

Rational asset pricing bubbles and portfolio constraints*

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Abstract

This article shows that portfolio constraints can give rise to rational asset pricing bubbles in equilibrium even if there are unconstrained agents in the economy who can benefit from the induced limited arbitrage opportunities. Furthermore, it is shown that bubbles can lead to both multiplicity and real indeterminacy of equilibria. The general results are illustrated by two explicitly solved examples where seemingly innocuous portfolio constraints make bubbles a necessary condition for the existence of an equilibrium.

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1 Introduction

The absence of arbitrages, defined as the possibility of simultaneously buying and selling at different prices two securities (or portfolios) that produce the same cash flows, is the cornerstone of modern finance. Yet, violations of this basic paradigm are frequently observed. In particular, over the past three decades numerous deviations from the fundamental value implied by no-arbitrage restrictions, so-called rational asset pricing bubbles, have been detected in financial markets across the world.¹ Despite this evidence and a clear need for insights into the origins, determinants, and welfare implications of rational asset pricing bubbles, neo-classical financial economics has had little to say about such phenomena because they are for the most part inconsistent with equilibrium in the frictionless framework that is the workhorse of modern asset pricing theory. Indeed, the results of (Santos and Woodford 1997) and (Loewenstein and Willard 2000, 2006) imply that rational asset pricing bubbles, defined as a wedge between the market price of a security and the lowest cost of a portfolio that produces the same or higher cash-flows,² cannot arise on positive net supply securities such as stocks as long as agents have to maintain nonnegative wealth.

The main contribution of this paper is to show that this need not be the case if some agents in the economy are subject to portfolio constraints. Specifically, I show that portfolio constraints can generate rational equilibrium bubbles on positive net supply assets even if the economy includes unconstrained agents who are only subject to a standard solvency condition that only requires them to maintain nonnegative wealth at all times. The intuition for this finding is that even though agents are price takers, the presence of constrained agents places an implicit liquidity provision constraint on the unconstrained agents through the market clearing conditions. Indeed, at times when the constraint binds, the unconstrained agent have to hold those securities that the constrained agent cannot, and this is where the mispricing finds its origin. Bubbles arise to incite unconstrained agents to provide a sufficient amount of liquidity, and they can persist in equilibrium because the nonnegative wealth constraint prevents unconstrained agents from indefinitely scaling their arbitrage position.

¹Examples include mispricing in equity carve-outs (Lamont and Thaler 2003a,b), dual class shares (Lamont and Thaler 2003a) and the simultaneous trading of shares from “Siamese twin” conglomerates such as Royal Dutch/Shell and Unilever NV/PLC. See among others (Rosenthal and Young 1990, Lamont and Thaler 2003a, Ashcraft, Garleanu, and Pedersen 2010, Garleanu and Pedersen 2011).

²By contrast, the literature on speculative or irrational bubbles (see e.g. (Miller 1977, Harrison and Kreps 1978, Scheinkman and Xiong 2003)) uses a different definition of the fundamental value that is not based on any cash-flow replication considerations and, therefore, cannot connect bubbles to the existence of limited arbitrage opportunities. Furthermore, these models are in general set in partial equilibrium as they assume the existence of a riskless technology in infinitely elastic supply.

To articulate this idea I consider a popular class of continuous-time equilibrium models with heterogeneous agents, multiple risky securities and portfolio constraints. I assume that the economy is populated by two groups of agents: unconstrained agents who are free to choose the composition of their portfolio subject to a standard solvency condition; and constrained agents who have logarithmic utility and are subject to convex portfolio constraints.³ Following the rational asset pricing bubble literature (see e.g. (Blanchard 1979, Blanchard and Watson 1982, Santos and Woodford 1997, Loewenstein and Willard 2000)), I define the bubble on a security as the difference between its market price and the smallest cost to an unconstrained agent of producing the same cash flows by using a dynamic trading strategy that maintains nonnegative wealth. This replication cost is referred to as the fundamental value of the security, and is uniquely determined by the trading opportunities available to unconstrained agents.

In this setting, I show that portfolio constraints can give rise to rational bubbles in equilibrium. Furthermore, I demonstrate that their presence can be assessed by studying the properties of a single economic state variable, the so-called weighting process, that is defined as the ratio of the agents' marginal utility of consumption.⁴ The optimality of the agents' decisions and the assumption of logarithmic utility jointly imply that the weighting process has no drift and, therefore, behaves like a martingale on time intervals of infinitesimal length. This does not mean, however that it is a martingale over the horizon of the model because integrability conditions are needed for a driftless process to be a martingale. This distinction may appear to be a technical subtlety, and is oftentimes overlooked, but it is in fact economically significant. In particular, this paper shows that the weighting process is a true martingale if and only if there are no bubbles in equilibrium. To illustrate this result I present two explicitly solved examples of economies with seemingly innocuous portfolio constraints in which the presence of bubbles is necessary for the existence of an equilibrium.

The first example I consider is a generalization of the restricted participation model of (Basak and Cuoco 1998) in which there is a single stock, agents have logarithmic utility and the constrained agent can neither short the stock nor invest more than a fixed fraction

³As shown by (Cvitanic and Karatzas 1992) the assumption of logarithmic utility is critical to obtain a tractable characterization of optimality under portfolio constraints. In a general equilibrium setting a similar assumption is imposed by (Detemple and Murthy 1997, Basak and Cuoco 1998, Basak and Croitoru 2000, 2006, Shapiro 2002, Gallmeyer and Hollifield 2008, Pavlova and Rigobon 2008, Soumare and Wang 2006) and (Schornick 2007) among many others.

⁴(Cuoco and He 1994) were the first to introduce this state variable in order to characterize equilibria in dynamic economies with incomplete markets. This construction has since become quite standard in the asset pricing literature, see (Basak and Cuoco 1998, Basak 2005, Basak and Croitoru 2000, 2006, Shapiro 2002, Gallmeyer and Hollifield 2008, Pavlova and Rigobon 2008, Soumare and Wang 2006) and (Schornick 2007) among others.

of his wealth into it. For an equilibrium to exist in this economy, the unconstrained agent must find it optimal to hold a leveraged position in the stock. As a result, the interest rate must be lower and the market price of risk must be higher than in an otherwise equivalent unconstrained economy. These local effects of the constraint go in the right direction, but they are not sufficient to reach an equilibrium. Indeed, I show that two conditions must be satisfied in equilibrium. First, the prices of the stock and the riskless asset must both include a bubble. Second, the bubble on the riskless must be larger in relative terms than that on stock. The intuition behind these findings is as follows: Since exploiting the bubble on one security generally means going long in the other, the unconstrained agent cannot benefit from both bubbles at the same time. Taking into account the fact that he must maintain nonnegative wealth, the unconstrained agent therefore exploits the limited arbitrage opportunity on the riskless asset as it requires less collateral per unit of initial profit. The fact that the stock also includes a strictly positive bubble increases its collateral value, and allows the unconstrained agent to increase his short position in the riskless asset to the level required by market clearing.

When the market consists in a single stock the equilibrium price of that security is simply given by the sum of the agents' wealth. When there are multiple stocks, the total value of the economy (i.e. the market portfolio) is still given by the sum of the agents' wealth but it is not clear *a priori* how this aggregate value should be split among the individual stocks. If the market portfolio is free of bubbles, then the existence of an equilibrium is sufficient to guarantee that the unconstrained agent's marginal utility can be used as a state price density to compute the individual equilibrium stock prices. On the contrary, if the market portfolio includes a bubble then one can no longer compute prices in this way. The second main contribution of this paper is to provide a way to compute the prices in such cases, and to show that there may exist a continuum of equilibria which correspond to different repartitions of the aggregate bubble among the stocks. Importantly, this indeterminacy is not only nominal but also real as different prices imply different optimal consumption paths. This striking implication of rational asset pricing bubbles is, to the best of my knowledge, novel to this paper.⁵

To illustrate the indeterminacies generated by bubbles and portfolio constraints, I consider an economy with two stocks and two logarithmic agents and assume that one of them faces a risk constraint that limits the volatility of his wealth. As in the limited

⁵While the role of portfolio constraints in expanding the set of equilibria has been recently pointed out by (Basak, Cass, Licari, and Pavlova 2008), it is important to note that the nature of the multiplicity in their model is different from that which occurs in mine. In their model there are several goods and multiplicity arises from the fact that agents can partially alleviate portfolio constraints by trading in the goods market. Furthermore, none of the equilibria they identify includes bubbles.

participation economy, the constraint prevents one agent from investing as much as desired in the stocks and thereby forces the other to hold a leveraged position. This implicit liquidity provision constraint makes rational bubbles on both the market portfolio and the riskless asset necessary for markets to clear and, relying on this result, I show that the economy admits a continuum of distinct equilibria.

If agents have collinear initial endowments (i.e. shares of the market portfolio) then the indeterminacy is only nominal in the sense that the consumption allocation, the interest rate and the market price of risk are fixed across the set of equilibria. On the contrary, if agents have non collinear endowments then the repartition of the bubble determines the initial distribution of wealth in the economy and, therefore, impacts the agents' consumption shares, the interest rate and the market prices of risk. I provide an explicit solution for the constrained agent's expected utility and show that as the share of the bubble that is attributed to a stock increases the agents' welfare move in opposite directions. To gain further insights into the set of equilibria I conduct a comparative static analysis of key equilibrium quantities in a model where the two stocks are ex-ante similar. My results show that despite this similarity the stock prices differ in all equilibria, and that variations across the set of equilibria can be quite significant. For example, when the model is calibrated to match the first two moments of the returns on the Standard and Poor's composite price index, the consumption share of the constrained agent varies from 35 to 70% depending on the repartition of the bubble among the stocks.

The rest of this paper is organized as follows. In Section 2 I present my main assumptions about the economy, the traded assets and the agents. In Section 3 I define the notion of asset pricing bubble that I use throughout the paper, and review some basic consequences of this definition. In Section 4 I derive conditions for the existence of equilibrium asset pricing bubbles and show how such bubbles can give rise to multiplicity and indeterminacy of equilibrium. Sections 5 and 6 contain the two examples and Section 7 concludes the paper. Appendices A and B contain all proofs.

2 The model

2.1 Information structure

I consider a continuous-time economy on the finite time span $[0, T]$ and assume that the uncertainty in the economy is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which carries a n -dimensional Brownian motion Z . All random processes are assumed to be adapted with respect to the usual augmentation of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ generated by the

Brownian motion, and all statements involving random quantities are understood to hold either almost surely or almost everywhere depending on the context.

2.2 Securities markets

Agents trade continuously in $n + 1$ securities: a locally riskless savings account in zero net supply and $n \geq 1$ risky assets, or stocks, in positive supply of one unit each. The price of the riskless asset evolves according to

$$S_{0t} = 1 + \int_0^t S_{0s} r_s ds$$

for some short rate process $r \in \mathbb{R}$ which is to be determined in equilibrium. On the other hand, stock i is a claim to a dividend process $e_i > 0$ that evolves according to

$$e_{it} = e_{i0} + \int_0^t e_{is} a_{is} ds + \int_0^t e_{is} v_{is}^\top dZ_s$$

for some exogenously specified drift and volatility processes $(a_i, v_i) \in \mathbb{R} \times \mathbb{R}^n$ where the notation $^\top$ denotes transposition. The vector of stock price processes is denoted by S and it will be shown that S_i evolves according to

$$S_{it} + \int_0^t e_{is} ds = S_{i0} + \int_0^t S_{is} \mu_{is} ds + \int_0^t S_{is} \sigma_{is}^\top dZ_s$$

for some initial value $S_{i0} \in \mathbb{R}_+$, drift $\mu_i \in \mathbb{R}$ and volatility $\sigma_i \in \mathbb{R}^n$ which are to be determined in equilibrium. To simplify the notation I denote by

$$e_t \equiv \sum_{i=1}^n e_{it} = e_0 + \int_0^t e_s a_s ds + \int_0^t e_s v_s^\top dZ_s$$

the aggregate dividend process, by μ the drift of the vector S and by σ the matrix obtained by stacking up the transpose of the individual stock volatilities.

2.3 Agents

The economy is populated by two agents who have homogenous beliefs about the state of the economy. The preferences of agent a are represented by

$$U_a(c) \equiv E \left[\int_0^T e^{-\rho s} u_a(c_s) ds \right]$$

for some utility functions $(u_a)_{a=1}^2$ and some constant $\rho \geq 0$ that represents the agents' common rate of time preference.⁶ In what follows, I assume that

$$u_2(c) \equiv \log(c)$$

and that u_1 satisfies textbook monotonicity and concavity assumptions as well as the Inada conditions $u_{1c}(0) = \infty$, $u_{1c}(\infty) = 0$. As a result, the marginal utility function u_{1c} admits an inverse which I will denote by I_1 .

Agent a is endowed with β_a units of the riskless asset and $0 \leq \alpha_{ai} \leq 1$ units of stock i . Leveraged positions are allowed as long as the agents' initial wealth levels

$$w_a \equiv \beta_a + \alpha_a^\top S_0 = \beta_a + \sum_{i=1}^n \alpha_{ai} S_{i0}$$

are strictly positive when computed at equilibrium prices. In what follows, I let $(\alpha, \beta) \equiv (\alpha_2, \beta_2)$ and set $\alpha_1 = 1 - \alpha$, $\beta_1 = -\beta$ so that markets clear.

2.4 Trading strategies and feasible plans

A trading strategy is a pair of processes $(\pi; \phi) \in \mathbb{R}^{1+n}$ satisfying

$$\int_0^T \|\sigma_t^\top \pi_t\|^2 dt + \int_0^T |\phi_t r_t + \pi_t^\top \mu_t| dt < \infty,$$

as well as $W_T \geq 0$, where

$$W_t = W_t(\pi; \phi) \equiv \phi_t + \pi_t^\top \mathbf{1}$$

is the corresponding wealth process, and $\mathbf{1} \in \mathbb{R}^n$ denotes a vector of ones. The scalar process ϕ represents the amounts invested in the riskless asset while the vector process π represents the amounts invested in each of the available risky assets. The constraint that the terminal wealth W_T is nonnegative is meant to guarantee that agents do not leave the market in debt at the terminal time.

⁶The assumption of homogenous beliefs and discount rates is imposed for simplicity of exposition and does not restrict the generality of the model. Under appropriate modifications, all the results of this paper can be shown to hold with heterogenous beliefs and/or discount rates.

A trading strategy $(\pi; \phi)$ is said to be self-financing for agent a given intermediate consumption at rate c if its wealth process satisfies

$$W_t = w_a + \int_0^t (\phi_s r_s + \pi_s^\top \mu_s - c_s) ds + \int_0^t \pi_s^\top \sigma_s dZ_s. \quad (1)$$

While the first agent is unconstrained in his portfolio choice I assume that the trading strategy of the second agent must belong to

$$\mathcal{C} \equiv \{(\pi; \phi) : \pi_t \in W_t(\phi, \pi)\mathcal{C}_t \text{ for all } t \in [0, T]\}$$

where $(\mathcal{C}_t)_{t \in [0, T]}$ is a family of closed convex sets which contain the origin. As is easily seen, this definition amounts to a constraint on the proportion of wealth invested the stocks and the property that the set \mathcal{C}_t contains the origin means that not investing in the stocks is always allowed. A wide variety of constraints, including short sales, collateral constraints and risk constraints can be modeled in this way, see (Cvitanic and Karatzas 1992) and Sections 5–6 below for various examples.

