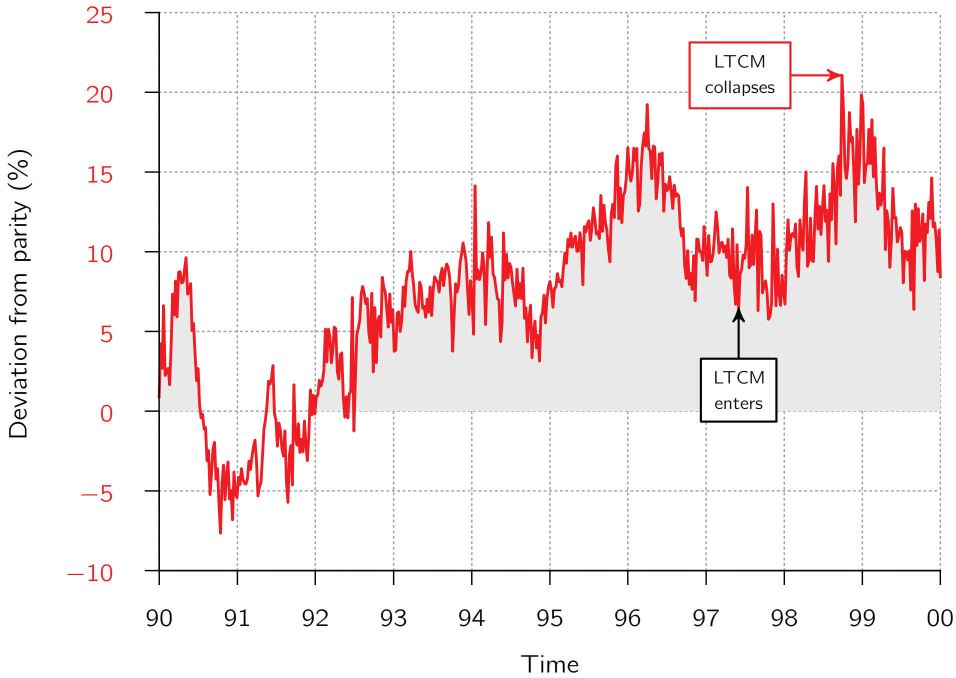


Theory

- Neo–classical theory has little to say:
  - The workhorse model of modern asset pricing is the representative agent model of Lucas (1974).
  - In this model mispricing on positive net supply assets is incompatble with the existence of an equilibrium.
- Most of the work on the origin of bubbles is behavioral
  - Common feature: **partial equilibrium setting**.
  - Different definition of the fundamental value which implies that bubbles are not connected to arbitrage activity.

Portfolio constraints

- There are some models where arbitrages arise endogenously due to portfolio constraints.
  - Common feature: all agents are constrained, **riskless arbitrage**
  - If the constraints are lifted for some agents then mispricing becomes inconsistent with equilibrium.
- This need not be the case with **risky arbitrage**: **portfolio constraints can generate bubbles** in equilibrium even if there are unconstrained arbitrageurs in the economy.

This paper

- Continuous-time model with two groups of agents:
  - Unconstrained agents,
  - Constrained agents with logarithmic utility.
- Necessary and sufficient conditions under which portfolio constraints generate **bubbles in equilibrium**.
- When there are multiple stocks, the presence of bubbles may give rise to **multiplicity** and **real indeterminacy**.
- **Examples** of innocuous portfolio constraints, including limited market participation, that generate bubbles in equilibrium.

Related literature

- Behavioral models:
  - Harrison and Kreps (1979), DeLong et al. (1990), Scheinkman and Xiong (2003), Abreu and Brunnermeier (2003).
- Equilibrium under constraints:
  - Basak and Cuoco (1997), Detemple and Murthy (1997), Shapiro (2002), Pavlova and Rigobon (2007), Garleanu and Pedersen (2010).
- Equilibrium mispricing:
  - Santos and Woodford (1997), Loewenstein and Willard (2000,2008), Basak and Croitoru (2000,2006), Grombs and Vayanos (2002),...
- Partial equilibrium:
  - Cox and Hobson (2005), Jarrow et al. (2008,2010),...

Outline

1. The model
2. Equilibrium bubbles
3. Limited participation
4. Multiplicity

The model

- Continuous–time economy on $[0, T]$.
- One perishable consumption good and $n+1$ traded securities:
  - A locally riskless asset in zero net supply,
  - n risky assets in positive net supply of one unit each.
- The price of the riskless asset evolves according to

$$\mathrm{d}S_{0t} = r_t S_{0t} \mathrm{d}t$$

  where the instantaneously risk free rate process r_t is to be determined endogenously in equilibrium

Risky assets

- Dividends evolve according to

$$\mathrm{d}\delta_t = \mathrm{diag}(\delta_t)\left(\mu_{\delta t}\mathrm{d}t + \sigma_{\delta t}\mathrm{d}B_t\right)$$

  for some exogenous $(\mu_\delta, \sigma_\delta)$ where B is a BM in $\mathbb{R}^n$.
- The stock prices evolve according to

$$\mathrm{d}S_t + \delta_t \mathrm{d}t = \mathrm{diag}(S_t)\left(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t\right).$$

  where the initial price S_0, the drift μ_t and the volatility σ_t are to be determined endogenously in equilibrium.

Agents

- Two agents indexed by $a = 1, 2$.
- The preferences of agent a are represented by

$$U_a(c) = E_0\left[\int_0^T e^{-\rho\tau} u_a(c_\tau)\mathrm{d}\tau\right]$$

where ρ is a nonnegative discount rate, $u_2 \equiv \log$ and u_1 is a utility function satisfying textbook regularity conditions.
- Agent 2 is initially endowed with β units of the riskless asset and a positive fraction α_i of the supply of stock i.

Trading strategies

- A **trading strategy** is a process $(\phi, \pi) \in \mathbb{R} \times \mathbb{R}^n$.
- The strategy (ϕ, π) is self financing for agent a given a consumption plan c if the corresponding **wealth process**

$$W_t = W_t(\phi, \pi) \equiv \phi_t + \mathbf{1}^*\pi_t$$

satisfies the dynamic budget constraint

$$W_t = w_a + \int_0^t (\phi_\tau r_\tau + \pi_\tau^*\mu_\tau - c_\tau)\mathrm{d}\tau + \int_0^t \pi_\tau^*\sigma_\tau \mathrm{d}B_\tau$$

where the constant w_a denotes the agent's initial wealth computed at equilibrium prices.

Portfolio constraints

- Agent 1 is unconstrained (except for $W_t \geq 0$)
- Agent 2 is **constrained**: I assume that the trading strategy that he chooses must satisfy

$$\text{Amount in stocks} = \pi_t \in W_t \mathscr{C}_t$$

as well as $W_t \geq 0$ where $\mathscr{C}_t \subseteq \mathbb{R}^n$ is a closed convex set.

- A wide variety of constraints, including constraints on short selling, collateral constraints, borrowing and participation constraints can be modeled in this way.

