

Asset pricing with arbitrage activity*

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Abstract

We study an economy populated by three groups of myopic agents: Constrained agents subject to a portfolio constraint that limits their risk-taking, unconstrained agents subject to a standard nonnegative wealth constraint, and arbitrageurs with access to a credit facility. Such credit is valuable as it allows arbitrageurs to exploit the limited arbitrage opportunities that emerge endogenously in reaction to the demand imbalance generated by the portfolio constraint. The model is solved in closed-form and we show that, in contrast to existing models with frictions and logarithmic agents, arbitrage activity has an impact on the price level and generates both excess volatility and the leverage effect. We show that these results are due to the fact that arbitrageurs amplify fundamental shocks by leveraging up in good times and deleveraging in bad times.

Keywords: Limits of arbitrage; Rational bubbles; Wealth constraints; Excess volatility; Leverage effect.

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1 Introduction

Textbook asset pricing theory asserts that arbitrage opportunities cannot exist in a competitive market because they would be instantly exploited, and thereby eliminated, by arbitrageurs. This basic principle is certainly valid for riskless arbitrage opportunities defined as trades that require no initial investment and whose value can only grow over time.¹ However, there is no reason to believe that it should hold for risky arbitrage opportunities, such as convergence trades,² that guarantee a sure profit at some future date but require capital to fund potential losses at interim dates. Indeed, the fact that arbitrageurs have limited capital and are subject to solvency requirements limits their ability to benefit from such risky arbitrage opportunities and implies that they may subsist in equilibrium. In such cases, the trading activity of arbitrageurs will not suffice to close the arbitrage opportunities but will nonetheless have an impact on the equilibrium, and the goal of this paper is to investigate the effect of this risky arbitrage activity on asset prices, volatilities, risk sharing and welfare.

To address these questions it is necessary to construct a general equilibrium model in which risky arbitrage opportunities exist in the first place. We achieve this by considering a model of an exchange economy similar to those of [Basak and Cuoco \(1998\)](#) and [Hugonnier \(2012\)](#). Specifically, we start from a continuous-time model that includes a riskless asset in zero net supply, a dividend-paying risky asset in positive supply and two groups of agents with logarithmic preferences. Agents in both groups are subject to a standard nonnegativity constraint on wealth,³ but while agents in the first group are unconstrained in their portfolio choice, we assume that agents in the second group are subject to a portfolio constraint that limits their risk-taking and, thereby, tilts their demand towards the riskless asset. This portfolio constraint generates excess demand for the riskless asset and captures in a simple way the global imbalance phenomenon pointed out by [Caballero \(2006\)](#), [Caballero, Farhi, and Gourinchas \(2008\)](#) and [Caballero](#)

¹Such an opportunity arises for example when two assets that carry the same exposure to risk offer different returns. See [Gromb and Vayanos \(2002\)](#) for a model where such arbitrage opportunities arise due to market segmentation, and [Basak and Croitoru \(2000\)](#) for a model where they arise due to the fact that securities are subject to different margin constraints.

²Examples of such arbitrage opportunities include mispricing in equity carve-outs [Lamont and Thaler \(2003a,b\)](#), dual class shares [Lamont and Thaler \(2003a\)](#) and the simultaneous trading of shares from Siamese twin conglomerates such as Royal Dutch and Shell. See [Rosenthal and Young \(1990\)](#), [Lamont and Thaler \(2003a,b\)](#), [Ashcraft, Gârleanu, and Pedersen \(2010\)](#) and [Gârleanu and Pedersen \(2011\)](#).

³Nonnegativity constraints on wealth were originally proposed by [Dybvig and Huang \(1988\)](#) as a realistic mechanism to preclude doubling strategies. They are widely used, and usually considered innocuous, in continuous-time models but are also introduced in discrete-time, infinite-horizon models. See for example [Kocherlakota \(1992\)](#) and [Magill and Quinzii \(1994\)](#) among others.

and Krishnamurthy (2009) among others. This excess demand naturally implies that the interest rate decreases and the market price of risk increases compared to a frictionless economy. But it also implies that the stock and the riskless asset are overvalued in that their equilibrium prices each include a strictly positive bubble.⁴ The intuition is that even though agents of both groups are price takers, the presence of constrained agents places an implicit liquidity provision constraint on unconstrained agents through the market clearing conditions: At times when the portfolio constraint binds, unconstrained agents have to hold the securities that constrained agents cannot, and this is where the mispricing finds its origin. Bubbles arise to incite unconstrained agents to provide a sufficient amount of liquidity, and they persist in equilibrium because the nonnegative wealth constraint prevents them from indefinitely scaling their positions.