If agents were allowed to use any self-financing strategy then doubling strategies would be feasible and, as a result, no equilibrium could exist. To circumvent this, one can either impose integrability conditions to guarantee that

$$\xi_t W_t + \int_0^t \xi_s c_s ds \quad (2)$$

is a martingale for some suitable strictly positive state price density process ξ ; or require that agents maintain nonnegative wealth at all times as in Harrison and Pliska (1981) and Dybvig and Huang (1988). In the present context both approaches lead to similar results.⁷ However, the second one is more realistic and allows for a wider set of strategies so it is the one I will follow. Accordingly, I define a consumption plan c to be feasible for agent 1 if there exists a trading strategy $(\pi; \phi)$ that is self-financing given consumption

⁷If the trading strategy $(\pi; \phi)$ is self-financing given consumption at rate c and such that the process of Eq. (2) is a martingale for some state price density process $\xi > 0$ then

$$W_t = \frac{1}{\xi_t} E_t \left[\xi_T W_T + \int_t^T \xi_s c_s ds \right] \geq 0$$

and it follows that this class of strategies is contained in the class of strategies which maintain nonnegative wealth. On the other hand, Propositions 2 and 3 below imply that in the class of strategies which maintain nonnegative wealth the optimizer is such that the process of Eq. (2) is a martingale for some suitable state price density and it follows that the optimal strategies, and hence the equilibrium, do not depend on which class is used to define the individual optimization problems.

at rate c and whose wealth process is nonnegative. Feasible plans for agent 2 are defined similarly with the added requirement that the trading strategy belongs to \mathcal{C} .

2.5 Equilibrium

The concept of equilibrium that I use is similar to that of equilibrium of plans, prices and expectations which was introduced by (Radner 1972):

Definition 1. *An equilibrium is a pair of security price processes (S^0, S) and a set $\{c_a, (\pi_a; \phi_a)\}_{a=1}^2$ of consumption plans and trading strategies such that:*

1. *Given (S, S^0) the consumption plan c_a maximizes U_a over the feasible set of agent a and is financed by the trading strategy $(\pi_a; \phi_a)$.*
2. *Markets clear: $\phi_1 + \phi_2 = 0$, $\pi_1 + \pi_2 = S$ and $c_1 + c_2 = e$.*

In the model there are as many risky assets as there are sources of risks. As a result, one naturally expects that markets will be complete for the unconstrained agent in equilibrium. Unfortunately, and as shown by (Cass and Pavlova 2004), (Berrada, Hugonnier, and Rindisbacher 2007) and Hugonnier, Malamud, and Trubowitz (2011), this need not be the case in general. To avoid such difficulties, and in order to facilitate the definition of bubbles in the next section, I will restrict the analysis to equilibria in which the stocks volatility matrix has full rank at all times. Since none of the stocks are redundant in such an equilibrium, I will refer to this class as that of non redundant equilibria.

3 Asset pricing bubbles

In order to motivate the analysis of later sections, I start by reviewing the definition and basic properties of asset pricing bubbles.

3.1 Definition

Let (S^0, S) denote the securities prices in a given non redundant equilibrium and assume that there are no trivial arbitrage opportunities for otherwise the market could not be in equilibrium. As is well-known (see for example (Duffie 2001)), this assumption implies that there exists a process $\theta \in \mathbb{R}^n$ such that

$$\mu_{it} - r_t = \sigma_{it}^\top \theta_t \quad \text{and} \quad \int_0^T \|\theta_t\|^2 dt < \infty.$$

This process is referred to as the market price of risk and is uniquely defined since the volatility matrix σ has full rank in a non redundant equilibrium. Now consider the state price density defined by

$$\xi_{1t} \equiv \frac{1}{S_{0t}} \exp \left(- \int_0^t \theta_s^\top dZ_s - \frac{1}{2} \int_0^t \|\theta_s\|^2 ds \right). \quad (3)$$

Loosely speaking, the strictly positive quantity $\xi_{1t}(\omega)d\mathbb{P}(\omega)$ gives the value of one unit of consumption at date t in state ω , and thus constitutes the continuous-time counterpart of a standard Arrow-Debreu security. The next proposition makes this statement rigorous by showing that ξ_1 can be used as a pricing kernel to compute the cost to the unconstrained agent of replicating a stream of cash flows.

Proposition 1. *If c is a nonnegative process then*

$$F_t(c) \equiv E_t \left[\int_t^T \frac{\xi_{1s}}{\xi_{1t}} c_s ds \right]$$

is the minimal amount that the unconstrained agent needs to hold at time t in order to replicate the cash flows of a security that pays dividends at rate c while maintaining nonnegative wealth at all times.

Applying the above proposition to the valuation of stock i shows that starting from the amount $F_{it} \equiv F_t(e_i)$ the unconstrained agent can find a strategy that is self financing given consumption at rate e_i and maintains nonnegative wealth. Since stock prices are nonnegative in the absence of trivial arbitrages, a similar result can also be achieved by buying the stock at its market price and then holding it indefinitely. If $S_{it} = F_{it}$ then this buy and hold strategy is actually the cheapest way to replicate the dividends of the stock. If, on the contrary, $F_{it} < S_{it}$ then there exists a strategy that produces the same cash flows but at a lower cost by dynamically trading in the available securities.

Following the rational bubble literature (see (Santos and Woodford 1997, Loewenstein and Willard 2000, 2006) and (Heston, Loewenstein, and Willard 2007)) I will refer to F_{it} as the fundamental value of stock i because it is the value that would be attributed to the stock by any rational and unconstrained agent; and to

$$B_{it} \equiv S_{it} - F_{it} = S_{it} - E_t \left[\int_t^T \frac{\xi_{1s}}{\xi_{1t}} e_{is} ds \right]$$

as the bubble on its price. Notice that, since the fundamental value is the minimal amount necessary to replicate the dividends of the stock, the bubble defined above is

always nonnegative or zero. Furthermore, it can be shown that if the bubble is zero at time t then it is zero ever after that time. In other words, rational bubbles as defined above can burst at any point in time but they cannot be born after the start of the model, see (Loewenstein and Willard 2000) for details.

An important feature of the above definition is that the assessment of whether a given stock has a bubble is relative to other securities. To illustrate this point, consider two stocks whose dividends satisfy $e_{1t} = \phi e_{2t}$ for all t and some $\phi > 0$ as in the Royal Dutch/Shell example mentioned in the introduction.⁸ In such a case there are at least two ways of replicating the dividends of stock 1: One can either buy stock 1 at a cost of S_{1t} , or buy ϕ units of stock 2 at a cost of ϕS_{2t} . Since the fundamental value is the minimal amount necessary to replicate the dividends this implies $F_{1t} \leq \min(S_{1t}, \phi S_{2t})$ and the market price of stock 1 includes has a non zero bubble as soon as it exceeds that of ϕ units of stock 2.

3.2 Bubbles and limited arbitrage

At first glance, it might seem that bubbles are inconsistent with optimal choice, and thus also with the existence of an equilibrium, since their presence implies that two assets with the same cash flows have different prices.

To see that this is not the case, assume that stock i has a bubble and consider the strategy which sells short $x > 0$ units of the stock, buys the portfolio that replicates the corresponding dividends and invests the remaining strictly positive amount $x(S_{i0} - F_{i0}) = xB_{i0}$ in the riskless asset. This strategy requires no initial investment and has terminal value $xB_{i0}S_{0T} > 0$ so it does constitute an arbitrage opportunity in the usual sense. However, this strategy cannot be implemented on a standalone basis by the unconstrained agent because its wealth process

$$W_t(x) \equiv x(F_{it} - S_{it} + B_{i0}S_{0t}) = x(B_{i0}S_{0t} - B_{it})$$

can take negative values with strictly positive probability. The reason for this is that the market price of the stock and its fundamental value may diverge further before they eventually converge at the terminal date.

To undertake the above arbitrage trade while maintaining nonnegative wealth the agent needs to hold enough collateral, in the form of cash or securities, to absorb the

⁸In the case of Royal Dutch and Shell the profits of the conglomerate were shared on a 60/40 basis so $\phi = 1.5$ under the assumption that one share of stock 1 represents one share of Royal Dutch and that one share of stock 2 represents one share of Shell.

potential interim losses. For example, if the agent already holds one unit of the stock at the initial date then he can implement the above arbitrage trade with $x = 1$ since the corresponding wealth process

$$S_{it} + W_t(1) = S_{it} + (B_{i0}S_{0t} - B_{it}) = F_{it} + B_{i0}S_{0t}$$

is nonnegative at all times. As illustrated by the outcome of this collateralized trade, a stock with a bubble is simply a dominated asset. Indeed, the above strategy starts from the same value as the stock and produces the same intermediate dividends but, contrary to the stock, it also generates a strictly positive terminal lump dividend. In the terminology of [Harrison and Pliska \(1981\)](#) buying a stock whose price includes a bubble is equivalent to investing in a suicide strategy that turns a strictly positive amount of wealth into nothing by the terminal date.

The above discussion implies that starting from some strictly positive initial wealth the agent will be able to implement the arbitrage trade up to a certain size but will not be able to indefinitely increase its scale. In other words, the presence of a bubble implies the existence of an arbitrage opportunity but the unconstrained agent cannot exploit it fully because he is required to maintain nonnegative wealth. Importantly, this shows that bubbles are not incompatible with the existence of an equilibrium.

3.3 Bubbles on the riskless asset

The above discussion has focused on stock market bubbles, but asset pricing bubbles may be defined on any security, including the riskless asset.

Indeed, over the time interval $[0, T]$ the riskless asset can be viewed as a derivative security that pays a single lumpsum dividend equal to S_{0T} at time T . By slightly modifying the proof of [Proposition 1](#), it can be shown that the fundamental value of such a derivative security is

$$F_{0t} \equiv E_t \left[\frac{\xi_{1T}}{\xi_{1t}} S_{0T} \right].$$

On the other hand, the market value of this security is simply S_{0t} and this naturally leads to defining the riskless asset bubble as

$$B_{0t} \equiv S_{0t} - F_{0t} = S_{0t} \left(1 - E_t \left[\frac{\xi_{1T} S_{0T}}{\xi_{1t} S_{0t}} \right] \right). \quad (4)$$

The presence of a bubble implies that the riskless asset is a dominated asset. To see this, assume that there is a bubble and consider a strategy that buys the replicating portfolio and invests the amount $B_{00} > 0$ into the riskless asset. This strategy has an initial cost equal to 1 and its terminal value

$$F_{0T} + B_{00}S_{0T} = S_{0T}(1 + B_{00})$$

is strictly larger than that of the riskless asset. Said differently, the presence of a bubble implies that it is possible to create a synthetic savings account whose rate of return over $[0, T]$ is strictly higher than that of the riskless asset.

As can be seen by appending to the above strategy a short position in one unit of the riskless asset, the existence of a bubble exposes an arbitrage opportunity. However, the strategy that exploits this mispricing entails the possibility of interim losses and, thus, cannot be implemented by the unconstrained agent unless he holds sufficient collateral. As was the case for stocks, bubbles on the riskless asset are therefore consistent with both optimal choice and the existence of an equilibrium if agents are required to maintain nonnegative wealth. In fact, the examples in Sections 5 and 6 show that, when constrained agents are present in the economy, bubbles on both the stocks and the riskless asset may be necessary for markets to clear.

Remark 1. Equation (4) shows that the riskless asset has a bubble if and only if the process $M_t \equiv S_{0t}\xi_{1t}$ satisfies $E[M_T] < M_0 = 1$. Since the stock volatility has full rank in a non redundant equilibrium, this process is the unique candidate for the density of the risk-neutral probability measure and it follows that the existence of a bubble on the riskless asset is equivalent to the non existence of the risk-neutral probability measure. See (Loewenstein and Willard 2000) and (Heston et al. 2007).

4 Equilibrium asset pricing bubbles

In this section I provide a characterization of non redundant equilibria and determine conditions under which prices include bubbles in equilibrium. Furthermore, I show that the presence of bubbles can potentially generate indeterminacy of equilibrium.

4.1 Individual optimality

Since agent 1 is unconstrained, it follows from Proposition 1 that his dynamic portfolio and consumption choice problem can be formulated as

$$\sup_{c \geq 0} E \left[\int_0^T e^{-\rho t} u_1(c_t) dt \right] \quad \text{s.t.} \quad F_0(c) = E \left[\int_0^T \xi_{1t} c_t dt \right] \leq w_1.$$

The solution to this static problem can be obtained by applying standard Lagrangian techniques and is reported in the following:

Proposition 2. *In equilibrium, the optimal consumption and wealth of the unconstrained agent are given by $c_{1t} = I_1(y_1 e^{\rho t} \xi_{1t})$ and $W_{1t} = F_t(c_1)$ for some $y_1 > 0$.*

When the agent's ability to trade is restricted by portfolio constraints, the problem is more difficult to solve since ξ_1 no longer identifies the unique arbitrage free state price density. However, combining the duality approach of (Cvitanic and Karatzas 1992) with the assumption of logarithmic utility allows to derive the solution of the constrained problem in closed form as if the agent faced the unique state price density of a fictitious unconstrained economy.

Proposition 3. *In equilibrium, the optimal consumption, wealth process and trading strategy of the constrained agent are given by*

$$\begin{aligned} c_{2t} &= 1 / (y_2 e^{\rho t} \xi_{2t}) = W_{2t} / \eta(t), \\ \sigma_t^\top \pi_{2t} / W_{2t} &= \theta_{2t} \equiv \Pi(\theta_t | \mathcal{D}_t), \end{aligned}$$

for some constant $y_2 > 0$ where

$$\xi_{2t} \equiv \xi_{1t} \exp \left(- \int_0^t (\theta_{2s} - \theta_s)^\top dZ_s + \frac{1}{2} \int_0^t \|\theta_{2s} - \theta_s\|^2 ds \right) \quad (5)$$

represents his implicit state price density, $\Pi(\cdot | \mathcal{D}_t)$ denotes the projection on the convex set $\mathcal{D}_t \equiv \sigma_t^\top \mathcal{C}_t$ and the deterministic function

$$\eta(t) \equiv \int_t^T e^{-\rho(s-t)} ds = \frac{1}{\rho} (1 - e^{-\rho(T-t)}) \quad (6)$$

represents the inverse of his marginal propensity to consume.

4.2 Characterization of equilibrium

Since one of the agents is subject to portfolio constraints, the usual construction of a representative agent as a linear combination of the individual utility functions with constant weights is impossible. Nevertheless, the aggregation of individual preferences remains possible if one allows for stochastic weights in the definition of the representative agent's utility function (see (Cuoco and He 1994)).

This construction is very useful from the computational point of view as it reduces the search for an equilibrium to the specification of the weights. However, one should be cautious with its interpretation because a no-trade equilibrium for the representative agent cannot be decentralized into an equilibrium for the two agents constrained economy in general. As shown in the next section, the reason for this discrepancy is precisely that the equilibrium prices of the two agents economy can include bubbles whereas those of the representative agent economy cannot.