Equilibrium

- An **equilibrium** is a collection of prices, consumption plans and trading strategies such that:
  (a) c_a maximizes U_a and is financed by (ϕ_a, π_a),
  (b) The securities and goods markets clear

$$\phi_1 + \phi_2 = 0,$$
$$\pi_1 + \pi_2 = S,$$
$$c_1 + c_2 = \mathbf{1}^* \delta \equiv e.$$

- I will restrict the analysis to the class of **non redundant equilibria** in which the stock volatility is invertible.

Rational stock bubbles

- A traded security is said to have a bubble if its market price differs from its fundamental value: $B_{it} \equiv S_{it} - F_{it}$.
- Since markets are complete for Agent 1, the fundamental value of a stock is unambiguously defined as

$$F_{it} = \frac{1}{\xi_t} E_t \left[\int_t^T \xi_\tau \delta_{i\tau} \mathrm{d}\tau \right]$$

where the process

$$\xi_t = \frac{1}{S_{0t}} \exp\left(- \int_0^t \theta_\tau^* \mathrm{d}B_\tau - \frac{1}{2} \int_0^t \|\theta_\tau\|^2 \mathrm{d}\tau \right)$$

is the SPD and θ is the market price of risk.

Basic properties

- A bubble is nonnegative and satisfies $B_{iT} = 0$.
- A bubble cannot be born: if $B_{it} = 0$ then $B_{i\tau} = 0$ for all $\tau \geq t$.
- A bubble is **not an arbitrage**: The strategy which
  - Sells the stock short,
  - Buys the replicating portfolio,
  - Invests the remainder in the riskless asset,

  has wealth process

$$W_t = B_{i0} S_{0t} - B_{it}$$

and thus is not admissible on its own (even if the positive wealth constraint is relaxed to allow for bounded credit).

Riskless asset bubble

- Over $[0, T]$ the riskless asset can be seen as a European derivative security with pay–off S_{0T} at the terminal time.
- The fundamental value of such a security is

$$F_{0t} = E_t \left[\frac{\xi_T}{\xi_t} S_{0T} \right] = S_{0t} \, E_t \left[\frac{M_T}{M_t} \right]$$

  where $M_t \equiv \xi_t S_{0t}$.
- The existence of a bubble on the riskless asset is **equivalent** to the non existence of the EMM.

The equilibrium SPD

- **Proposition.** In equilibrium

$$\xi_t = e^{-\rho t} \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)}$$

  where e_t is the aggregate dividend process, λ_t is the ratio of the agents' marginal utilities and

$$u(e, \lambda_t) = \max_{c_1 + c_2 = e} \left\{ u_1(c_1) + \lambda_t u_2(c_2) \right\}.$$

- Since the allocation is inefficient, λ **is not a constant** but a stochastic process that acts as an endogenous state variable.

Bubble on the market portfolio

$$\sum_{i=1}^{n} B_{it} = \sum_{i=1}^{n} S_{it} - E_t\left[\int_t^T e^{-\rho(\tau-t)}\frac{u_c(e_\tau,\lambda_\tau)}{u_c(e_t,\lambda_t)}e_\tau d\tau\right]$$

$$= W_{1t} + W_{2t} - E_t\left[\int_t^T e^{-\rho(\tau-t)}\frac{u_c(e_\tau,\lambda_\tau)}{u_c(e_t,\lambda_t)}e_\tau d\tau\right]$$

$$= W_{2t} - E_t\left[\int_t^T e^{-\rho(\tau-t)}\frac{u_c(e_\tau,\lambda_\tau)}{u_c(e_t,\lambda_t)}c_{2\tau} d\tau\right]$$

$$= E_t\left[\int_t^T e^{-\rho(\tau-t)}\frac{u_c(e_\tau,\lambda_\tau)}{u_c(e_t,\lambda_t)}\left(\frac{\lambda_t}{\lambda_\tau}-1\right)c_{2\tau} d\tau\right]$$

$$= \frac{1}{u_c(e_t,\lambda_t)}E_t\left[\int_t^T e^{-\rho(\tau-t)}(\lambda_t-\lambda_\tau) d\tau\right] \qquad (u_2 = \log)$$

Equilibrium bubbles

- **Proposition.** In equilibrium,

$$\lambda_t = \lambda_0 - \int_0^t \lambda_\tau\left(\theta_\tau - \Pi\left(\theta_\tau|\sigma_\tau^*\mathscr{C}_\tau\right)\right)^* dB_\tau$$

where Π is the projection operator and θ solves

$$\theta_t = \sigma_{et}R_t + s_t R_t\left(\theta_t - \Pi(\theta_t|\sigma_t^*\mathscr{C}_t)\right)$$

with

$$R_t = -\frac{u_{cc}(e_t,\lambda_t)}{u_c(e_t,\lambda_t)}e_t, \qquad s_t = \frac{c_{2t}}{e_t} = \frac{\lambda_t}{u_c(e_t,\lambda_t)}.$$

The weighting process is a local martingale and it **is a martingale if and only if** the stock prices do not include bubbles.

Limited participation

- Consider the following specification:
  - There is a single stock,
  - Both agents have logarithmic utility,
  - The dividend is a GBM with drift μ_δ and volatility σ_δ,
  - $\mathscr{C}_t = [0, 1-\varepsilon]$ for some $0 \leq \varepsilon \leq 1$.
- Assume $\beta < (1-\alpha)\delta_0 T$ to guarantee that the unconstrained agent is not so deeply in debt that he can never repay.
- **Special cases** include
  - Unconstrained economy ($\varepsilon = 0$).
  - Restricted participation model of Basak and Cuoco ($\varepsilon = 1$).

Equilibrium

- **Proposition.** Let λ denote the unique solution to

$$\lambda_t = \frac{w_2}{w_1} - \int_0^t \lambda_\tau(1+\lambda_\tau)\sigma_\lambda \mathrm{d}B_\tau$$

with $\sigma_\lambda = \varepsilon\sigma_\delta$. In the **unique** equilibrium, the consumption plans and trading strategies are given by

$$\phi_{1t} = -\varepsilon\lambda_t W_{1t}, \qquad \pi_{1t} = (1+\varepsilon\lambda_t)W_{1t}, \qquad c_{1t} = \frac{e_t}{1+\lambda_t},$$
$$\phi_{2t} = \varepsilon W_{2t}, \qquad \pi_{2t} = (1-\varepsilon)W_{2t}, \qquad c_{2t} = \frac{e_t\lambda_t}{1+\lambda_t},$$

and the stock price is $S_t/e_t = \int_t^T e^{-\rho(\tau-t)}d\tau \equiv \eta(t)$.

Equilibrium bubbles

- The weighting process is a **strict local martingale!**
- **Proposition.** The riskless asset and the stock both include bubble components that are given by

$$\frac{B_t}{S_t} = b(t, s_t) \leq b_0(t, s_t) = \frac{B_{0t}}{S_{0t}}$$

where the bounded process

$$s_t = \frac{c_{2t}}{e_t} = \frac{\lambda_t}{1 + \lambda_t}$$

represents the constrained agent's share of aggregate consumption and b, b_0 are known functions.