To study the impact of arbitrage activity on equilibrium outcomes we then introduce a third group of agents that we refer to as arbitrageurs. These agents have logarithmic utility and are unconstrained in their portfolio choice, but they differ from unconstrained agents along two important dimensions. First, these agents initially hold no capital and thus will only be able to consume if they can exploit the risky arbitrage opportunities that arise due to the presence of constrained agents. Second, these agents have access to a credit facility that enhances their trading opportunities by allowing them to weather transitory periods of negative wealth. This facility should be thought of as a reduced-form for various types of uncollateralized credit such as unsecured financial commercial paper⁵ (see e.g. Kacperczyk and Schnabl (2009), Adrian, Kimbrough, and Marchioni (2010)), implicit lines of credit (see e.g. Sufi (2009)), or loan guarantees. To capture the fact that the availability of arbitrage capital tends to be procyclical (see e.g. Ang, Gorovyy, and Van Inwegen (2011) and Ben-David, Franzoni, and Moussawi (2012)) we assume that this credit facility is proportional to the market portfolio.

We derive the unique equilibrium in closed form in terms of aggregate consumption and an endogenous state variable that measures the consumption share of constrained agents. Importantly, because the portfolio constraint acts as a partial hedge against bad fundamental shocks this state variable is negatively correlated with aggregate consump-

⁴The bubble on the price of a security is the difference between the market price of the security and its fundamental value defined as the minimal amount of capital that an unconstrained agent needs to hold in order to replicate the cash flows of the security while maintaining nonnegative wealth. See Santos and Woodford (1997), Loewenstein and Willard (2000), Hugonnier (2012) and Section 2.5 below for a precise definition and a discussion of the basic properties of asset pricing bubbles.

⁵Commercial paper is among the largest source of short term funding for both financial and non financial institutions. For example, over the period 2010–2013 the average amount of commercial paper outstanding was 1.04 trillion dollars, and about half of that amount is accounted for by unsecured paper issued by financial institutions, see Fred (2014).

tion. The analysis of the equilibrium sheds light on the disruptive role of arbitrageurs in the economy, and reveals that risky arbitrage activity results in an amplification of fundamental shocks that may help understand empirical regularities such as excess volatility and the leverage effect. The main implications can be summarized as follows. First, we show that arbitrage activity brings the equilibrium prices of both securities closer to their fundamental values and simultaneously has a negative impact on the equilibrium stock price level. The latter finding is unique to our setting and stands in stark contrast to what happens in exchange economies with exogenous dividends and logarithmic preferences where the impact of frictions is entirely captured by the interest rate and market price of risk, see for example [Detemple and Murthy \(1997\)](#), [Basak and Cuoco \(1998\)](#) and [Basak and Croitoru \(2000\)](#).

Second, and related, we show that the trading of arbitrageurs pushes the stock volatility above that of the underlying dividend process. This excess volatility is self-generated within the system, and comes from the fact that arbitrageurs amplify fundamental shocks by optimally leveraging up their positions in good times, and deleveraging in bad times. The excess volatility component implied by our model is quantitatively significant, and increases with both the size of the arbitrageurs' credit facility and the consumption share of constrained agents. Since the latter is negatively correlated with aggregate consumption our model implies that volatility tends to increase when the stock price falls, and it follows that risky arbitrage activity is consistent with the leverage effect documented by [Black \(1976\)](#), [Schwert \(1989\)](#), [Mele \(2007\)](#) and [Ait Sahalia, Fan, and Li \(2013\)](#) among others. Furthermore, we show that because constrained agents are partially shielded from bad fundamental shocks the market price of risk is negatively correlated with aggregate consumption and it follows that, in line with the evidence in [Mehra and Prescott \(1985, 2003\)](#) and [Fama and French \(1989\)](#), our model also produces a countercyclical equity premium.

Third, we show that, because arbitrageurs are always levered in equilibrium, arbitrage activity mitigates the portfolio imbalance induced by constrained agents and results in an increase of the interest rate and a decrease of the market price of risk compared to the model where arbitrageurs are absent. This liquidity provision however comes at a cost as we show that arbitrage activity has a negative impact on the consumption share of constrained agents and their welfare. This occurs through two channels: Arbitrage activity reduces the stock price and hence the initial wealth of constrained agents, but it also increases the stock volatility and therefore tightens the portfolio constraint that limits the risk-taking of constrained agents.