Consider the representative agent with utility function

$$u(c, \lambda_t) \equiv \max_{c_1+c_2=c} (u_1(c_1) + \lambda_t u_2(c_2))$$

where λ is a nonnegative process that evolves according to

$$d\lambda_t = \lambda_t m_t dt + \lambda_t \Gamma_t^\top dZ_t$$

for some drift m and volatility Γ that are to be determined in equilibrium. Since consuming the aggregate dividend is optimal for the representative agent, the process of marginal rates of substitution

$$\xi_{1t} = e^{-\rho t} \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} \tag{7}$$

identifies the unconstrained state price density. Furthermore, the individual plans must solve the representative agent's allocation problem and it follows that

$$c_{1t} = I_1(y_1 e^{\rho t} \xi_{1t}) = I_1(u_c(e_t, \lambda_t)), \quad c_{2t} = \frac{1}{y_2 e^{\rho t} \xi_{2t}} = \frac{\lambda_t}{u_c(e_t, \lambda_t)}. \tag{8}$$

for some strictly positive constants (y_1, y_2) . Combining these expressions with the results of Propositions 2 and 3 shows that the weighting process is

$$\lambda_t = \frac{u_{1c}(c_{1t})}{u_{2c}(c_{2t})} = \lambda_0 \frac{\xi_{1t}}{\xi_{2t}}. \tag{9}$$

Applying Itô's lemma to the unconstrained state price density in Eq. (7) and comparing the result with Eq. (3) allows to pin down the interest rate and the market price of risk as functions of the unknown coefficients m and Γ . On the other hand, using the above identity in conjunction with Proposition 3 allows to solve for m and Γ and putting everything back together yields the following:

Proposition 4. *In a non redundant equilibrium, the market price of risk and the interest rate are given by*

$$\theta_t = R_t(v_t - s_t\Gamma_t), \quad (10)$$

$$r_t = \rho + a_t R_t + s_t(P_t - R_t)\Gamma_t^\top \theta_t + \frac{1}{2} P_t R_t (\|s_t\Gamma_t\|^2 - \|v_t\|^2). \quad (11)$$

where $s_t \equiv c_{2t}/e_t$ and (R_t, P_t) denote the relative risk aversion and relative prudence of the representative agent at the point (e_t, λ_t) . Furthermore, the volatility of the equilibrium weighting process solves

$$\Gamma_t = \Pi(\theta_t|\mathcal{D}_t) - \theta_t = \Pi(v_t R_t - s_t R_t \Gamma_t | \mathcal{D}_t) - R_t(v_t - s_t \Gamma_t) \quad (12)$$

and its the drift is given by $m \equiv 0$

The structure uncovered by the above proposition is typical of equilibrium models with portfolio constraints, see (Detemple and Murthy 1997, Cuoco 1997, Basak and Cuoco 1998) and (Shapiro 2002) among others. In particular, it follows from Eq. (10) that expected stock returns satisfy a generalized consumption-based CAPM in which the weighting process acts as a second factor. This process accounts for the presence of portfolio constraints and encapsulates the differences in wealth across agents.

In order to complete the characterization of the equilibrium it is necessary to compute the equilibrium prices. To this end, the first step consists in determining whether or not there are bubbles in the price system as this will allow to pin down the relation between the stock prices and the state variables e and λ that drive the equilibrium.

4.3 Conditions for equilibrium bubbles

Combining Eq. (8) with the results of Propositions 2 and 3 shows that in equilibrium the agents' wealth are given by

$$W_{1t} = F_t(c_1) = E_t \left[\int_t^T e^{-\rho(s-t)} \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} I_1(u_c(e_s, \lambda_s)) ds \right] \quad (13)$$

for the unconstrained agent, and

$$W_{2t} = \eta(t)c_{2t} = E_t \left[\int_t^T \frac{\xi_{2s}}{\xi_{2t}} c_{2s} ds \right] = E_t \left[\int_t^T e^{-\rho(s-t)} \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} \frac{\lambda_t}{u_c(e_s, \lambda_s)} ds \right] \quad (14)$$

for the constrained agent. Since the sum of the agents' wealth equals the sum of the stock prices in equilibrium, the above expressions imply that the equilibrium price of the market portfolio is given by

$$\begin{aligned} \bar{S}_t &\equiv \mathbf{1}^\top S_t = E_t \left[\int_t^T e^{-\rho(s-t)} \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} \left(I_1(u_c(e_s, \lambda_s)) + \frac{\lambda_t}{u_c(e_s, \lambda_s)} \right) ds \right] \\ &= E_t \left[\int_t^T e^{-\rho(s-t)} \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} \left(e_s + \frac{\lambda_t - \lambda_s}{u_c(e_s, \lambda_s)} \right) ds \right] \end{aligned} \quad (15)$$

where the last equality follows from the goods market clearing condition. On the other hand, since the market portfolio can be seen as a security that pays dividends at rate e , it follows from Proposition 1 and Eq. (7) that its fundamental value is

$$\bar{F}_t \equiv F_t(e) = E_t \left[\int_t^T e^{-\rho(s-t)} \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} e_s ds \right].$$

Comparing the two previous expressions shows that in equilibrium the price of the market portfolio includes a bubble that is given by

$$\bar{B}_t = E_t \left[\int_t^T e^{-\rho(s-t)} \frac{\lambda_t - \lambda_s}{u_c(e_t, \lambda_t)} ds \right] = \int_t^T e^{-\rho(s-t)} \frac{\lambda_t - E_t[\lambda_s]}{u_c(e_t, \lambda_t)} ds. \quad (16)$$

The results of Proposition 4 imply that in a non redundant equilibrium the weighting process evolves according to

$$\lambda_t = \lambda_0 + \int_0^t \lambda_s \Gamma_s^\top dZ_s$$

for some volatility Γ that can be obtained by solving Eq. (12). This shows that the weighting process is a stochastic integral with respect to Brownian motion and one might therefore be tempted to conclude that it is a martingale in which case the aggregate stock market bubble vanishes. Despite its natural appeal, this conclusion is erroneous in general. Indeed, the fact that the weighting process is driftless implies that it is a local martingale, which means that it behaves like a martingale over time intervals

of infinitesimal length, but additional conditions are required to guarantee that it is a true martingale. This distinction may appear to be a technical subtlety but it is in fact economically significant as it determines whether or not stock price bubbles arise in equilibrium.

Theorem 1. *The equilibrium stock prices are free of bubbles if and only if the weighting process is a true martingale.*

Since the weighting process is a positive local martingale, it is a supermartingale (see (Karatzas and Shreve 1998, p.36)) and will be a martingale if and only if it is constant in expectation.⁹ Thus, the above theorem shows that bubbles arise in equilibrium if and only if the weight of the constrained agent is strictly decreasing in expectation. This suggests that bubbles are related to the opportunity costs that the portfolio constraint imposes on agent 2. To confirm this intuition observe that

$$\bar{B}_t = \bar{S}_t - F_t(e) = W_{2t} - F_t(c_2) = E_t \left[\int_t^T \left(\frac{\xi_{2s}}{\xi_{2t}} - \frac{\xi_{1s}}{\xi_{1t}} \right) c_{2s} ds \right] \quad (17)$$

where the second equality follows from Proposition 2 and the clearing of the goods market. The constrained agent's preferences being strictly increasing, his wealth can be interpreted as the minimal amount needed to replicate his consumption with a constrained portfolio. Thus, the above identity shows that bubbles arise if and only if the cost of replicating the constrained agent's consumption is strictly higher for him than for the unconstrained agent. In other words, bubbles signal that there is "money left on the table" in the sense that, at the equilibrium prices, both agents can be made strictly better off by delegating the management of all wealth to the unconstrained agent.¹⁰

Remark 2 (Unconstrained economies and heterogenous beliefs). Absent constraints, the weighting process is automatically a martingale since it is constant. Thus, a direct implication of Theorem 1 is that in unconstrained economies with complete markets there can be no equilibrium bubble on positive net supply securities.

Similarly, if agents are unconstrained but have heterogenous beliefs about the state of the economy then it follows from well-known results (see e.g. (Basak 2005)) that the

⁹A local martingale which is not a true martingale is said to be a *strict* local martingale, see (Elworthy, Li, and Yor 1999). Apart from the study of asset pricing bubbles strict local martingales play an important role in stochastic volatility models (see (Sin 1998)) and in the modeling of relative arbitrages (see (Fernholz, Karatzas, and Kardaras 2005) and (Fernholz and Karatzas 2010))

¹⁰Note that, in contrast to the result of Theorem 1, this interpretation of the aggregate stock market bubble and the validity of Eq. (17) only require that the constrained agent's preferences are strictly increasing and therefore does not depend on the assumption of logarithmic utility.

equilibrium weighting process is constant although agent 1 now has a state dependent utility function given by $k_t u_1(c)$ where $k_t \equiv E_t[d\mathbb{P}_1/d\mathbb{P}_2]$ is the density of his subjective probability measure with respect to that of agent 2 taken as a reference. Being constant, the weighting process is a martingale and it follows that there can be no stock market bubbles in unconstrained economies with heterogenous beliefs and complete markets. Importantly, this conclusion does not depend on the way in which agents form their anticipations and, therefore, applies indifferently to models in which agents are Bayesian learners and to models in which they have boundedly rational beliefs (see (Kogan, Ross, Wang, and Westerfield 2006, Berrada 2009) and (Dumas, Kurshev, and Uppal 2009)).

4.4 Bubbles and multiplicity of equilibria

Having identified the conditions under which the price system includes bubbles, I now turn to the determination of the equilibrium stock prices. In particular, the following proposition gives necessary and sufficient condition for a process to be an equilibrium stock price process.

Proposition 5. *Let $S \in \mathbb{R}^n$ be a nonnegative process, assume that its volatility matrix σ is invertible and set*

$$\lambda_t = \lambda_0 \exp \left(-\frac{1}{2} \int_0^t \|\Gamma_s\|^2 ds + \int_0^t \Gamma_s^\top dZ_s \right)$$

where the process Γ solves Eq. (12) and the constant $\lambda_0 > 0$ solves

$$\beta + \alpha^\top S_0 = \frac{\lambda_0 \eta(0)}{u_c(e_0, \lambda_0)}. \quad (18)$$

Then the process S is the stock price in a non redundant equilibrium if and only if it satisfies the aggregate restriction

$$\mathbf{1}^\top S_t = E_t \left[\int_t^T e^{-\rho(s-t)} \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} e_s ds + \int_t^T e^{-\rho(s-t)} \frac{\lambda_t - \lambda_s}{u_c(e_t, \lambda_t)} ds \right], \quad (19)$$

and the nonnegative process

$$N_{it} \equiv e^{-\rho t} \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} S_{it} + \int_0^t e^{-\rho s} \frac{u_c(e_s, \lambda_s)}{u_c(e_0, \lambda_0)} e_{is} ds$$

is a local martingale for each i .

The conditions of the above proposition can be explained as follows. First, the requirement that the process N_i has no drift implies that the candidate stock prices offer the market price of risk of Proposition 4. In conjunction with the definition of the weighting process this guarantees the optimality of the equilibrium consumption allocation and implies that the agents' wealth processes are given by Eqs. (13) and (14). The aggregate restriction of Eq. (19) then implies that the sum of these wealth processes coincides with the market portfolio and guarantees that the market for the riskless asset clears at all times. This further implies that the agents' optimal portfolios satisfy $\sigma_t^\top (\pi_{1t} + \pi_{2t} - S_t) = \mathbf{0}$ and it follows that the stock market clears since the volatility matrix is assumed to be invertible.

If the weighting process fails to be a martingale then Proposition 5 pins down the market portfolio but it does not allow to uniquely determine the individual prices since at least one of the stocks includes a bubble. This has two important consequences. First, bubbles can potentially give rise to multiple equilibria that correspond to different repartitions of the aggregate bubble among the stocks. Second, bubbles can potentially generate real indeterminacy, and thereby have an impact on the agents' welfare, since the weighting process depends on the equilibrium stock prices and determines the equilibrium allocation, market price of risk and interest rate.

These two implications will be illustrated in Section 6 below where I present an explicitly solved example of a multiple stocks economy in which the presence of a risk related portfolio constraint generates both nominal and real indeterminacy of equilibria through the occurrence of bubbles.

5 Limited participation

In this section I study a single stock economy that generalizes the restricted stock market participation model of (Basak and Cuoco 1998). Using the results of the previous sections, I show that the equilibrium of this economy is unique and includes bubbles on both the stock and the riskless asset.

5.1 The economy

Consider an economy with a single stock whose dividend evolves according to

$$e_t = e_0 + \int_0^t e_s a ds + \int_0^t e_s v dZ_s.$$

for some constants $e_0 > 0$, $a \in \mathbb{R}$ and $v > 0$. Agents have homogenous logarithmic preferences,¹¹ and I assume that the portfolio constraint set is given by

$$\mathcal{C}_t = \mathcal{C}_0 \equiv [0, 1 - \varepsilon]$$

for some constant $\varepsilon \in (0, 1]$. This is a participation constraint which implies that the agent cannot short the stock and must keep at least $\varepsilon\%$ of his wealth in the riskless asset at all times. In particular, the case $\varepsilon = 1$ coincides with the restricted stock market participation model proposed by (Basak and Cuoco 1998).¹²

To complete the description of the economy, I assume that the initial wealth of the constrained agent is given by $w_2 = \beta + \alpha S_0$ for some $\alpha \in [0, 1]$, $\beta \geq 0$ such that

$$\beta < \eta(0)(1 - \alpha)e_0.$$

This restriction guarantees that the unconstrained agent does not start so deeply in debt that he can never pay back from the dividend supply. As in (Basak and Cuoco 1998) this condition is necessary and sufficient for the existence of an equilibrium.

5.2 The equilibrium

Under the assumption of homogenous logarithmic utility, the representative agent's utility function is explicitly given by

$$u(c, \lambda) = (1 + \lambda) \log c + \lambda \log \lambda - (1 + \lambda) \log(1 + \lambda). \quad (20)$$

Differentiating the right hand side and substituting the result into Eqs. (8), (13) and (14) shows that in equilibrium the agents' consumption and wealth processes are explicitly given by

$$c_{2t} = s_t e_t = \frac{\lambda_t e_t}{1 + \lambda_t} = \frac{W_{2t}}{\eta(t)}, \quad c_{1t} = \frac{e_t}{1 + \lambda_t} = \frac{W_{1t}}{\eta(t)} \quad (21)$$

¹¹This assumption is imposed for simplicity of exposition. In particular, the model of this section can be solved with similar, albeit less explicit, conclusions under the assumption that the unconstrained agent has a power utility function. See (Prieto 2011).

¹²The limiting case $\varepsilon = 0$ corresponds to a situation in which agent 2 can neither borrow nor short the risky asset. Since agents have homogenous preferences, they would not invest in the riskless asset absent portfolio constraint and it follows that this case leads to an unconstrained equilibrium.

Since there is a single traded stock in the economy, its equilibrium price is given by the sum of the agents' wealth processes. This gives

$$S_t = W_{1t} + W_{2t} = \eta(t)(c_{1t} + c_{2t}) = \eta(t)e_t. \quad (22)$$

and it follows that, as usual in models with logarithmic preferences, the volatility of the stock equals that of the aggregate dividend. Using this volatility in conjunction with Eqs. (12) and (18) shows that the weighting process evolves according to

$$\lambda_t = \frac{w_2}{\eta(0)e_0 - w_2} - \int_0^t \lambda_s(1 + \lambda_s)v_\lambda dZ_s \quad (23)$$

with $v_\lambda \equiv \varepsilon v$. Finally, inserting the volatility of the weighting process into the formulas of Proposition 4 shows that the equilibrium market price of risk and interest rate are given in closed form by

$$\theta_t = v(1 + \varepsilon\lambda_t), \quad (24)$$

$$r_t = \rho + a - \|v\|^2(1 + \varepsilon\lambda_t). \quad (25)$$

The stochastic differential Eq. (23) completely identifies the weighting process and hence the equilibrium. In particular, the existence and uniqueness of a strictly positive solution to this equation implies the existence and uniqueness of the equilibrium, and the properties of this solution determine whether equilibrium prices include bubbles.

Proposition 6. *Equation (23) admits a unique strictly positive solution which is a local martingale but not a martingale. Consequently,*

1. *There exists a unique equilibrium that is given by Eqs. (21), (22) and (25) where λ_t is the unique solution to Eq. (23).*
2. *In the unique equilibrium, the prices of the stock and the riskless asset both include bubbles that are given by*

$$\frac{B_t}{S_t} = b(t, s_t) \leq b_0(t, T, s_t) = \frac{B_{0t}}{S_{0t}} \quad (26)$$

where $s_t = \lambda_t/(1 + \lambda_t)$ represents the constrained agent's consumption share, the functions b and b_0 are defined by

$$b_0(t, T, s) \equiv s^{-1/\varepsilon} H(T - t, s; a_0),$$

$$b(t, s) \equiv \frac{1}{\rho\eta(t)} H(T - t, s; a_1) + \frac{\eta'(t)}{\rho\eta(t)} H(T - t, s; 1),$$

for some constants a_0, a_1 given in the appendix and

$$H(\tau, s; a) \equiv s^{\frac{1+a}{2}} \Phi(d_+(\tau, s; a)) + s^{\frac{1-a}{2}} \Phi(d_-(\tau, s; a)),$$

$$d_{\pm}(\tau, s; a) \equiv \frac{1}{\|v_{\lambda}\|\sqrt{\tau}} \log s \pm \frac{a}{2} \|v_{\lambda}\|\sqrt{\tau},$$

where Φ denotes the standard normal cumulative distribution function.