Bubbles

- The bubbles are explicitly given by

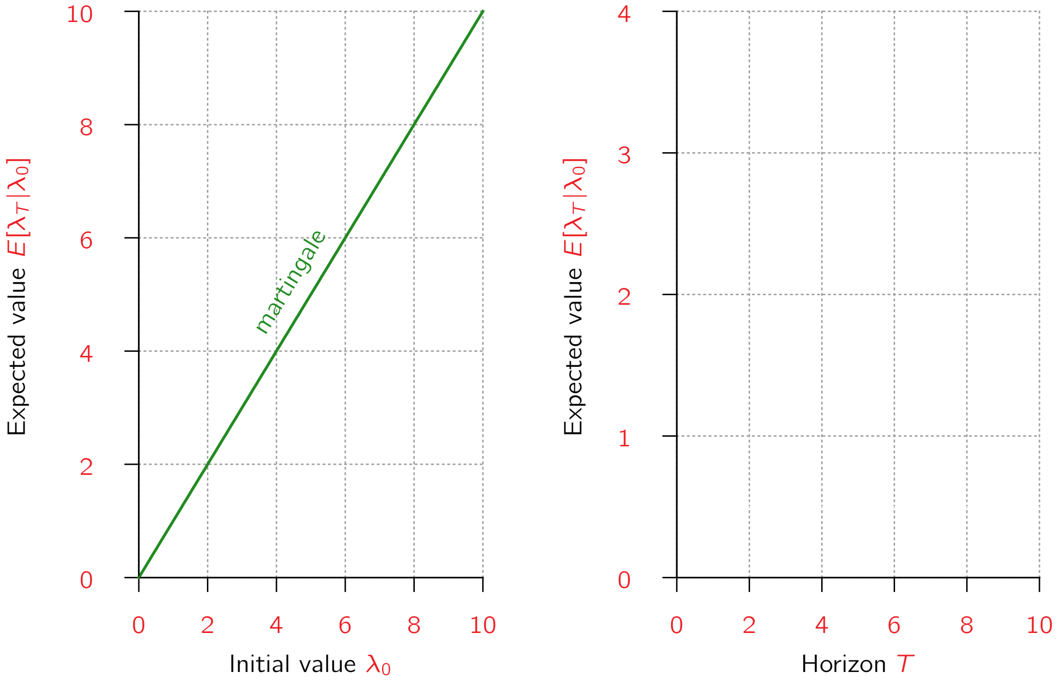
$$b_0(t, T, s) \equiv s^{-1/\varepsilon} H(T - t, s; a_0),$$
$$b(t, s) \equiv \frac{1}{\rho\eta(t)} H(T - t, s; a_1) + \frac{\eta'(t)}{\rho\eta(t)} H(T - t, s; 1),$$

where a_0, a_1 are constants

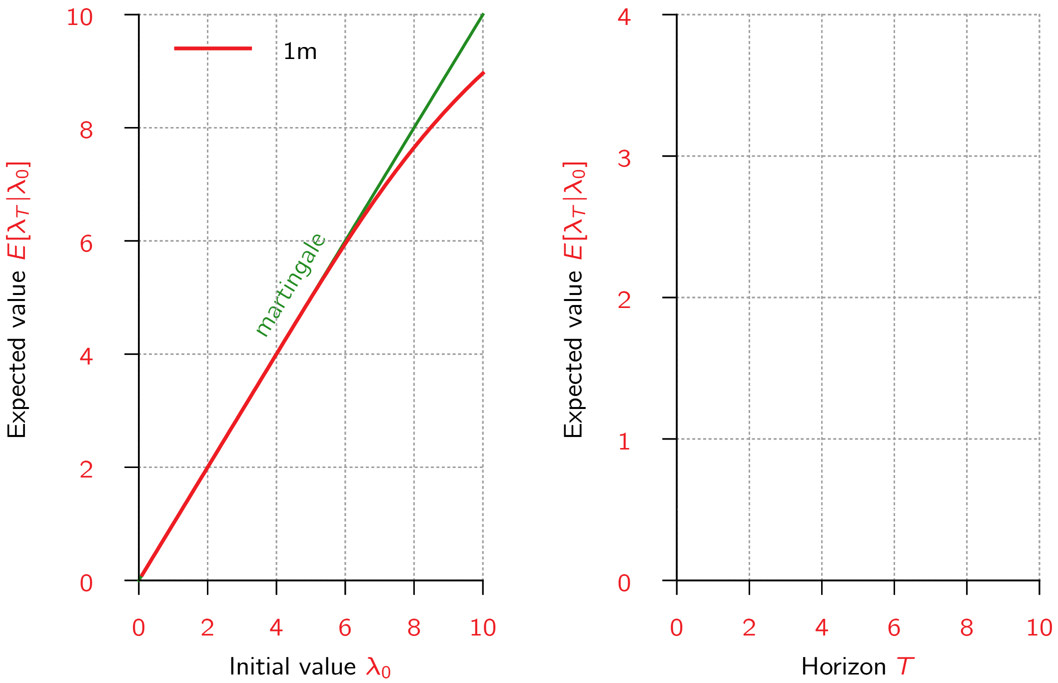
$$H(\tau, s; a) \equiv s^{\frac{1+a}{2}} \Phi(d_+(\tau, s; a)) + s^{\frac{1-a}{2}} \Phi(d_-(\tau, s; a)),$$
$$d_\pm(\tau, s; a) \equiv \frac{1}{\|v_\lambda\|\sqrt{\tau}} \log s \pm \frac{a}{2}\|v_\lambda\|\sqrt{\tau},$$

and Φ denotes the normal cdf.