Equations (24) and (25) show that limited participation always implies a higher market price of risk and a lower interest rate than in an unconstrained economy. To understand this feature, note that in the absence of portfolio constraints the agents do not trade in the riskless asset since they have homogenous preferences. In the constrained economy, however, the second agent is forced to invest a strictly positive fraction of his wealth in the riskless asset. The unconstrained agent must therefore be induced to become a net borrower in equilibrium and it follows that the interest rate must decrease and the market price of risk must increase compared to an unconstrained economy.

These local effects of the constraint go in the right direction but they are not sufficient to reach an equilibrium. Indeed, the second part of Proposition 6 shows that the equilibrium prices of both the stock and the riskless asset include bubbles. As the unconstrained agent cannot benefit from both bubbles simultaneously, the question is to determine which of the two bubbles he chooses to exploit. Taking into account the nonnegative wealth constraint, one naturally expects the unconstrained agent to arbitrage the bubble on the riskless asset because, as shown by Eq. (26), it requires less collateral per unit of initial profit. This intuition will be confirmed in the next section where I compute the dynamic trading strategies that allow to benefit from the bubbles and show that the equilibrium strategy of the unconstrained agent can be seen as the combination of an all equity portfolio and a continuously resettled arbitrage position that exploits the bubble on the riskless asset.

Remark 3 (Infinite horizon economies). The result of Proposition 6 remains qualitatively unchanged if the economy has an infinite horizon. In particular, it can be shown that the infinite horizon economy admits a unique equilibrium that is given by Eqs. (21), (22)

and (25) but with the constant $1/\rho$ in place of the function $\eta(t)$ and in which the prices of both securities include bubbles.

The bubble component in the equilibrium price of the stock can be computed by taking the limit of the corresponding quantity in Proposition 6. This gives

$$\lim_{T \rightarrow \infty} B_t = \frac{e_t}{\rho} \sqrt{s_t^{1+a_1}} = \frac{e_t}{\rho} \left(\frac{\lambda_t}{1 + \lambda_t} \right)^{\frac{1}{2}(1+a_1)}$$

where λ_t is the unique solution to Eq. (23). On the other hand, since the functional form of the market price of risk does not depend on the horizon of the economy it follows from Proposition 6 that over some fixed time interval $[0, \tau]$ the price of the riskless asset includes a bubble component that is explicitly given by

$$\frac{B_{0t}(\tau)}{S_{0t}} = 1 - E_t \left[\frac{\xi_{1\tau} S_{0\tau}}{\xi_{1t} S_{0t}} \right] = b_0(t, \tau, s_t), \quad t \leq \tau.$$

The only thing that changes when going from a finite to an infinite horizon is the comparison between the bubbles. Indeed, with an infinite horizon, the bubbles can no longer be compared over the lifespan of the stock and it becomes necessary to compute the stock bubble over a finite investment horizon before it can be compared to that on the riskless asset. With this modification it can be shown that the conclusion of Proposition 6 carries over to the infinite horizon setting in the sense that the relative bubble on the riskless asset dominates that on the stock over any finite investment horizon.

5.3 Arbitrage strategies

Since the equilibrium prices of the stock and the riskless asset both include bubbles, it follows from the results of Section 3 that the unconstrained agent benefits from limited arbitrage opportunities. To determine how the unconstrained agent exploits these opportunities in equilibrium I start by computing the trading strategies that allow to benefit from the presence of bubbles on the stock and the riskless asset.

Consider first the case of the stock. According to Proposition 6 the fundamental value of the stock is given by

$$F_t = S_t - B_t = (1 - b(t, s_t))S_t < S_t. \quad (27)$$

As explained in Section 3 this inequality means that there exists a synthetic asset that produces the same cash flows as the stock but at a strictly lower market price. To

arbitrage this difference starting at some fixed date $\tau \in [0, T)$ the unconstrained agent must sell the stock short, go long in the dynamic portfolio implied by the fundamental value and invest the strictly positive proceeds of these operations in the riskless asset.¹³ This dynamic trading strategy requires no initial investment at date τ and its value process is given by

$$A_t(\tau) \equiv S_{0t}(B_\tau/S_{0\tau}) - S_t + F_t = S_{0t}(B_\tau/S_{0\tau}) - b(t, s_t)S_t.$$

The following proposition relies on Itô's lemma, the definition of the fundamental value process and the results of Proposition 6 to identify the stock and riskless asset positions required to manage the portfolio until the terminal time where it generates the strictly positive arbitrage profit $A_T(\tau) = S_{0T}(B_\tau/S_{0\tau})$.

Proposition 7. *The value of the stock bubble arbitrage is given by*

$$A_t(\tau) = W_t(\pi(\tau); \phi(\tau)), \quad \tau \leq t \leq T,$$

where the trading strategy

$$\begin{aligned} \pi_t(\tau) &\equiv \left[\varepsilon s_t \frac{\partial b}{\partial s}(t, s_t) - b(t, s_t) \right] S_t, \\ \phi_t(\tau) &\equiv A_t(\tau) - \pi_t(\tau) = S_{0t}(B_\tau/S_{0\tau}) - \varepsilon s_t \frac{\partial b}{\partial s}(t, s_t) S_t, \end{aligned}$$

is self-financing given no intermediate consumption.

The above trading strategy produces a strictly positive pay-off without requiring any investment and therefore constitutes an arbitrage opportunity but this arbitrage cannot be implemented by the unconstrained agent on an arbitrary scale. Indeed, the market value and the fundamental value may diverge further before they converge, and thereby generate interim losses at any time prior to maturity. It follows that the unconstrained agent needs to hold a sufficient amount of collateral to weather these losses while maintaining nonnegative wealth. This imposes a limit on the size of the arbitrage that he can implement and explains why bubbles can persist.

¹³An arbitrage also obtains if instead of being invested in the riskless asset the proceeds are invested in a portfolio that is self-financing given no consumption and maintains nonnegative wealth. I present the details only for the case where the proceeds are invested in the riskless asset because this is the type of strategy that is employed by the unconstrained agent in equilibrium, see Eq. (29) below.

Abstracting from the investment of the initial proceeds of the trade in the riskless asset, the above dynamic strategy requires the agent to take a position in

$$\frac{\pi_t(\tau)}{S_t} = \varepsilon s_t \frac{\partial b}{\partial s}(t, s_t) - b(t, s_t)$$

units of the stock and to finance this position by borrowing or lending at the riskless rate. The direction of these positions, i.e. whether the agent is net long or short in

Insert Figure 1 here

each of the two securities, is not constant and depends on both the consumption share of constrained agents and the remaining time until the terminal date. Specifically, the left panel of Figure 1 reveals that for typical parameter values the strategy implies a net short position in the stock if the time to maturity is sufficiently long and/or the consumption share of the constrained agent is sufficiently large, and a net long position otherwise.

Consider now the riskless asset, and recall from Proposition 6 that the fundamental value of the riskless asset over the horizon of the economy is

$$F_{0t} = S_{0t} - B_{0t} = (1 - b_0(t, T, s_t))S_{0t} < S_{0t}.$$

To arbitrage the difference between this replication value and the market value of the riskless asset starting at some fixed date $\tau \in [0, T)$ the unconstrained agent must borrow $S_{0\tau}$, go long in the replicating portfolio associated with the fundamental value and invest the strictly positive proceeds of these operations in the riskless asset. This trade requires no initial investment at date τ and its value process is given by

$$A_{0t}(\tau) \equiv S_{0t}(B_{0\tau}/S_{0\tau}) - S_{0t} + F_{0t} = S_{0t}(B_{0\tau}/S_{0\tau}) - b_0(t, T, s_t)S_{0t}.$$

In particular, this trade generates a strictly positive pay-off at the terminal time and therefore constitutes an arbitrage opportunity as it does not require any initial investment. The following proposition relies on arguments similar to those of Proposition 7 in order to derive the corresponding stock and riskless asset positions.

Proposition 8. *The value of the riskless bubble arbitrage is given by*

$$A_{0t}(\tau) = W_t(\pi_0(\tau); \phi_0(\tau)), \quad \tau \leq t \leq T,$$

where the trading strategy

$$\begin{aligned}\pi_{0t}(\tau) &\equiv \varepsilon s_t \frac{\partial b_0}{\partial s}(t, T, s_t) S_{0t}, \\ \phi_{0t}(\tau) &\equiv A_{0t}(\tau) - \pi_{0t}(\tau) = \left[b_0(\tau, T, s_t) - b_0(t, T, s_t) - \varepsilon s_t \frac{\partial b_0}{\partial s}(t, T, s_t) \right] S_{0t}\end{aligned}\tag{28}$$

is self-financing given no intermediate consumption.

As was the case for the stock arbitrage of Proposition 7 the above dynamic trading strategy constitutes an arbitrage opportunity that cannot be implemented to an arbitrary scale by the unconstrained agent. Indeed, this strategy can have negative value at any time prior to maturity and therefore requires the agent to hold sufficient collateral to absorb the induced losses while maintaining nonnegative wealth.

Abstracting from the investment of the initial proceeds in the riskless asset, the above dynamic trading strategy requires the agent to hold a long position in

$$\frac{\pi_{0t}(\tau)}{S_t} = \varepsilon s_t \frac{\partial b_0}{\partial s}(t, T, s_t) \frac{S_{0t}}{S_t} > 0$$

units of the stock and to finance this investment by borrowing at the riskless rate. While the direction of these positions is constant, their size is not. Specifically, the right panel of Figure 1 shows that the size of the stock position is bell-shaped as a function of both the duration of the trade and the consumption share of the constrained agent.

To determine how the unconstrained agent exploits these arbitrage opportunities in equilibrium, consider his optimal trading strategy which, according to Proposition 2 and Eq. (24), is explicitly given by

$$(\pi_{1t}; \phi_{1t}) \equiv (W_{1t}(\theta_t/v); W_{1t}(1 - \theta_t/v)) = ((1 + \varepsilon \lambda_t)W_{1t}; -\varepsilon \lambda_t W_{1t}).$$

Combining this expression with Eq. (28) and the definition of the consumption share process shows that this optimal trading strategy can be decomposed as

$$(\pi_{1t}; \phi_{1t}) = (W_{1t}; 0) + k_t (\pi_{0t}(t); \phi_{0t}(t))\tag{29}$$

with

$$\frac{1}{k_t} \equiv \frac{S_t}{S_{0t}} \frac{\partial b_0}{\partial s}(t, T, s_t) > 0.$$

Since agents have identical preferences it follows from standard risk sharing results that they would not find it optimal to invest in the riskless asset if none of them was subject to portfolio constraints. As a result, the first part of the above decomposition can be interpreted as the all equity strategy that the agent would have implemented in the absence of portfolio constraints. The second part, on the other hand, is a continuously resettled strategy that exploits the bubble on the riskless asset by going short in that asset and simultaneously long in the stock.

As explained above and in Section 3.2, the latter strategy is not admissible on a standalone basis as it entails the possibility of interim losses. However, when coupled with a large enough collateral investment in the stock this strategy becomes admissible and allows the unconstrained agent to benefit from the limited arbitrage opportunity induced by the bubble on the riskless asset. The fact that the stock also has a bubble increases its collateral value, and thereby allows the agent to increase the size of his short position in the riskless asset to the level required by market clearing.

5.4 Comparative statics

In order to illustrate the properties of the equilibrium, I now analyze the impact of the model horizon and the weight of constrained agents on key equilibrium quantities. To facilitate the economic interpretation, I will express all quantities as functions of time, the dividend and the endogenous state variable

$$s_t \equiv s(\lambda_t) = \frac{\lambda_t}{1 + \lambda_t} = \frac{c_{2t}}{c_{1t} + c_{2t}} = \frac{W_{2t}}{W_{1t} + W_{2t}}$$

which represents both the consumption share of the constrained agent and the fraction of the total wealth that he holds.

Since the function s is increasing and concave, it follows from Jensen's inequality and Proposition 6 that the process s_t is a supermartingale. This implies that the consumption share of the constrained agent is expected to decline over time. Because the weighting process is a local martingale this would be the case even if there was no bubble on the stock, but one expects that the presence of a bubble increases the speed at which the constrained agent's consumption share decreases.

This intuition is confirmed by Figure 2 which relies on the results of the appendix (see Lemma A.4) to plot the expected consumption share of the constrained agent as a function of the horizon. In particular, the middle curve in the figure shows that starting from an initial share of 50% the constrained agent is expected to consume only about a quarter of

Insert Figure 2 here

the total endowment after twenty years.¹⁴ This rapid decrease of the constrained agent's consumption share shows that explanations of asset pricing puzzles based on limited participation should be taken with care as they can only be transitory. For example, (Basak and Cuoco 1998) report that, with $\varepsilon = 1$, $v = 0.0357$ and $\rho = 0.001$ the fraction of constrained agents that is needed to match the average interest rate and market price of risk estimated by (Mehra and Prescott 1985) is approximately equal to 88%. But this calibration is really short lived since, with these parameters, the fraction of constrained agents is expected to decrease by approximately 11% over the first fifteen years.

As explained above, the bubbles arise because the unconstrained agent must find it optimal to hold a leveraged position. On the other hand, Eq. (29) and the definition of the constrained agent's consumption share show that the risky part of the unconstrained agent's equilibrium portfolio can be written as

$$\pi_{1t} = W_{1t} + \varepsilon W_{2t} = W_{1t} + \varepsilon s_t \eta(t) e_t.$$

It follows that the optimal leverage increases with the consumption share of the constrained agent, the horizon of the economy and the tightness of the constraint. Since the optimal leverage is determined by the contribution of the bubbles to the equilibrium prices, this suggests that the size of the bubbles in percentage of the underlying prices should be increasing functions of both s and T .

This intuition is confirmed by Figure 3 which plots the relative contribution of the bubbles to the equilibrium prices of the stock and the riskless asset as functions of the

Insert Figure 3 here

model horizon and the constrained agent's consumption share. The figure also shows that the bubble component can be quite large. For example, the right panel shows that with $s_0 = 50\%$ of constrained agents and a relatively short horizon of twenty five years

¹⁴This figure should be compared to those found in the literature studying the effect of bounded rationality on asset prices, see e.g. (Kogan et al. 2006), (Berrada 2009) and (Dumas et al. 2009). These models show that the consumption share of irrational traders decreases over time but the speed of this decline is usually rather slow. For example, (Berrada 2009) finds that when both agents have logarithmic utility it takes about 80 years for the consumption share of the boundedly rational agent to decrease from 50% to 25%.

the bubbles account for 20% of the equilibrium stock price and almost 60% of the riskless asset price.

6 Indeterminacy in a model with two stocks

In this section I study an economy with two stocks in which one of the agent is subject to a constraint that limits the volatility of his wealth.¹⁵ Relying on previous results I establish that any non redundant equilibrium for this economy includes a bubble on the price of the market portfolio and show that this gives rise to both multiplicity and real indeterminacy of equilibrium.

6.1 The economy

Consider an economy with two stocks, and assume that the aggregate dividend process evolves according to

$$e_t = e_0 + \int_0^t e_s a ds + \int_0^t e_s v^\top dZ_s$$

for some constants $(e_0, a) \in \mathbb{R}_+ \times \mathbb{R}$ and $v \in \mathbb{R}^2$ where Z is a two dimensional Brownian motion. Rather than modeling the individual dividends of the stocks, I assume that the dividend share x_{1t} of the first stock evolves according to

$$x_{1t} = x_{10} + \int_0^t x_{1s}(1 - x_{1s})v_x^\top dZ_s$$

for some constants $x_{10} \in (0, 1)$, $v_x \in \mathbb{R}^2$ and set $x_{2t} = 1 - x_{1t}$. In order to simplify the presentation of the results of this section I further assume that the vectors v and v_x are orthogonal.¹⁶ This cash flow model is a special case of that used by (Menzly, Santos, and Veronesi 2004) in their study of stock return predictability and I refer to them for details on the properties of the induced dividend processes.

Agents have homogenous logarithmic preferences with discount rate $\rho \geq 0$ and I assume that the portfolio constraint set is given by

$$\mathcal{C}_t \equiv \{p \in \mathbb{R}^2 : \|\sigma_t^\top p\| \leq (1 - \varepsilon)\|v\|\} \quad (30)$$

¹⁵The number of stocks in this example is limited to two for simplicity of exposition but the results easily generalize to economies with an arbitrary number of stocks.

¹⁶This specification is adopted for analytical convenience only. The crucial properties needed for the validity of the results of this section are that the volatility of the dividend share be linearly independent from that of the aggregate dividend and that the latter be non zero.

for some constant $\varepsilon \in (0, 1)$ where σ_t denotes the endogenous stock volatility matrix. This is a risk constraint which implies that the volatility of the agent's wealth cannot exceed a fixed threshold.¹⁷ Since agents are myopic their consumption-to-wealth ratios are deterministic at the optimum, and it follows that the price-dividend ratio of the market portfolio is deterministic in equilibrium. This in turn implies that the equilibrium volatility of the market portfolio is equal to that of the aggregate dividend and it follows that the above constraint forces the agent to choose a portfolio whose returns are less volatile than those of the stock market as a whole.

In order to complete the description of the economic environment, I assume that the initial wealth of the constrained agent is given by $w_2 = \beta + \alpha_1 S_{10} + \alpha_2 S_{20}$ for some constants $\alpha_i \in [0, 1]$ and $\beta \geq 0$ such that

$$\beta < (1 - \max(\alpha_1, \alpha_2)) \eta(0) e_0 \quad (31)$$

where the deterministic function η is defined by Eq. (6). As in the previous example, this restriction implies that the unconstrained agent does not start so deeply in debt that he can never repay from the dividend supply, and is sufficient to guarantee that the set of non redundant equilibria is non empty.

Remark 4 (Proportional dividends). If the volatility $\|v_x\|$ of the dividend share is set to zero then the two stocks pay proportional dividends as in the Royal Dutch/Shell example discussed in Section 3.1 and any deviation from parity immediately implies the existence of a bubble on one of the stocks. The results of this section extend to this limiting case with a few caveats. Specifically, markets are now automatically complete with respect to the filtration generated by the single Brownian source of risk given by

$$\hat{Z}_t = \frac{1}{\|v\|} \left(\int_0^t \frac{de_s}{e_s} - at \right) = \frac{v^\top Z_t}{\|v\|}$$

and all processes of interest are instantaneously perfectly correlated. In particular, the two stocks are redundant in equilibrium and, as a result, the agents' optimal portfolios are no longer uniquely defined.

¹⁷Alternatively, it can be shown that the portfolio constraint that is implied by the set \mathcal{C} of Eq. (30) is equivalent to a risk constraint that limits the expected shortfall, or equivalently the conditional value at risk, associated with the agent's portfolio over a fixed horizon. See (Cuoco, He, and Isaenko 2008) and (Prieto 2011) for details.

6.2 Existence and indeterminacy of equilibrium

Since agents have logarithmic preferences, the representative agent's utility function can be computed explicitly as in Eq. (20). As a result, the agents' equilibrium consumption and wealth processes are given by Eq. (21) and it follows that the equilibrium price of the market portfolio is

$$\bar{S}_t = S_{1t} + S_{2t} = W_{1t} + W_{2t} = \eta(t)e_t. \quad (32)$$

On the other hand, since the modified constraint set

$$\mathcal{D}_t = \sigma_t^\top \mathcal{C}_t = \{x \in \mathbb{R}^2 : \|x\| \leq (1 - \varepsilon)\|v\|\}$$

does not depend on the stock volatility it follows from Proposition 4 that the volatility of the weighting process can be computed independently of the individual stock prices by solving Eq. (12). Combining this property with the results of Propositions 4 and 6 yields the following proposition.

Proposition 9. *In a non redundant equilibrium the weighting process solves*

$$\lambda_t = \lambda_0 - \int_0^t \lambda_s (1 + \lambda_s) v_\lambda^\top dZ_s \quad (33)$$

for some constant $\lambda_0 > 0$, the market price of risk and interest rate are given by

$$\begin{aligned} \theta_t &= (1 + \varepsilon\lambda_t)v, \\ r_t &= \rho + a - (1 + \varepsilon\lambda_t)\|v\|^2, \end{aligned}$$

and, unless $v_\lambda \equiv \varepsilon v = \mathbf{0}$, the prices of both the market portfolio and the riskless asset include bubbles that are given by

$$\bar{B}_t = b(t, s_t)\bar{S}_t \quad \text{and} \quad B_{0t} = b_0(t, T, s_t)S_{0t}$$

where $s_t = s(\lambda_t)$ represents the constrained agent's share of aggregate consumption and the functions b_0 and $b \leq b_0$ are defined as in Proposition 6.

The first part of the proposition shows that the equilibrium weighting process follows the same dynamics as in the limited participation example and, therefore, fails to be a martingale despite the fact that it has no drift. More importantly, the second part shows that the equilibrium prices include bubbles as soon as there are portfolio constraints

($\varepsilon \neq 0$) and randomness at the aggregate level ($v \neq 0$). This implies that, unless the economy is unconstrained or deterministic at the aggregate level, the presence of bubbles is a necessary condition for equilibrium.

Proposition 9 pins down the market price of risk, interest rate and consumptions as functions of the processes e and λ (or equivalently e and s) so all that remains to do in order to obtain an equilibrium is to construct a stock price process. Using the unconstrained equilibrium state price density

$$\xi_{1t} = e^{-\rho t} \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} = e^{-\rho t} \frac{e_0(1-s_0)}{e_t(1-s_t)}$$

in conjunction with the assumed independence between e and x it can be shown that the fundamental value of stock i is given by

$$F_{it} = E_t \left[\int_t^T \frac{\xi_{1s}}{\xi_{1t}} x_{is} e_s ds \right] = x_{it} (1 - b(t, s_t)) \bar{S}_t \quad (34)$$

where \bar{S}_t is the price of the market portfolio as defined in Eq. (32). On the other hand, since both the discounted aggregate bubble

$$\xi_{1t} \bar{B}_t = e^{-\rho t} \eta(t) e_0 \left(\frac{1-s_0}{1-s_t} \right) b(t, s_t) \quad (35)$$

and the nonnegative process

$$\xi_{1t} F_{it} + \int_0^t \xi_{1s} x_{is} e_s ds = E_t \left[\int_0^T \xi_{1s} x_{is} e_s ds \right] = N_{it} - \xi_{1t} B_{it} \quad (36)$$

are by construction driftless, Proposition 5 suggest that any constant repartition of the aggregate bubble among the stocks leads to a non redundant equilibrium. To confirm this intuition let ϕ denote an arbitrary vector in the unit simplex $\mathcal{S} \subset \mathbb{R}_+^2$, assume that the constrained agent's consumption share starts from some initial value $s_0 \in [0, 1)$ and denote by

$$S_t(\phi, s_0) \equiv F_t + \phi \bar{B}_t = \bar{S}_t (x_t + b(t, s_t) (\phi - x_t)) \quad (37)$$

the vector of candidate equilibrium prices that obtains when the aggregate bubble is spread among the stocks according to the constant sharing rule ϕ .

Proposition 10. For each $\phi \in \mathcal{S}$ let $s_0 = s_0(\phi)$ denote the unique solution to

$$g(\phi, s_0) \equiv \beta + \alpha^\top S_0(\phi, s_0) - s_0 \bar{S}_0 = 0 \quad (38)$$

in the unit interval. Then the nonnegative process

$$S_t(\phi) \equiv S_t(\phi, s_0(\phi))$$

is an equilibrium price process for every sharing rule $\phi \in \mathcal{S}$. Furthermore, the initial equilibrium price $S_{i0}(\phi)$ of stock i is an increasing function of the share ϕ_i of the aggregate bubble that is included in its price.

The above proposition shows that the presence of volatility constraints generates multiple equilibria which correspond to different repartitions of the aggregate bubble among the stocks.¹⁸ The nature of this multiplicity, however, can be quite different depending on the agents' endowments.

To see this, assume first that agents have collinear endowments in the sense that their stock holdings amount to a fraction of the market portfolio. In this case Eq. (38) is independent from the bubble sharing rule and, as a result, the initial value of the constrained agent's consumption share is uniquely defined. Since the equilibrium weighting process is autonomous this implies that the path of s is independent from the bubble sharing rule, and it follows that the market price of risk, the interest rate and the consumption shares are all constant across the set of equilibria. In other words, with collinear initial endowments the indeterminacy of equilibrium caused by bubbles is only nominal and, thus, has no impact on the agents' welfare.

On the contrary, if the agents' initial endowments are non collinear then each bubble sharing rule ϕ is associated to a different initial value $s_0(\phi)$ and, therefore, a different path of the process s . Since the market price of risk, the interest rate and the consumption plans are expressed as functions of s this implies that all the equilibrium quantities vary through the set of equilibria: With non collinear endowments the indeterminacy of equilibrium caused by bubbles is not only nominal but also real. In particular, the consumption allocation now fluctuates with the bubble sharing rule and this implies that bubbles have an impact on the agents' welfare.

The next proposition allows to quantify these welfare effects by providing an explicit expression for the equilibrium expected utility of the constrained agent.

¹⁸The equilibria in Proposition 10 are based on constant bubble sharing rules and therefore do not allow bubbles to burst. This can be remedied by constructing equilibria in which the bubble sharing rule can be time and state dependent. See Proposition 12 below for details.

Proposition 11. *In the equilibrium associated with the bubble sharing rule $\phi \in \mathcal{S}$ the expected utility of the constrained agent is*

$$U_2(\phi) \equiv E \left[\int_0^T e^{-\rho t} \log(s_t e_t) dt \right] = L(s_0(\phi))$$

for some function L defined in the appendix. In particular, the constrained agent's welfare is increasing in ϕ_1 when $\alpha_1 \geq \alpha_2$ and decreasing otherwise.

The second part of the proposition shows that the constrained agent's welfare increases with the share of the aggregate bubble that is attributed to the largest stock in his portfolio. To understand the intuition behind this finding, recall from Proposition 10 that an increase in ϕ_1 implies an increase in the price of stock 1 and a decrease in the price of stock 2. If the constrained agent initially holds more of stock 1 ($\alpha_2 \leq \alpha_1$) then the first effect dominates and an increase in ϕ_1 triggers an increase in the initial consumption share of the constrained agent. Since the path of s depends positively on its initial value¹⁹ this in turn implies an increase in the consumption share of the constrained agent all along the path and, hence, an increase in his expected utility. Symmetrically, if the constrained agent initially holds less of stock 2 then an increase in ϕ_1 results in a smaller consumption share at all times and triggers a decrease in utility.

When the equilibrium consumption share of one agent increases that of the other simultaneously decreases by the same amount. Thus, the above results imply that, unless initial endowments are collinear, the agents' welfare vary in opposite directions as functions of the bubble sharing rule. An important implication of this finding is that agents cannot agree to coordinate on an equilibrium. In particular, any equilibrium selection device must be extraneous to the model.

6.3 Comparative statics

In order to illustrate the indeterminacies generated by volatility constraints I now analyze the impact of the bubble sharing rule on key equilibrium quantities. To facilitate the discussion I start by fixing the parameters of the model.

Following (Basak and Cuoco 1998) I assume that the agents' common discount rate is $\rho = 0.001$ and set the growth rate and volatility of the aggregate dividend process

¹⁹This monotonicity property follows from the definition of the consumption share process and the fact that the derivative $\delta_t \equiv \partial \lambda_t / \partial \lambda_0$ of the weighting process with respect to its initial value is given by the unique solution to the linear stochastic differential equation $-d\delta_t / \delta_t = (1 + 2\lambda_t) v_\lambda^\top dZ_t$ with initial value equal to one. See (Protter 2004, Theorem 39 p.305).

to $a = 0.0825$ and $v = (0.1654 \ 0)^\top$ so as to match the mean and standard deviation of the return on the Standard and Poor's composite index estimated by (Mehra and Prescott 1985). I assume that the volatility of the dividend shares is $v_x = (0 \ 0.2)^\top$ and set $x_{10} = x_{20} = 0.5$. This last assumption implies that the two stocks are ex-ante similar in the sense that the statistical properties of their dividends are the same as seen from the initial date of the model. In particular, it follows from Eq. (34) that under this assumption the stocks have the same fundamental value at time zero in any non redundant equilibrium and, hence, would have the same initial value in an unconstrained equilibrium. Finally, I assume that the economy has an horizon of $T = 50$ years, that $\varepsilon = 0.75$ and that $\alpha_1 = 1$, $\beta = \alpha_2 = 0$ so that the constrained agent initially owns the whole supply of stock 1 and nothing in stock 2.²⁰

The left panel of Figure 4 shows that as ϕ_1 increases the consumption share of the constrained agent increases while that of the unconstrained agent decreases. To understand this feature recall from Proposition 10 that an increase in ϕ_1 triggers an increase in the price of stock 1 and a symmetric decrease in the price of stock 2. Since the unconstrained agent holds the whole supply of stock 2 his wealth must absorb the

Insert Figure 4 here

entire decrease in its price and it follows that his consumption decreases. Symmetrically, the constrained agent stands alone to benefit from the increase in the price of stock 1 and it follows that his initial consumption increases with ϕ_1 . In accordance with Proposition 11, the right panel shows that this increase occurs not only at date zero but all along the path and thereby triggers an increase in the constrained agent's welfare.

Turning to the equilibrium prices, the left panel of Figure 5 shows that, as predicted by Proposition 10, an increase in ϕ_1 triggers an increase in the price of stock 1 and a symmetric decrease in the price of stock 2 since the value of the market portfolio is independent from ϕ_1 . In addition, this panel shows that as ϕ_1 increases the common fundamental value of the stocks decreases while the relative importance of the bubbles

Insert Figure 5 here

in the equilibrium prices increases for stock 1 and decreases for stock 2. Both of these

²⁰Equation (31) is sufficient to guarantee that an equilibrium exists for all $\phi \in \mathcal{S}$ but it is not necessary. Indeed, it can be shown that a fixed $\phi \in \mathcal{S}$ gives rise to a non redundant equilibrium if and only if $g(\phi, 1) < 0$. For the parameter values of this section $g(\phi, 1) = \eta(0)e_0(\phi_1 - 1)$ and it follows that an equilibrium exists for all $\phi \in \mathcal{S}$ such that $\phi_1 < 1$.

effects are due to the fact that as ϕ_1 increases the consumption share of the constrained agent $s_0(\phi)$, and therefore also the size of the aggregate bubble $b(0, s_0(\phi))\bar{S}_0$, increases. Finally, this panel illustrates the fact that two stocks which have the same fundamental value need not have the same price in equilibrium as these prices might include the aggregate bubble in different proportions.