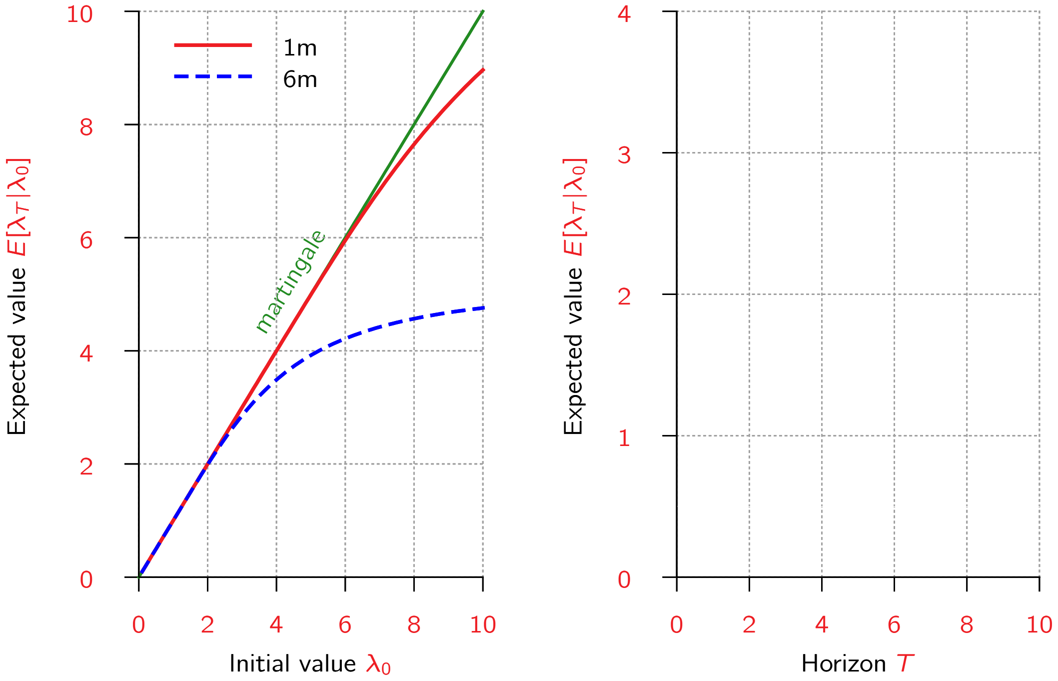
Strict local martingale



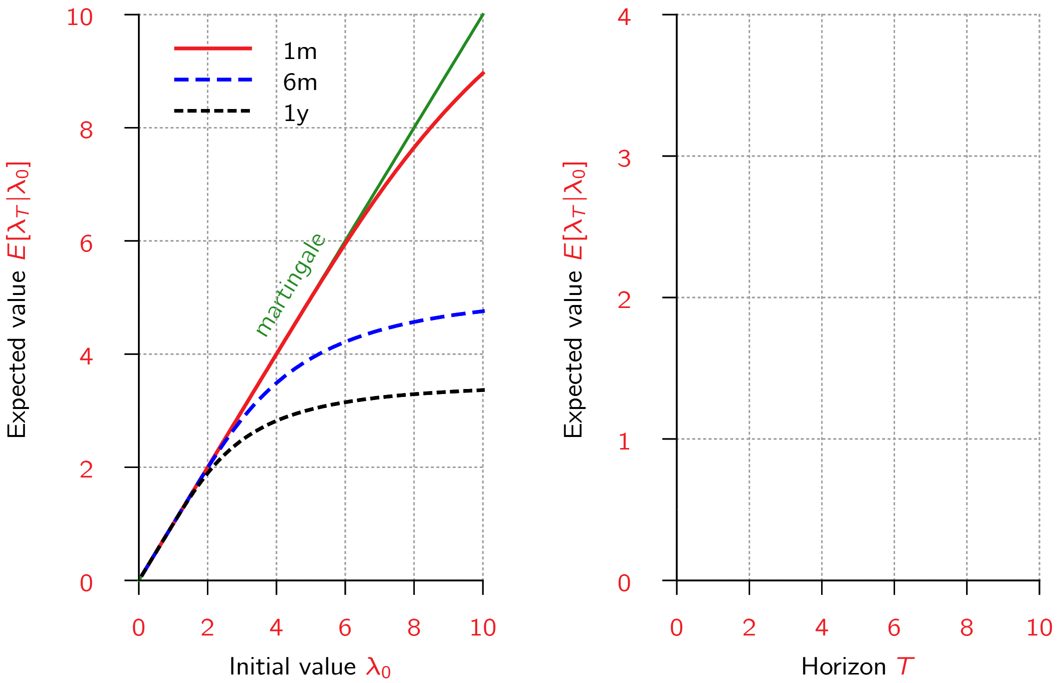
Strict local martingale



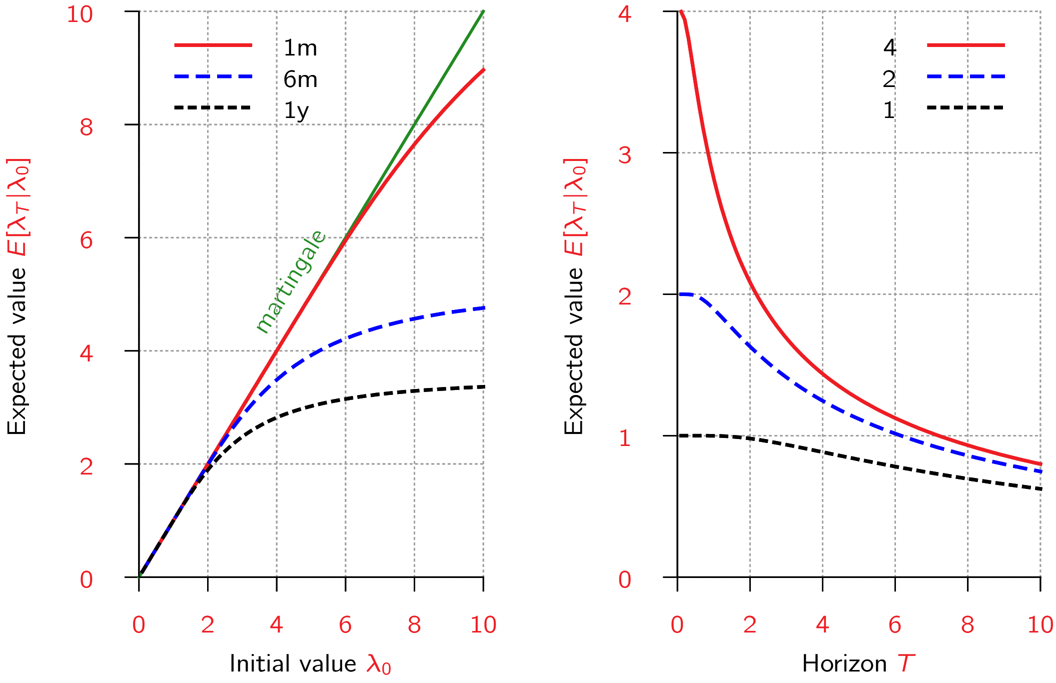
Strict local martingale



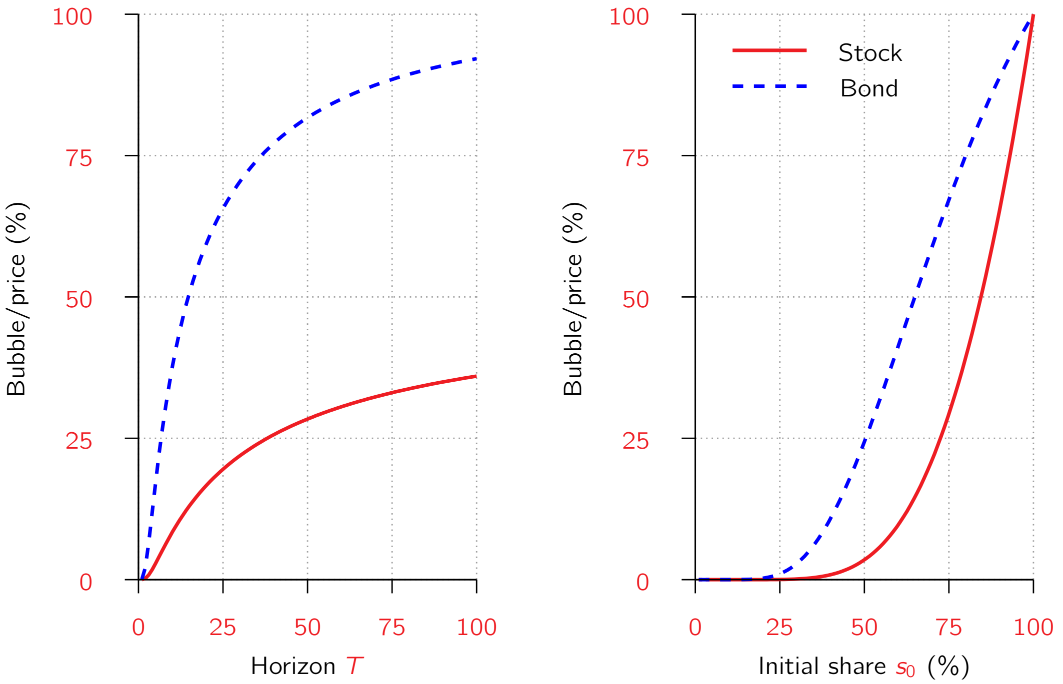
Strict local martingale



Strict local martingale



Equilibrium bubbles



Mechanism

- Agent 2 must keep some wealth in the bank.
- Agent 1 must find it optimal to hold a leveraged position.
- This implies that the **short rate must decrease** and the **market price of risk must increase**. Indeed:

$$r_t = \rho + \mu_\delta - (1 + \varepsilon\lambda_t)|\sigma_\delta|^2 = r_t^{\text{nc}} - \varepsilon\lambda_t|\sigma_\delta|^2,$$
$$\theta_t = (1 + \varepsilon\lambda_t)\sigma_\delta = \theta_t^{\text{nc}} + \varepsilon\lambda_t\sigma_\delta.$$

- But this is **not sufficient** to entice Agent 1 to hold the highly leveraged portfolio necessary to clear markets.

Equilibrium portfolio

- The equilibrium portfolio of Agent 1 can be decomposed into: A **short position** of size

$$m_t \equiv \frac{S_t}{1/(\varepsilon s_t) + \partial_s \log b^0(t, s_t)} > 0$$

in the riskless asset bubble and a long position in the stock.
- The first part is an **arbitrage strategy** with negative value
  - This strategy is not admissible by itself,
  - The bubble on the stock raises its collateral value and allows the agent to scale his position to the required level.

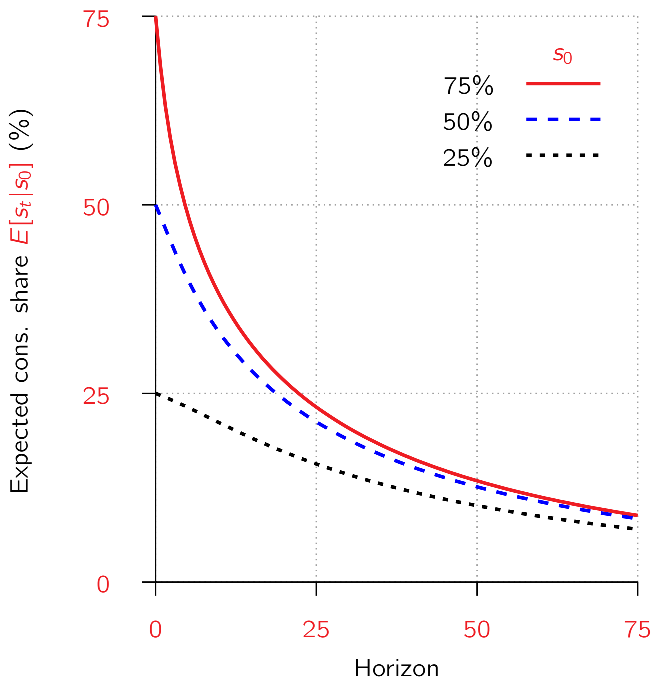
Consumption share

- The equilibrium consumption share of the constrained agent can be explicitly computed as

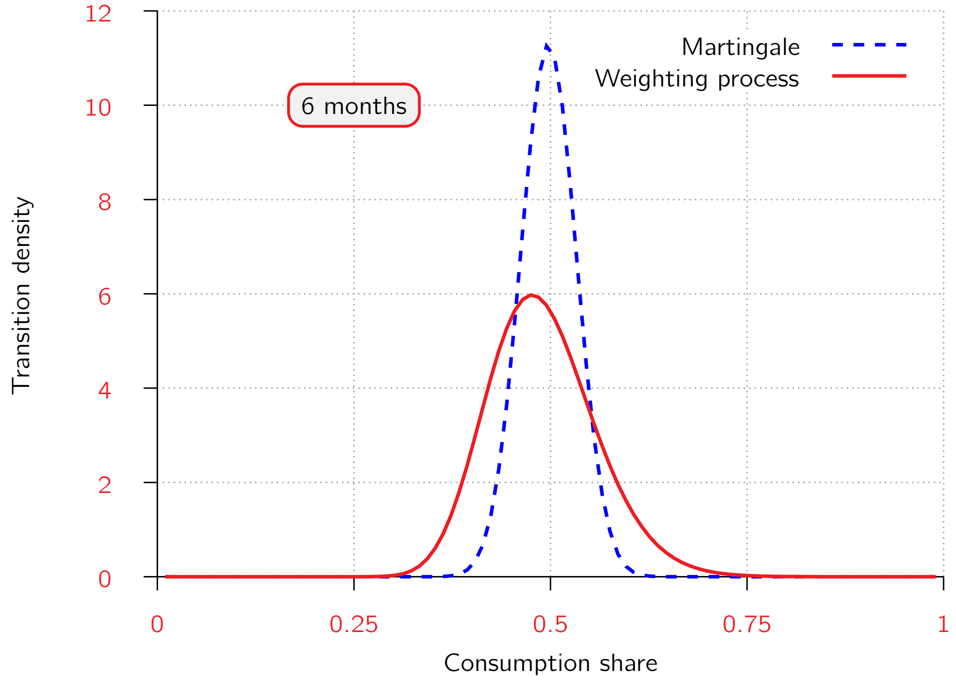
$$s_t = \frac{c_{2t}}{c_{1t} + c_{2t}} = \frac{\lambda_t}{1 + \lambda_t} \equiv \mathfrak{s}(\lambda_t).$$

- Since the weighting process is a nonnegative local martingale and the function $\mathfrak{s}$ is increasing and concave, the consumption share is a supermartingale and is thus **expected to decrease**.
- This would be the case even if the weighting process λ_t was a true martingale (comp. heterogenous beliefs) but the presence of **bubbles increases the speed** at which s decreases.