As a final illustration the right panel of Figure 5 shows that, as the fraction of the aggregate bubble that is attributed to stock 1 increases, the market price of risk on aggregate shocks, θ_{10} , increases while the risk free rate decreases.²¹ To understand this feature recall from Figure 4 that the weight of the constrained agent in the economy, as measured by his consumption share, increases with ϕ_1 . This implies that the size of the leveraged position that the unconstrained agent must hold increases as ϕ_1 increases and it follows that the interest rate must decrease and the market price of risk must increase. Note that, due to the same effect, the market price of risk and the interest rate are respectively higher and lower in the constrained economy than in an otherwise equivalent unconstrained economy.

6.4 Other equilibria

The equilibria of Proposition 10 are based on a constant repartition of the bubble among the stocks. Since the aggregate stock bubble is always strictly positive this implies that a given stock either always has a bubble or never does. In other words, these equilibria do not allow for bubbles to burst. To remedy this problem one has to construct equilibria in which the repartition of the aggregate bubble among the stocks can be time and state dependent. This is the purpose of the next proposition.

Proposition 12. *For any \mathcal{S} -valued process ϕ , let the constant $s_0 = s_0(\phi)$ be the unique solution to $g(\phi_0, s_0) = 0$ in the unit interval and define*

$$S_t(\phi) \equiv \bar{S}_t(x_t + b(t, s_t(\phi))(\phi_t - x_t))$$

where $s_t(\phi)$ denotes the corresponding path of the consumption share. Then $S(\phi)$ is an equilibrium stock price process if and only if

$$d\phi_{1t} = \psi_{1t}^\top dZ_t + \varepsilon s_t(\phi) \left(\frac{1}{1 - s_t(\phi)} + \frac{\partial \log b}{\partial s}(t, s_t(\phi)) \right) v^\top \psi_{1t} dt$$

²¹In accordance with Proposition 4 the market price of risk θ_2 associated with the second Brownian motion is constantly equal to zero in equilibrium as this source of risk influences neither the aggregate consumption nor the weighting process due to the fact that $v_2 \equiv 0$ by assumption.

for some diffusion coefficient ψ_1 such that

$$b(t, s_t(\phi)) \det(v, \psi_{1t}) + x_{1t}(1 - x_{1t})(1 - b(t, s_t(\phi))) \det(v, v_x) \neq 0. \quad (39)$$

In particular, $S(x) = x\bar{S}$ is an equilibrium stock price process.

The second condition in the proposition guarantees that the volatility matrix of the candidate price process is non singular. On the other hand, an application of Itô's lemma shows that the first condition is equivalent to the requirement that

$$M_t(\phi) \equiv e^{-\rho t} \frac{u_c(e_t, \lambda_t(\phi))}{u_c(e_0, \lambda_0(\phi))} \phi_t b(t, s_t(\phi)) \bar{S}_t$$

be a local martingale. Since the normalized marginal utility of the representative agent gives the unconstrained state price density, the vector process $M(\phi)$ can be viewed as the discounted value of the bubble components in the candidate prices. Given this interpretation, the result of Proposition 12 shows that a bubble share process gives rise to an equilibrium if and only if the induced bubble components, $B_{it} = \phi_{it} b(t, s_t(\phi)) \bar{S}_t$, are arbitrage free in that they offer the market risk premium.

The characterization of equilibrium stock prices in Proposition 12 is a lot richer than that of the previous section. In particular, it allows for the possibility of finitely lived bubbles on the risky assets. In order to construct a simple example where this phenomenon occurs let

$$\tau_1^* = \inf\{t \geq 0 : x_{1t} \leq x^*\}$$

denote the first time at which the dividend share of the first stock falls below some fixed level $x^* \in (0, 1)$, and define the bubble sharing rule by setting

$$\phi_{1t} \equiv E_t [1_{\{\tau_1^* > t\}}].$$

The fact that the dividend share is a Markov process implies that

$$\phi_{1t} = 1_{\{\tau_1^* > t\}} f(t, x_{1t})$$

for some non decreasing function and, since $v^\top v_x = 0$ by assumption, it follows that the two conditions of Proposition 12 hold with

$$\psi_{1t} \equiv 1_{\{\tau_1^* > t\}} \frac{\partial f(t, x_{1t})}{\partial x} x_{1t} (1 - x_{1t}) v_x.$$

Consequently, $S(\phi)$ is an equilibrium stock price process and since $\phi_{1t} = 0$ for all $t \geq \tau_1^*$ I conclude that in this equilibrium the price of the first stock includes a bubble only up to the stopping time $\min\{\tau_1^*; T\}$ while the price of the second stock includes a strictly positive bubble over the entire horizon of the economy.

More generally, if the process $\phi_1 \in [0, 1]$ is a martingale that is independent from the aggregate dividend process and such that Eq. (39) holds then $S(\phi)$ is an equilibrium price process. Furthermore, in this equilibrium the bubble on the first stock bursts at the first time that the process ϕ_1 reaches zero while the bubble on the second stock vanishes at the first time that it reaches one.²²

Remark 5 (Sunspots). Propositions 10 and 12 can be generalized to introduce extrinsic uncertainty, or sunspots, into the model. The simplest way of doing so is to let $\phi_0 = \mu$ where μ is an \mathcal{F}_0 -measurable random variable that is independent from the Brownian motion. As usual with sunspots, this additional layer of uncertainty should be thought of as a selection device that allows to coordinate on a particular equilibrium. Note however that this device is in some sense static because the value of μ is drawn at the initial time and is never changed after that. Introducing dynamic sunspots by setting $\phi_t = \mu_t$ for some process μ that depends on an extrinsic Brownian motion is not possible because it would change the local risk structure of the economy.

7 Conclusion

In this paper, I study a continuous-time, pure exchange economy populated by two agents. One of the agents has logarithmic utility and faces portfolio constraints while the other has arbitrary utility and is unconstrained apart from a standard solvency condition which requires him to maintain nonnegative wealth.

The first main contribution of this paper is to show that in this setting portfolio constraints may give rise to rational asset pricing bubbles in equilibrium even though there are unconstrained agents who can exploit the corresponding limited arbitrage opportunity. Furthermore, I show that the presence of bubbles can be assessed by analyzing the behavior of a single variable that is given by the ratio of the agents' marginal utilities. I illustrate these results by studying a limited participation model *à la* (Basak and Cuoco 1998). In this model, the unconstrained agent must find it optimal to hold a leveraged

²²Since bubbles cannot be born, the bubble shares ϕ_i cannot reach the value of zero or one and then continue to change. This implies that the boundaries 0 and 1 are necessarily absorbing for an equilibrium bubble share process.

position and I show that this forces the equilibrium prices of both the stock and riskless asset to include bubbles.

The second main contribution of this paper is to show that when there are multiple risky assets the presence of a bubble in the market portfolio can lead to both multiplicity and real indeterminacy of equilibrium. I illustrate this implication by studying an economy with two stocks where some agents face a risk constraint that limits the volatility of their wealth. In this setting, I prove the existence of a continuum of equilibria and show that the variations of key quantities, such as consumption shares, expected utilities, stock prices, interest rates and market prices of risk can be substantial.

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Appendix

A Technical results

This appendix serves as a basis for the proofs in Appendix B and is devoted to the study of the strict local martingale that plays the role of the weighting process in the examples of Sections 5 and 6.

Lemma A.1. *Let $(\lambda_0, v_\lambda) \in (0, \infty) \times \mathbb{R}^n$. The stochastic differential equation*

$$\lambda_t = \lambda_0 - \int_0^t \lambda_s (1 + \lambda_s) v_\lambda^\top dZ_s \quad (\text{A.1})$$

admits a unique strong solution. This solution is a strictly positive local martingale but not a true martingale unless $v_\lambda = \mathbf{0}$.

Proof. The existence and uniqueness of a strictly positive strong solution follows directly from Lemma 1 of [Basak and Cuoco \(1998\)](#).

To establish the second part I argue by contradiction. Assume that the solution is a true martingale so that $\mathbb{Q}(A) \equiv E [1_{\{A\}}(\lambda_T/\lambda_0)]$ defines an equivalent probability measure. Girsanov's theorem then implies that

$$W_t \equiv Z_t + \int_0^t (1 + \lambda_s) v_\lambda ds$$

is a Brownian motion under the probability measure \mathbb{Q} and it follows that process λ solves the stochastic differential equation

$$d\lambda_t = \lambda_t(1 + \lambda_t)^2 \|v_\lambda\|^2 dt - \lambda_t(1 + \lambda_t) v_\lambda^\top dW_t. \quad (\text{A.2})$$

Let $p(x) = 1 - 1/x$ denote the scale function associated with this stochastic differential equation and consider the function

$$v(\lambda) \equiv \int_1^\lambda p'(x) \int_1^x \frac{2/p'(y)}{y^2(1+y)^2 \|v_\lambda\|^2} dy dx.$$

According to [Karatzas and Shreve \(1998, Theorem 5.5.29\)](#) the solution to equation (A.2) explodes with strictly positive probability under \mathbb{Q} since $v(\infty) < \infty$. On the other hand, since λ is a nonnegative \mathbb{P} -supermartingale it is almost surely finite under \mathbb{P} . This contradicts the equivalence between \mathbb{P} and \mathbb{Q} and establishes the desired result. \blacksquare

Lemma A.2. *The expectation function of the process λ is*

$$E_t[\lambda_{t+\tau}] = \lambda_t (1 - s_t^{-1} H(\tau, s_t; 1))$$

where $s_t = s(\lambda_t)$, the function H is defined by

$$H(\tau, s; a) \equiv s^{\frac{1+a}{2}} \Phi(d_+(\tau, s; a)) + s^{\frac{1-a}{2}} \Phi(d_-(\tau, s; a)),$$

$$d_{\pm}(\tau, s; a) \equiv \frac{1}{\|v_{\lambda}\| \sqrt{\tau}} \log s \pm \frac{a}{2} \|v_{\lambda}\| \sqrt{\tau}$$

and the function Φ is the standard normal cdf.

Proof. The results of [Elworthy et al. \(1999\)](#) show that

$$E_t[\lambda_{t+\tau}] = \lambda_t - \lim_{m \rightarrow \infty} m \mathbb{P}_t \left[\sup_{s \leq \tau} \lambda_{t+s} \geq m \right]. \quad (\text{A.3})$$

Denote by $p(\tau, \lambda_t, m)$ the conditional probability on the right hand side of the above expression. Well-known results on one dimensional diffusions (see for example [Borodin and Salminen \(2002, II.10\)](#)) show that

$$\int_0^{\infty} e^{-\alpha t} p(t, \lambda, m) dt = \frac{1}{\alpha} E \left[e^{-\alpha T_m} \mid \lambda_0 = \lambda \right] = \frac{1}{\alpha} \frac{\phi(\lambda)}{\phi(m)}$$

where T_m denotes the first hitting time of the level m and the function ϕ is the unique increasing solution to the Sturm–Liouville problem

$$\frac{1}{2} \|v_{\lambda}\|^2 \lambda^2 (1 + \lambda)^2 \phi''(\lambda) = \alpha \phi(\lambda)$$

with $\phi(0) = 0$. Solving this ordinary differential equation, I obtain

$$\phi(\lambda) = \lambda \left(\frac{\lambda}{1 + \lambda} \right)^{\frac{1}{2} \sqrt{1 + \frac{8\alpha}{\|v_{\lambda}\|^2} - \frac{1}{2}}}$$

Laplace transform inversion formulae (see for example [Erdelyi \(1954\)](#)) and tedious algebra then show that

$$p(t, \lambda, m) = \frac{1 + \lambda}{1 + m} H \left(t, \frac{\lambda}{m} \frac{1 + m}{1 + \lambda}; 1 \right).$$

Multiplying both sides of this expression by m , plugging the result into equation (A.3) and letting m go to infinity gives the desired result. ■

Lemma A.3. *If $\tau \geq 0$ is a constant then*

$$\frac{\rho}{1 + \lambda_t} E_t \left[\int_t^{t+\tau} e^{-\rho(s-t)} (\lambda_t - \lambda_s) ds \right] = H(\tau, s_t; a_1) - e^{-\rho\tau} H(\tau, s_t; 1)$$

where the function H is defined as in Lemma A.2 and $a_1 \equiv \sqrt{1 + 8\rho/\|v_\lambda\|^2}$.

Proof. This follows from Lemma A.2, the Markov property of the solution to equation (A.1) and tedious algebra, I omit the details. \blacksquare

Lemma A.4. *The expectation of the process $s_t = s(\lambda_t)$ is*

$$E_t [s_{t+\tau}] = \frac{s_t}{1 - s_t} \left(1 - e^{\|v_\lambda\|^2 t} s_t \right) + \frac{1}{1 - s_t} \left(e^{\|v_\lambda\|^2 t} H(\tau, s_t; 3) - H(\tau, s_t; 1) \right)$$

where the function H is defined as in Lemma A.2.

Proof. Let $\gamma > 0$ and consider the bounded function

$$g(\lambda_t) \equiv E_t \left[\int_t^\infty e^{-\gamma(u-t)} s_u du \right] = E_t \left[\int_t^\infty e^{-\gamma(u-t)} \frac{\lambda_u}{1 + \lambda_u} du \right].$$

Using Itô's lemma and equation (A.1) it can be shown that the function g is the unique bounded solution to the Sturm–Liouville problem

$$\frac{1}{2} \lambda^2 (1 + \lambda)^2 \|v_\lambda\|^2 g''(\lambda) + \frac{\lambda}{1 + \lambda} = \gamma g(\lambda)$$

with $g(0) = 0$. Solving this differential equation gives

$$g(\lambda) = \frac{\lambda/\gamma}{1 + \lambda} \left(1 + \frac{\lambda \|v_\lambda\|^2}{\|v_\lambda\|^2 - \gamma} + \left(\frac{\lambda}{1 + \lambda} \right)^{\frac{\sqrt{1+\alpha}-1}{2}} \frac{(1 + \lambda) \|v_\lambda\|^2}{\gamma - \|v_\lambda\|^2} \right)$$

with $\alpha = 8\gamma/\|v_\lambda\|^2$. Using standard formulae (see Erdelyi (1954)) to invert this Laplace transform and simplifying the resulting expression gives the desired conclusion. \blacksquare

Lemma A.5. *Let $\alpha \in \mathbb{R}$ be an arbitrary constant. Then the expectation function of the nonnegative local martingale*

$$Y_t(\alpha) = 1 - \int_0^t Y_s(\alpha) (\alpha + \lambda_s) v_\lambda^\top dZ_s \tag{A.4}$$

is explicitly given by

$$E_t [Y_{t+\tau}(\alpha)] = Y_t(\alpha) (1 - s_t^{-\alpha} H(\tau, s_t; 2\alpha - 1)) \quad (\text{A.5})$$

where the function H is defined as in Lemma A.2. In particular, the unique solution to equation (A.4) is a strictly positive local martingale but it not a true martingale.

Proof. Consider the equivalent probability measure defined by

$$\left. \frac{d\mathbb{P}^\alpha}{d\mathbb{P}} \right|_{\mathcal{F}_t} = M_t(\alpha) \equiv e^{(1-\alpha)v_\lambda^\top Z_t - \frac{1}{2}(1-\alpha)\|v_\lambda\|^2 t}, \quad 0 \leq t < \infty.$$

Using this definition in conjunction with Bayes rule and Itô's lemma shows that the expectation function of the process $Y(\alpha)$ is given by

$$E_t \left[\frac{Y_T(\alpha)}{Y_t(\alpha)} \right] = E_t^\alpha \left[\frac{Y_T(\alpha)M_t(\alpha)}{Y_t(\alpha)M_T(\alpha)} \right] = E_t^\alpha \left[e^{-\int_t^T g(\lambda_s)ds} \frac{\lambda_T}{\lambda_t} \right] \quad (\text{A.6})$$

where $g(\lambda) \equiv (\alpha - 1)(1 + \lambda)\|v_\lambda\|^2$. Furthermore, the dynamics of λ under the new probability measure are given by

$$d\lambda_t = \lambda_t g(\lambda_t) dt - \lambda_t (1 + \lambda_t) v_\lambda^\top dZ_t^\alpha.$$

This implies that the expectation on the right hand side of equation (A.6) can be computed as $f(T - t, \lambda_t)$ for some function f and since

$$N_t = e^{-\int_0^t g(\lambda_s)ds} \lambda_t$$

is a nonnegative local martingale under \mathbb{P}^α it follows from arguments similar to those of Fernholz and Karatzas (2010) that the function

$$h(\lambda) \equiv \int_0^\infty e^{-\gamma\tau} f(\tau, \lambda) d\tau = E^\alpha \left[\int_0^\infty e^{-\int_0^\tau (\gamma + g(\lambda_s))ds} \lambda_\tau d\tau \right]$$

is the smallest nonnegative solution to

$$\lambda g(\lambda) h'(\lambda) + \frac{1}{2} \lambda^2 (1 + \lambda)^2 \|v_\lambda\|^2 h''(\lambda) = (\gamma + g(\lambda)) h(\lambda) - \lambda$$

such that $h(\lambda) \leq \lambda/\gamma$. The general solution to this non homogenous second order differential equation is

$$h(\lambda) = \frac{\lambda}{\gamma} - \lambda A \left(\frac{\lambda}{1+\lambda} \right)^{\Theta_+} - \lambda B \left(\frac{\lambda}{1+\lambda} \right)^{\Theta_-} \quad (\text{A.7})$$

where A, B are constants to be determined and

$$\Theta_{\pm} \equiv \frac{1}{2} - \alpha \pm \sqrt{\frac{2\gamma}{\|v_\lambda\|^2} + \left(\frac{1}{2} - \alpha \right)^2}.$$

To satisfy the boundary condition $h(0) = 0$ it must be that $B \equiv 0$ for else $|h(0)| = \infty$ since $\Theta_- \leq 0$ by definition. On the other hand, since $\Theta_+ \geq 0$ it can be shown that the solution takes values in the interval $[0, \lambda/\gamma]$ if and only if $A \in [0, 1/\gamma]$ and, since h is the smallest such solution, I conclude that $A \equiv 1/\gamma$. Substituting these constants into equation (A.7) and inverting the Laplace transform gives the formula in equation (A.5) and the strict local martingale property follows by noting that $E[Y_T(\alpha)] < 1$. ■

B Proofs

Proof of Proposition 1. Without loss of generality I consider the replication problem only at the initial time. Assume that the trading strategy (ϕ, π) is self-financing given consumption at rate c and such that

$$W_t \equiv W_t(\phi, \pi) \geq 0, \quad \forall t \leq T.$$

An application of Itô's lemma shows that

$$\xi_{1t} W_t + \int_0^t \xi_{1s} c_s ds = W_0 + \int_0^t \xi_{1s} (\sigma_s^\top \pi_s - W_s \theta_s)^\top dZ_s$$

is a local martingale. Since c, ξ_1 and W are nonnegative I have that this local martingale is a nonnegative supermartingale and it follows that

$$F_0(c) = E \left[\int_0^T \xi_{1s} c_s ds \right] \leq E \left[\xi_{1T} W_T + \int_0^T \xi_{1s} c_s ds \right] \leq W_0 = \phi_0 + \pi_0^\top \mathbf{1}$$

where the first inequality results from the nonnegativity of $\xi_{1T} W_T$. This implies that the minimal amount necessary to replicate the consumption plan is larger than $F_0(c)$. To establish the reverse inequality assume that $F_0(c) < \infty$ for otherwise there is nothing to

prove and consider the nonnegative process

$$\hat{W}_t \equiv \frac{1}{\xi_{1t}} E_t \left[\int_t^T \xi_{1s} c_s ds \right].$$

Combining the martingale representation theorem with Itô's lemma shows that there exists a predictable process φ such that

$$\hat{W}_t = F_0(c) + \int_0^t (r_s \hat{W}_s - c_s) ds + \int_0^t (\varphi_s + \hat{W}_s \theta_s)^\top (dZ_s + \theta_s ds).$$

Since the stock volatility is non singular in a non redundant equilibrium this immediately implies that \hat{W} is the wealth process of the trading strategy defined by

$$\begin{aligned} \hat{\pi}_t &= (\sigma_t^\top)^{-1} (\varphi_t + \hat{W}_t \theta_t), \\ \hat{\phi}_t &= \hat{W}_t - \hat{\pi}_t^\top \mathbf{1}, \end{aligned}$$

and it follows that the minimal amount necessary to replicate the consumption plan c is smaller or equal to $F_0(c)$. ■

Proof of Proposition 2. As a result of Proposition 1 I have that a consumption plan c is feasible for agent 1 if and only if

$$F_0(c) = E \left[\int_0^T \xi_{1t} c_t dt \right] \leq w_1$$

and it follows that his portfolio and consumption choice problem, which is necessarily well-defined in equilibrium, can be reformulated as

$$\inf_{y>0} \sup_{c \geq 0} E \left[\int_0^T e^{-\rho t} u_1(c_t) dt + y \left(w_1 - \int_0^T \xi_{1t} c_t dt \right) \right]$$

where the Lagrange multiplier y enforces the agent's static budget constraint. The first order conditions of this concave problem require that

$$e^{-\rho t} u_{1c}(c_{1t}) - y_1 \xi_{1t} = w_1 - F_0(c_1) = 0$$

and it follows that the optimal consumption is given by $c_{1t} = I_1(y_1 e^{\rho t} \xi_{1t})$ for some strictly positive constant that saturates the agent's static budget constraint. ■

Proof of Proposition 3. If the market is in equilibrium then the value function of agent 2 is finite and it follows from [Cvitanic and Karatzas \(1992\)](#) that the optimal consumption and portfolio are given as in the statement but with

$$d\xi_{2t} = -\xi_{2t} (r_t + \delta_t(\theta_{2t} - \theta_t)) dt - \xi_{2t} \theta_{2t}^\top dZ_t,$$

and

$$\theta_{2t} = \theta_t + \operatorname{argmin}_{\nu \in \mathcal{M}_t} \left(\delta_t(\nu) + \frac{1}{2} \|\theta_t + \nu\|^2 \right)$$

where δ_t is the support function of the set $-\mathcal{D}_t = -\sigma_t^\top \mathcal{C}_t$ and \mathcal{M}_t denotes its effective domain. Using Fenchel's identity (see [Hiriart-Urruty and Lemaréchal \(2001, p.228\)](#)) in conjunction with the definition of the support function it can be shown that

$$\begin{aligned} \min_{\nu \in \mathcal{M}_t} \left(\delta_t(\nu) + \frac{1}{2} \|\theta_t + \nu\|^2 \right) &= \frac{1}{2} \|\theta_t\|^2 - \min_{k \in \mathcal{D}_t} \frac{1}{2} \|\theta_t - k\|^2, \\ \theta_t + \operatorname{argmin}_{\nu \in \mathcal{M}_t} \left(\delta_t(\nu) + \frac{1}{2} \|\theta_t + \nu\|^2 \right) &= \operatorname{argmin}_{k \in \mathcal{D}_t} \frac{1}{2} \|\theta_t - k\|^2. \end{aligned}$$

The second equality shows that $\theta_{2t} = \Pi(\theta_t | \mathcal{D}_t)$ as claimed in the statement. On the other hand, combining the two equalities shows that

$$\delta_t(\theta_{2t} - \theta_t) = \frac{1}{2} \|\theta_t\|^2 - \frac{1}{2} \|\theta_t - \theta_{2t}\|^2 - \frac{1}{2} \|\theta_{2t}\|^2 = \theta_{2t}^\top (\theta_t - \theta_{2t})$$

and the desired result now follows by plugging this expression into the dynamics of the process ξ_2 and then applying Itô's lemma to the ratio ξ_2/ξ_1 . ■

Proof of Proposition 4. Equations (5) and (9) imply that

$$\lambda_t = \lambda_0 \exp \left(-\frac{1}{2} \int_0^t \|\theta_{2s} - \theta_s\|^2 ds + \int_0^t (\theta_{2s} - \theta_s)^\top dZ_s \right)$$

and an application of Itô's lemma shows that $m = 0$. Taking the volatility of the weighting process as given and applying Itô's lemma to the definition of ξ_1 then gives the market price of risk and interest rate reported in the text. Finally, combining the above equation with Proposition 3 gives

$$\Gamma_t = \theta_t - \theta_{2t} = \theta_t - \Pi(\theta_t | \mathcal{D}_t).$$

Substituting the market price of risk of the statement into the above expression shows that Γ_t solves equation (12) and completes the proof. ■

Proof of Theorem 1. Assume that the equilibrium stock prices are free of bubbles so that $\bar{S}_t = \bar{F}_t$. Comparing this identity with equation (15) shows that

$$e^{-\rho t} u_c(e_t, \lambda_t) \bar{B}_t = \int_t^T e^{-\rho s} (\lambda_t - E_t[\lambda_s]) ds = 0.$$

According to Proposition 4 the weighting process is a nonnegative local martingale and hence a supermartingale (Karatzas and Shreve (1998, p.18)). Combining this property with the above equality shows that

$$\int_0^T e^{-\rho s} |\lambda_t - E_t[\lambda_s]| ds = 0$$

and it follows that λ is a martingale. Conversely, if λ is a martingale then equation (16) shows that there is no bubble on the market portfolio and hence no bubbles on the stocks since $B_{it} \geq 0$ for each i and $\sum_{i=1}^n B_{it} = \bar{B}_t = 0$. ■

Proof of Proposition 5. Assume that S is the stock price process in a non redundant equilibrium and let λ be defined as in the statement. The invertibility of the volatility matrix, the definition of the risk premium and Proposition 4 imply

$$dS_{it} = S_{it} r_t dt + S_{it} \sigma_{it}^\top (dZ_t + \theta_t dt) - e_{it} dt.$$

where (r, θ) are defined as in equations (10)–(11). Combining the above dynamics with the definition of the unconstrained state price density and applying Itô's lemma shows that the process N_i is a local martingale for each i . On the other hand, Propositions 2, 3, 4 and the definition of λ_0 imply

$$\sum_{a=1}^2 W_{at} = E_t \left[\int_t^T e^{-\rho(s-t)} \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} \left(e_s + \frac{\lambda_t - \lambda_s}{u_c(e_s, \lambda_s)} \right) ds \right]$$

and equation (19) now follows from the market clearing conditions.

Conversely, if S satisfies the conditions of the statement then N_i is driftless and it thus follows from the invertibility of σ_t that S offers the market price of risk of equation (10). Combining this with Propositions 2, 3 and the definition of λ_0 shows that the allocation

of equation (8) is optimal. Together with equation (19) this implies

$$\sum_{a=1}^2 W_{at} = \sum_{i=1}^n S_{it}$$

and it follows that the market for the riskless asset clears. Finally, applying Itô's lemma on both sides of the previous equality and equating the volatility terms shows that the optimal portfolios satisfy $\sigma_t^\top (\pi_{1t} + \pi_{2t} - S_t) = \mathbf{0}$ and the assumed invertibility of σ_t implies that the stock market clears. ■

Proof of Proposition 6. Equations (20), (21) and (22) imply that the initial value of the weighting process is uniquely given by $\lambda_0 = w_2/(\eta(0)e_0 - w_2)$. On the other hand, since the volatility of the stock is equal to that of the aggregate dividend it follows from Proposition 4 and the definition of \mathcal{C}_t that

$$v + \frac{\Gamma_t}{1 + \lambda_t} = \left(v - \frac{\lambda_t \Gamma_t}{1 + \lambda_t} \right)^+ - \left(v_\lambda - \frac{\lambda_t \Gamma_t}{1 + \lambda_t} \right)^+.$$

Solving this equation gives $\Gamma_t = -(1 + \lambda_t)v_\lambda$ and it now follows from Proposition 4 that the equilibrium weighting process is a solution to equation (23). The result of Lemma A.1 shows that such a solution is unique and it follows that there exists a unique equilibrium which is given by equations (21), (24) and (25).

Appealing once again to Lemma A.1 we have that the unique solution to equation (23) is a strict local martingale. In conjunction with Theorem 1 this implies that the equilibrium stock price includes a bubble that is given by

$$\frac{B_t}{S_t} = \frac{1}{\eta(t)} E_t \left[\int_t^T e^{-\rho(s-t)} \frac{\lambda_t - \lambda_s}{1 + \lambda_t} ds \right] = b(t, s_t) \quad (\text{B.1})$$

where the last equality follows from Lemma A.3. On the other hand, Remark 1 and the definition of the market price of risk imply that the equilibrium price of the riskless asset includes a bubble if and only if the process

$$M_t \equiv S_{0t} \xi_{1t} = 1 - \int_0^t M_s \theta_s dZ_s = 1 - \int_0^t M_s (1 + \varepsilon \lambda_s) v dZ_s$$

is a strict local martingale and that in this case the bubble is given by

$$\frac{B_{0t}}{S_{0t}} = 1 - E_t \left[\frac{M_T}{M_t} \right]. \quad (\text{B.2})$$

Observing that $M = Y(1/\varepsilon)$ where $Y(\alpha)$ is defined as in equation (A.4) and applying the result of Lemma A.5 shows that the equilibrium price of the riskless asset includes a strictly positive bubble that is given by

$$\frac{B_{0t}}{S_{0t}} = 1 - E_t \left[\frac{M_T}{M_t} \right] = 1 - E_t \left[\frac{Y_T(1/\varepsilon)}{Y_t(1/\varepsilon)} \right] = s_t^{-1/\varepsilon} H(T - t, s_t; 2/\varepsilon - 1)$$

and the desired result follows by setting $a_0 = 2/\varepsilon - 1$. To complete the proof it now only remains to show that $b \leq b_0$. This easily follows from equations (B.1), (B.2), the supermartingale property of the weighting process and the fact that, as can be checked by direct differentiation, the mapping

$$\alpha \mapsto E_t \left[\frac{Y_T(\alpha)}{Y_t(\alpha)} \right] = 1 - s_t^{-\alpha} H(T - t, s_t; 2\alpha - 1)$$

is strictly decreasing. I omit the details. ■

Proof of Proposition 9. Using Proposition 4 in conjunction with the definition of the set \mathcal{D}_t I obtain that the volatility of the weighting process solves

$$v + (1 - s_t)\Gamma_t = (v - s_t\Gamma_t) \min \left(1; \frac{(1 - \varepsilon)\|v\|}{\|v - s_t\Gamma_t\|} \right).$$

It is easily checked that $\Gamma_t = -(1 + \lambda_t)v_\lambda$ is a solution and since such a solution is unique it follows that the weighting process evolves according to equation (33) in any non redundant equilibrium. Apart from some notational changes the rest of the proof is similar to that of Proposition 6 and therefore is omitted. ■

Proof of Proposition 10. In order to facilitate the proof I start by establishing the existence, uniqueness and basic properties of the solution to equation (38).