Expected consumption share



Consumption share



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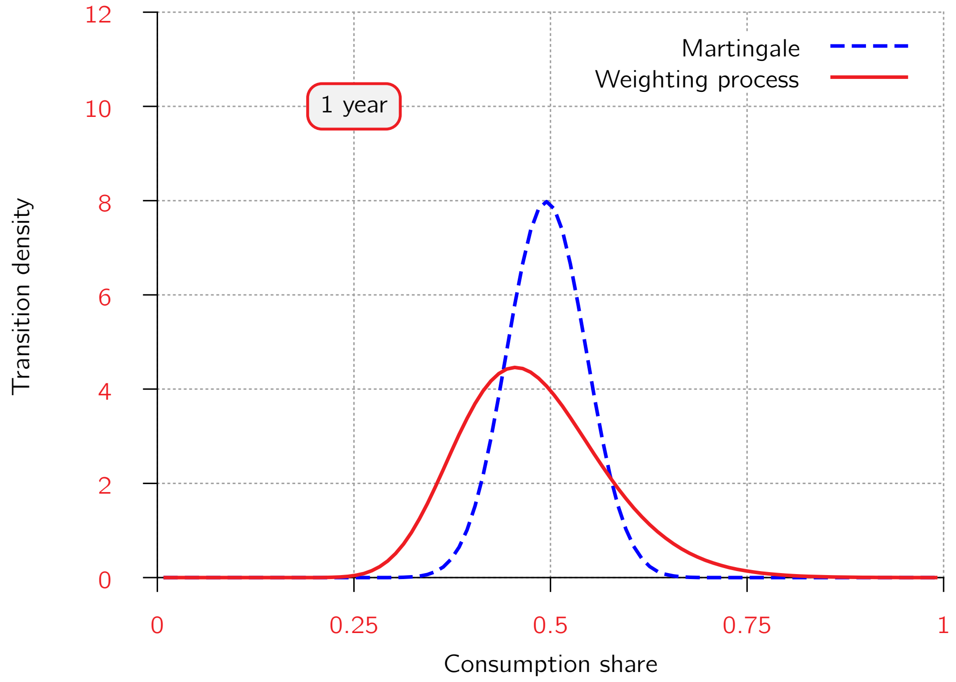

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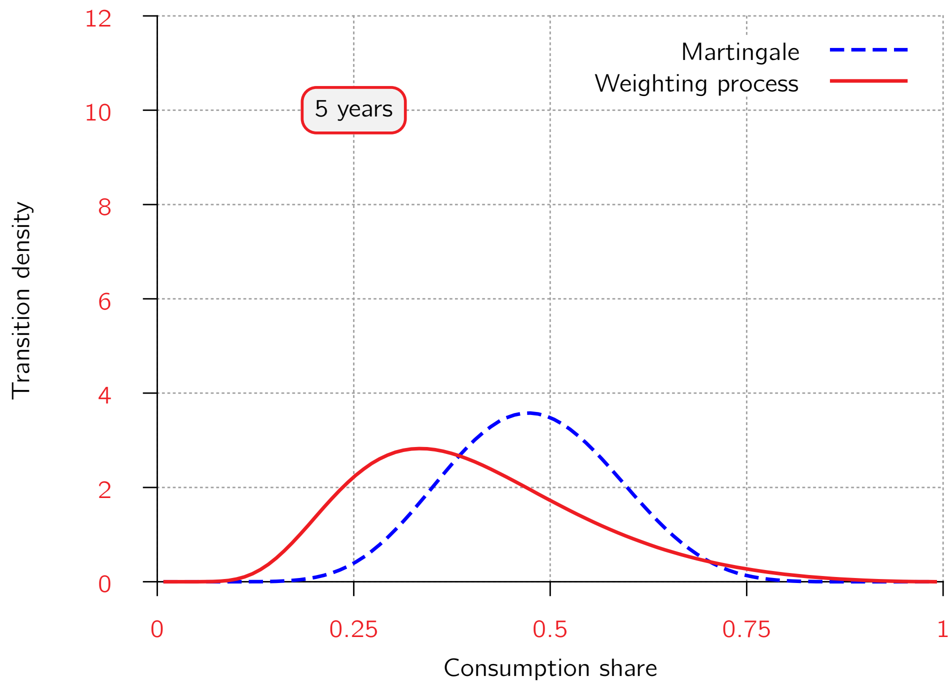

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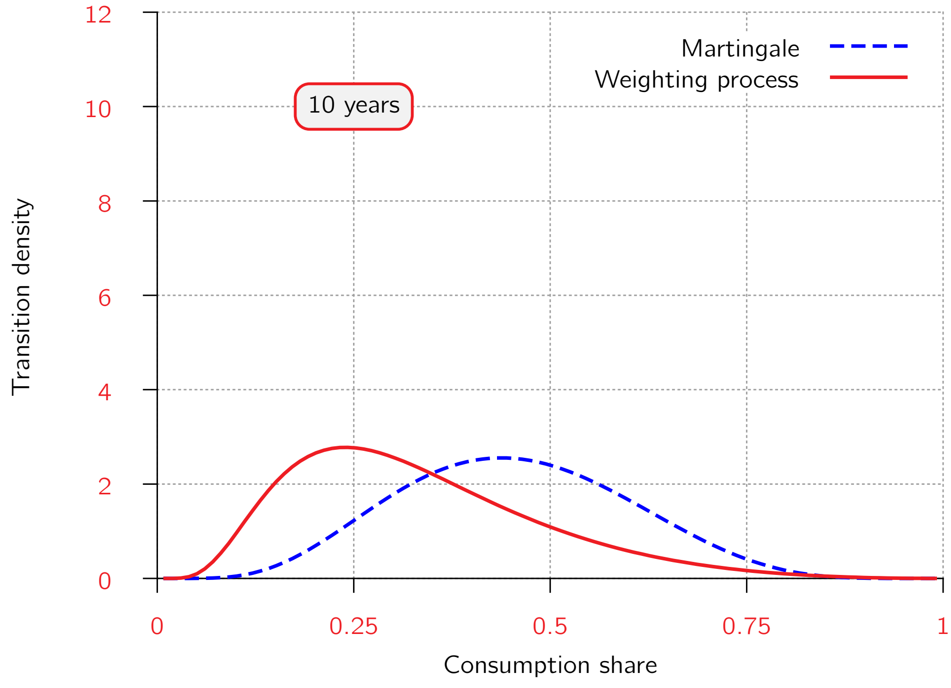

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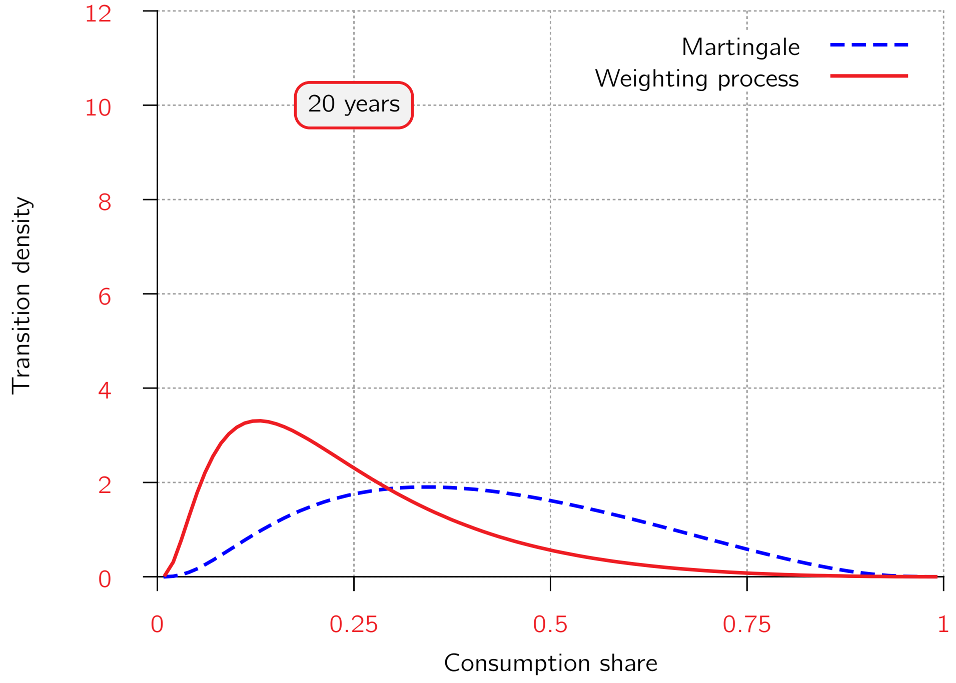