Using assumption (31) in conjunction with the fact that $\beta \geq 0$, $\alpha \in \mathcal{S}$, $x_0 \in \text{int } \mathcal{S}$ and $\phi \in \mathcal{S}$ I deduce that $g(\phi, 0) > 0$, $g(\phi, 1) < 0$ and the continuity of g implies that there exists at least one solution to equation (38) in the open interval $(0, 1)$. To prove uniqueness let $\phi \in \mathcal{S}$ be fixed and observe that

$$g_s(\phi, s) \equiv \frac{\partial g(\phi, s)}{\partial s} = \bar{S}_0 \left((\alpha_1 - \alpha_2)(\phi_1 - x_{10}) \frac{\partial b(0, s)}{\partial s} - 1 \right).$$

Assume first that $\Theta \equiv (\alpha_1 - \alpha_2)(\phi_1 - x_{10}) \leq 0$. In this case $g_s(\phi, \cdot)$ is negative throughout the unit interval since the function b is increasing and it follows that the solution to equation (38) is unique. Now assume that $\Theta \geq 0$. In this case the derivative $g_s(\phi, \cdot)$

starts out negative and is monotone increasing since the function b is convex. This implies that $g_s(\phi, \cdot)$ changes sign at most once and, since $g(\phi, 1) < 0$, it follows that the solution to equation (38) is unique. In particular, notice for later use that the above arguments imply

$$g_s(\phi, s_0(\phi)) = \Theta \bar{S}_0 \left(\frac{\partial b(0, s_0(\phi))}{\partial s} - \frac{1}{\Theta} \right) < 0 \quad (\text{B.3})$$

irrespective of the sign of the constant $\Theta \equiv \Theta(\alpha, \phi, x_0)$.

In order to prove that $S(\phi)$ gives rise to a non-redundant equilibrium it suffices to check that it satisfies the conditions of Proposition 5. Equations (35), (36) and the fact that ϕ is constant imply that N_i is a local martingale for each i . On the other hand, the definition of $S(\phi)$ implies that $S_{1t}(\phi) + S_{2t}(\phi) = \bar{S}_t$ and it now only remains to verify that the volatility of $S_t(\phi)$ is invertible. Using the dynamics of (e, x) in conjunction with Itô's lemma and the definition of b I obtain

$$\det \sigma_t(\phi) = \frac{x_{1t}(1 - x_{1t})(1 - b(t, s_t))(\bar{S}_t)^2}{S_{1t}(\phi)S_{2t}(\phi)} \det \begin{pmatrix} v_{x1} & v_1 \\ v_{x2} & v_2 \end{pmatrix}.$$

Since $x_t \in \text{int } \mathcal{S}$, $b(t, s) < 1$ for all $s \in (0, 1)$ and the vectors v and v_x are linearly independent by assumption, the above expression implies that the matrix-valued process $\sigma(\phi)$ is non-singular and the conclusion follows.

To complete the proof it remains to show that $S_{i0}(\phi)$ is increasing with respect to ϕ_i . Differentiating equation (37) I obtain

$$\frac{\partial S_{i0}(\phi)}{\partial \phi_i} = \frac{\partial S_{i0}(\phi, s_0(\phi))}{\partial \phi_i} + \frac{\partial s_0(\phi)}{\partial \phi_i} \frac{\partial S_{i0}(\phi, s_0(\phi))}{\partial s} = -\frac{(\eta(0)e_0 b(0, s_0(\phi)))^2}{g_s(\phi, s_0(\phi))}.$$

and the desired result now follows equation (B.3). ■

Proof of Proposition 11. Applying Itô's lemma to the process $s_t = s(\lambda_t)$ and using the dynamics of the equilibrium weighting process I obtain

$$\log s_t = \log s_0 - \frac{\|v_\lambda\|^2}{2} \int_0^t (1 + 2\lambda_s) ds - v_\lambda^\top Z_t.$$

Combining this with Fubini's theorem and the result of Lemma A.2 shows that the constrained agent's welfare is given by

$$U_2(\phi) = A(s_0(\phi)) - \|v_\lambda\|^2 E \left[\int_0^T e^{-\rho t} \eta(t) (1 + \lambda_t) dt \right]$$

where the function A is defined by

$$\begin{aligned} A(s) &\equiv E \left[\int_0^T e^{-\rho t} \left(\log(se_t) + \frac{1}{2} \|v_\lambda\|^2 t \right) dt \right] \\ &= \eta(0) \log(se_0) + \frac{1}{\rho} \left(a + \frac{\varepsilon^2 - 1}{2} \|v\|^2 \right) (\eta(0) + T\eta'(0)). \end{aligned}$$

Using the result of Lemma A.2 together with integration by parts and the definition of the functions η and H then gives

$$B(s) \equiv E \left[\int_0^T e^{-\rho t} \eta(t) (1 + \lambda_t) dt \right] = \frac{\eta(0) (1 - b(0, s)) + \eta'(0) (T - b^*(0, s))}{\rho(1 - s)}$$

where

$$b^*(0, s) = \lim_{\rho \rightarrow 0} \eta(0) b(0, s) = TH(T, s; 1) + \frac{4}{\|v_\lambda\|^2} \frac{\partial H(T, s; 1)}{\partial a}$$

and the desired result follows letting $L \equiv A - \|v_\lambda\|^2 B$. Using the chain rule in conjunction with Protter (2004, Theorem 39 p.305)) and the definition of $s_0(\phi)$ I obtain

$$\frac{\partial s_t}{\partial \phi_1} = \frac{\delta_t}{(1 - s_0(\phi))^2} \frac{\partial s_0(\phi)}{\partial \phi_1} = \frac{(\alpha_1 - \alpha_2) \eta(0) e_0 b(0, s_0(\phi))}{(1 - s_0(\phi))^2 |g_s(\phi, s_0(\phi))|} \delta_t$$

where the function g_s is defined as in equation (B.3) and $\delta_t = \partial \lambda_t / \partial \lambda_0$ solves the linear stochastic differential equation

$$\delta_t = 1 - \int_0^t \delta_s (1 + 2\lambda_s) v_\lambda^\top dZ_s.$$

Since the unique solution to this equation is nonnegative, the above identity shows that $s_t(\phi)$ is increasing as a function of ϕ_1 if and only if $\alpha_1 > \alpha_2$ and the result now follows from the definition of the agent's welfare. ■

Proof of Proposition 7. Proposition 1 implies that $F = W(\pi^*; \phi^*)$ for some trading strategy $(\pi^*; \phi^*)$ that is self-financing given consumption at rate e . Therefore, it follows

from equation (1) and Proposition 6 that

$$dF_t = r_t F_t dt + \pi_t^* v(dZ_t + \theta_t dt) - e_t dt. \quad (\text{B.4})$$

On the other hand, applying Itô's lemma to (27) and using (23) together with the definition of the consumption share I obtain that

$$\begin{aligned} dF_t &= (\dots) dt + \left[1 - b(t, s(\lambda_t)) + \varepsilon \lambda_t (1 + \lambda_t) s'(\lambda_t) \frac{\partial b}{\partial s}(t, s(\lambda_t)) \right] S_t v dZ_t \\ &= (\dots) dt + \left[1 - b(t, s_t) + \varepsilon s_t \frac{\partial b}{\partial s}(t, s_t) \right] S_t v dZ_t. \end{aligned}$$

Comparing this expression with equation (B.4) then shows that the replicating strategy for the stock is explicitly given by

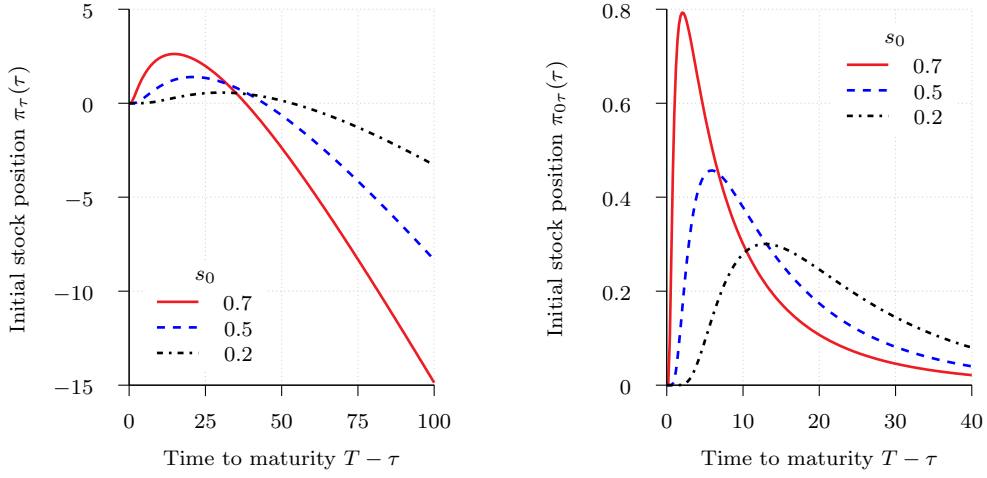
$$(\pi_t^*; \phi_t^*) = \left(1 - b(t, s_t) + \varepsilon s_t \frac{\partial b}{\partial s}(t, s_t); -\varepsilon s_t \frac{\partial b}{\partial s}(t, s_t) \right) S_t$$

and the desired result follows from the definition of the value process $A(\tau)$ and the linearity of the set of self-financing strategies. ■

Proof of Proposition 8 . The proof of this proposition is similar to that of Proposition 7 and, therefore, is omitted. ■

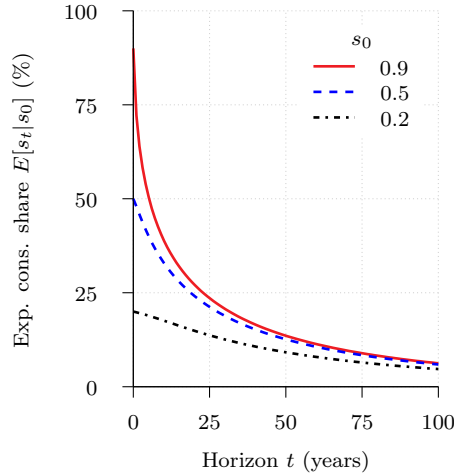
Proof of Proposition 12. This result follows from Proposition 5 and arguments similar to those used in the proof of Proposition 10. I omit the details. ■

Figure 1: Arbitrage strategies



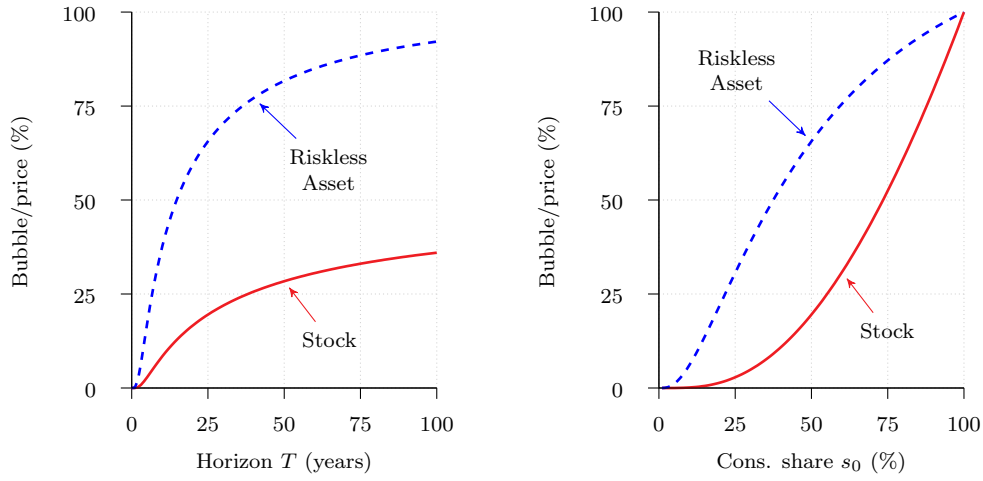
Notes. This figure plots the initial stock position required to arbitrage the bubble on the stock (left panel) and on the riskless asset (right panel) as functions of the trade duration $T - \tau$ for different values of s_0 . In both panels the initial value of the dividend is $e_\tau = 1$ and the other parameters are given by $v_\lambda = 0.20$, $\varepsilon = 0.5$, and $\rho = 0.001$.

Figure 2: Expected consumption share



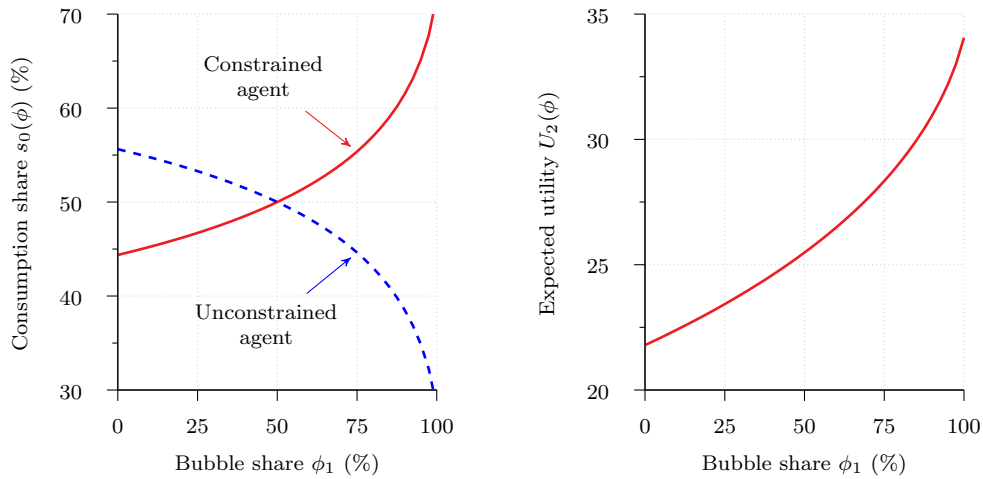
Notes. This figure plots the expected consumption share of the constrained agent $E[s_t|s_0]$ as a function of the horizon t for different values of s_0 and with $v_\lambda = 0.20$.

Figure 3: Relative size of the bubbles



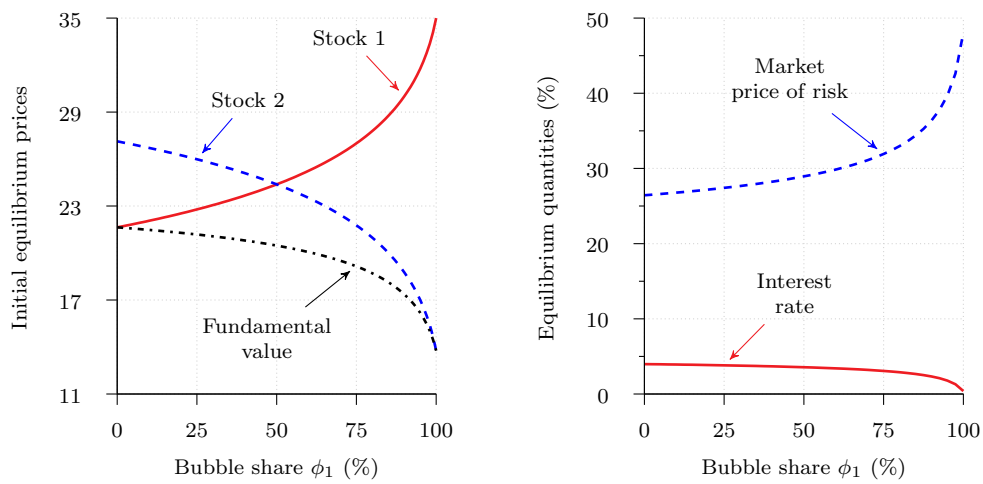
Notes. This figure plots the relative size of the bubbles as functions of the horizon and the initial consumption share of the constrained agent. The constraint weighted volatility is $v_\lambda = 0.20$, the discount rate is $\rho = 0.001$, the model has an horizon of 25 years and the initial consumption share of the constrained agent is $s_0 = 0.50$.

Figure 4: Equilibrium consumption and welfare



Notes. This figure plots the initial consumption share of both agents and the expected utility of the constrained agent as functions of the share of the aggregate bubble that is attributed to the first stock. The initial aggregate dividend is to $e_0 = 1$.

Figure 5: Indeterminacy of equilibrium



Notes. This figure plots the initial stock prices, fundamental values, risk free rate and market price of risk on aggregate shocks as functions of the share of the aggregate bubble that is attributed to the first stock. The initial aggregate dividend is $e_0 = 1$.