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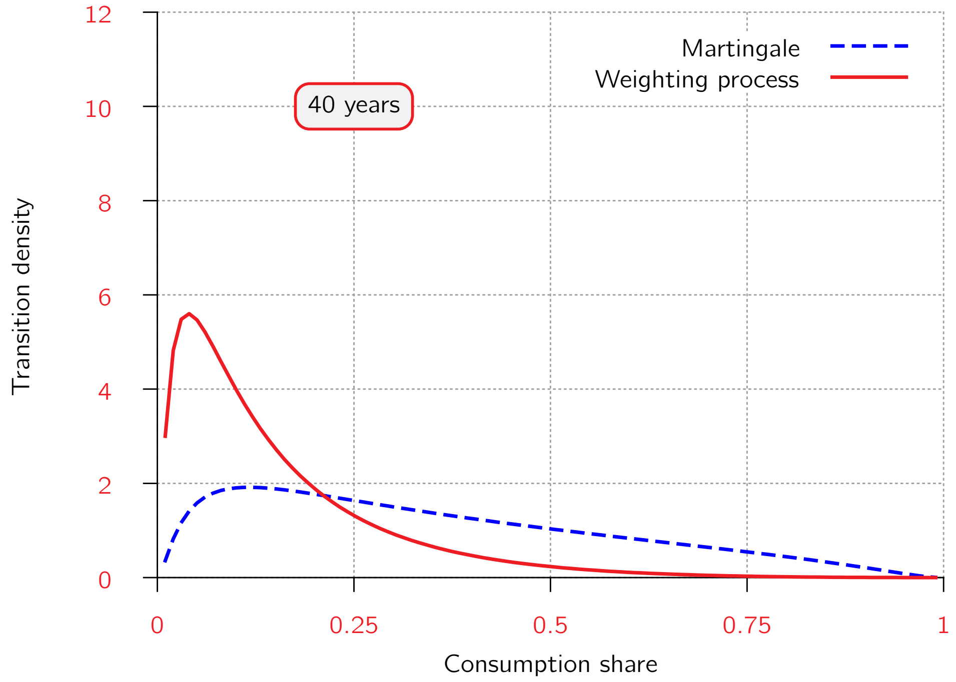

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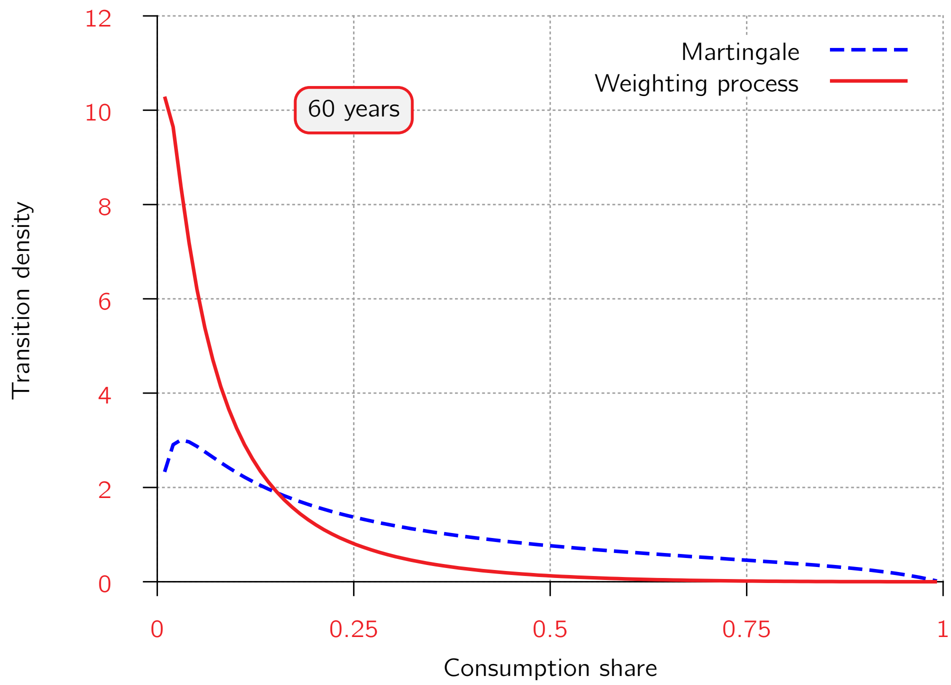


Multiple risky assets

- **If there is no bubble** in the market portfolio, then the stock prices are given by the familiar formula

$$S_t \equiv F_t = E_t\left[\int_t^T e^{-\rho(\tau-t)}\frac{u_c(e_\tau,\lambda_\tau)}{u_c(e_t,\lambda_t)}\delta_\tau \mathrm{d}\tau\right].$$

- The existence of a bubble-free equilibrium is thus equivalent to the existence of a solution to a FBSDE.
- If such a solution does not exists, then only the value of the market portfolio is uniquely determined.

Multiplicity

- **Proposition.** A process $S \in \mathbb{R}^n_+$ with invertible volatility matrix σ is an equilibrium price process if and only if

$$\sum_{i=1}^{n} S_{it} = E_t\left[\int_t^T e^{-\rho(\tau-t)} \frac{u_c(e_\tau, \lambda_\tau)e_\tau + \lambda_t - \lambda_\tau}{u_c(e_t, \lambda_t)} \mathrm{d}\tau\right]$$

  and the discounted process

$$e^{-\rho t} \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} S_t + \int_0^t e^{-\rho \tau} \frac{u_c(e_\tau, \lambda_\tau)}{u_c(e_0, \lambda_0)} \delta_\tau \mathrm{d}\tau$$

  is a nonnegative local martingale.
- For **risk constraints** of the form $\mathscr{C}_t = (\sigma_t^*)^{-1}\mathscr{C}_t^o$ the weighting process can be determined independently of the prices.

Volatility constraints

- Consider the following specification:
  - There are **two stocks**,
  - Agents have logarithmic utility,
  - The aggregate dividend is a GBM with drift μ_e and volatility σ_e,
  - The dividend share $x_{1t} = \delta_{1t}/e_t$ is a martingale that is independent from the aggregate dividend process.
  - The portfolio constraint set is

$$\mathscr{C}_t = \left\{p \in \mathbb{R}^2 : \|\sigma_t^* p\| \leq (1-\varepsilon)\|\sigma_e\|\right\}.$$

- This constraint restricts the volatility of the agent's wealth to be less than a fixed fraction of that of the market.

Equilibrium

- **Proposition.** Define λ as the unique solution to

$$\lambda_t = \frac{w_2}{w_1} - \int_0^t \lambda_\tau (1+\lambda_\tau)\hat{\sigma}^* \mathrm{d}B_\tau .$$

In equilibrium, the short rate, the risk premia, the fundamental value of the stocks and the value of the market are

$$r_t = \rho + \mu_e - (1+\varepsilon\lambda_t)\|\sigma_e\|^2, \qquad F_{it} = \delta_{it}\eta(t)(1-b(t,s_t)),$$
$$\theta_t = (1+\varepsilon\lambda_t)\sigma_e, \qquad \overline{S}_t = e_t\eta(t).$$

Furthermore, **bubbles** account for a fraction $b_0(t,s_t)$ of the riskless asset and $b(t,s_t)$ of the market portfolio.

Equilibrium prices

- **Proposition.** Let $s_0 = s_0(\phi) \in [0,1]$ solve

$$\beta + e_0\eta(0)\alpha^* \left(x_0 + (\phi - x_0)b(0,s_0)\right) = s_0 e_0 \eta(0).$$

and denote by $s_t(\phi)$ the corresponding path of the consumption share process. Then the nonnegative process

$$S_t(\phi) = e_t\eta(t)\left(x_t + (\phi - x_t)b(t,s_t)\right)$$

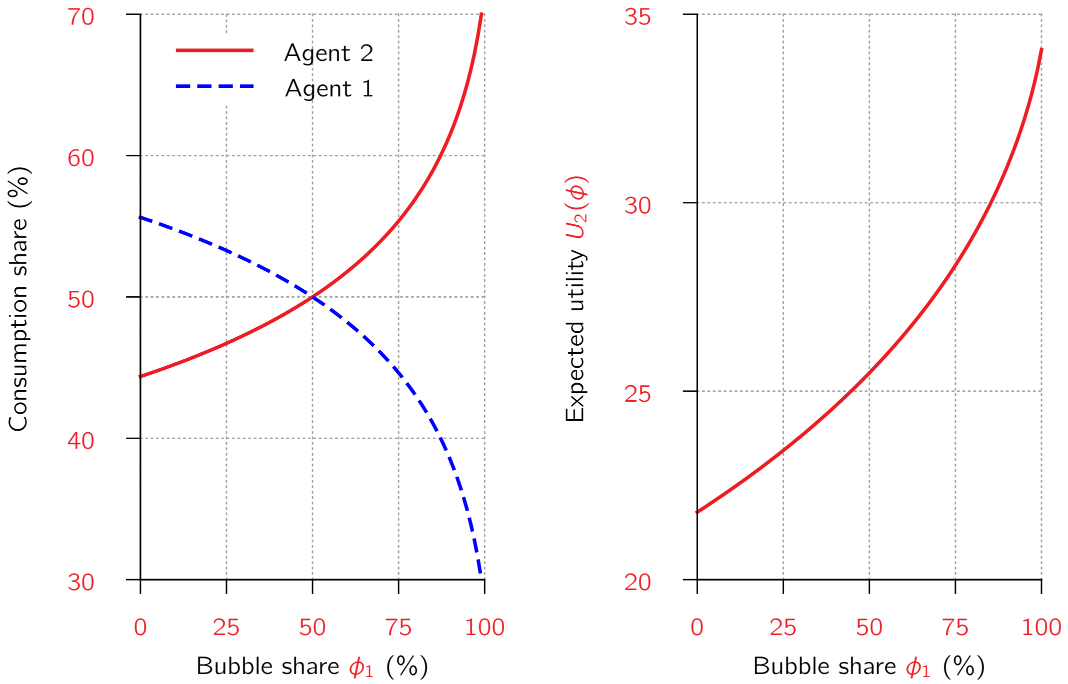
is an equilibrium price process for each $\phi \in \Delta^2$. In particular, the set of non redundant equilibria is non empty.

- Since all equilibria are Markovian this shows that we have not only multiplicity but also **real indeterminacy if** $(\alpha_1 \neq \alpha_2)$.

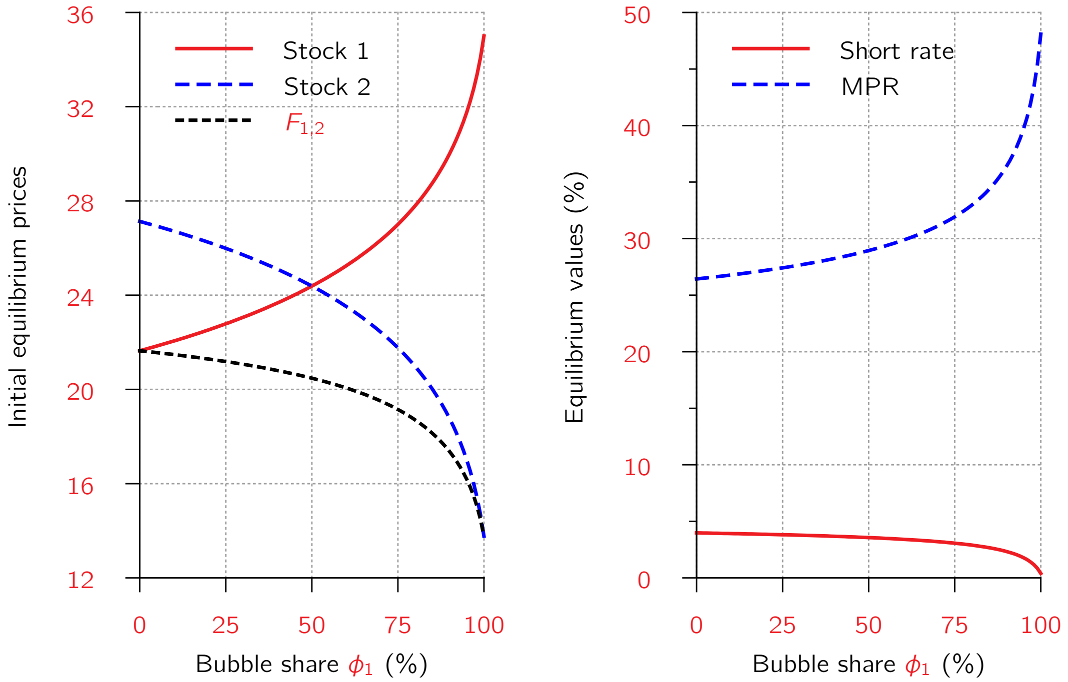
Parameter values

Symbol	Name	Value
μ_e	Market return	8.25%
σ_e	Market volatility	16.64%
σ_x	Vol. dividend share	20.00%
x_{10}	Initial dividend share	50.00%
β	Initial position in bank	0.00%
α_1	Initial position in S_1	100.00%
α_2	Initial position in S_2	0.00%

Real indeterminacy



Nominal indeterminacy



Some extensions

- **CRRA** utility for Agent 1: bubbles persist if $\gamma \geq 1$
- **Uncollateralized borrowing** (Hugonnier and Prieto (2010)):
  - Equilibrium fails if bound formulated in terms of S_{0t}
  - Equilibrium exists if bound formulated in terms of the market portfolio.
- Other types of constraint: Prieto (2010) shows that certain risk-based constraints also give rise to bubbles.
- Bubbles also arise in general equilibrium models with proportional **transaction costs** (Cujean (2011))

Thank you!