Our work is related to several recent contributions in the asset pricing literature. [Basak and Croitoru \(2000\)](#) show that mispricing can arise between two securities that carry the same risk if all agents are subject to a portfolio constraint that prevents them from exploiting the corresponding riskless arbitrage opportunity. If the constraint is removed even for a small fraction of the population then the unconstrained are able to close the arbitrage opportunity and mispricing becomes inconsistent with equilibrium. By contrast, we build a model where risky arbitrage opportunities persist in equilibrium despite the presence of unconstrained agents because they require agents to hold enough capital to sustain interim losses. [Basak and Croitoru \(2006\)](#) consider a production economy version of [Basak and Croitoru \(2000\)](#) in which they introduce risk-neutral arbitrageurs who hold no wealth and are subject to a portfolio constraint that sets an exogenous limit on the size of their position. As in this paper, the activity of these arbitrageurs brings prices closer to fundamentals but the linear production technology that the authors use determines the stock price and its volatility exogenously. On the contrary, all prices in our model are endogenously determined and we show that arbitrage activity generates excess volatility. Other key differences are that while the arbitrageurs in [Basak and Croitoru \(2006\)](#) saturate their constraint and instantaneously consume all profits, the arbitrageurs in this paper are risk-averse, accumulate wealth over time and never actually exhaust their credit limit.

Our findings are also related to those of [Gromb and Vayanos \(2002\)](#) who investigate the welfare implications of financially constrained arbitrage in a segmented market setting with zero net supply securities and an exogenous interest rate. In their model arbitrageurs exploit the riskless arbitrage opportunities that arise across markets and, thereby, allow Pareto improving trade to occur. By contrast, the arbitrageurs in our paper do not alleviate constraints and their trading activity may hinder the welfare of both constrained and unconstrained agents. In a related contribution [Gârleanu, Panageas, and Yu \(2013\)](#) study a model with endogenous entry where a continuum of investors, assets and financial intermediaries are located on a circle. In their model participation costs and collateral constraints prevent agents from trading in all assets. This implies that diversifiable risk is priced and exposes riskless arbitrage opportunities that cannot be eliminated due to prohibitive participation costs. By contrast, we show that the presence of constrained agents generates risky arbitrage opportunities that persist in equilibrium because unconstrained agents and arbitrageurs are subject to wealth constraints and study the feedback effects of arbitrage activity on the equilibrium outcomes.

Most of the existing models with risky arbitrage opportunities are cast in partial equilibrium. For example, [Liu and Longstaff \(2004\)](#) study the portfolio choice problem of a risk averse arbitrageur subject to margin constraints and who can trade in a riskless asset with a constant interest rate and an exogenous arbitrage opportunity modeled as a Brownian bridge.⁶ [Kondor \(2009\)](#) considers a model in which a risk neutral arbitrageur exploits an exogenous arbitrage opportunity whose duration is distributed exponentially. As in this paper, the price gap between the mispriced securities may diverge before it converges and thereby inflict interim losses on arbitrageurs. By contrast, we study the role of credit in enlarging the set of strategies available to arbitrageurs and its impact on equilibrium quantities such as prices, volatilities and interest rates within a model where mispricing is microfounded and all markets clear.

In a recent contribution [Haddad \(2013\)](#) also considers a model where some agents are levered and bear more aggregate risk but the mechanism is different. In his setting, agents choose dynamically whether to be levered in the stock in which case they actively participate in the firm's management. The collective activism of levered agents improves the growth rate of dividends, and these agents are remunerated for this service in a competitive way that makes the firm indifferent to the level of active capital. The analysis of [Haddad \(2013\)](#) also highlights the impact of deleveraging risk on equilibrium outcomes but, in contrast to this paper, the equilibrium of his economy features neither risky arbitrage opportunities nor excess volatility.

Finally, we highlight the connections with the literature on rational asset pricing bubbles.⁷ [Santos and Woodford \(1997\)](#) and [Loewenstein and Willard \(2000\)](#) show that in frictionless economies with complete markets bubbles may exist on zero net supply securities, such as options and futures, but not on positive net supply securities such as stocks. [Hugonnier \(2012\)](#) shows that the presence of portfolio constraints may generate bubbles also on positive net supply securities even if some agents are unconstrained, and [Prieto \(2012\)](#) extends these conclusions to economies where agents have heterogenous risk aversion and beliefs. In addition to these contributions, there are papers that analyze the properties of bubbles in partial equilibrium settings. In particular, [Cox and Hobson \(2005\)](#)

⁶In the same spirit [Jurek and Yang \(2007\)](#) and [Liu and Timmermann \(2012\)](#) study portfolio choice problems in which the value of the arbitrage opportunity follows a mean-reverting process so that the amount of time necessary to generate a profit is random.

⁷The literature on speculative bubbles (see e.g. [Miller \(1977\)](#), [Harrison and Kreps \(1978\)](#) and [Scheinkman and Xiong \(2003\)](#)) uses a different definition of the fundamental value that is not based on any cash-flow replication considerations and, therefore, cannot connect bubbles to the existence of arbitrage opportunities. Furthermore, these models are in general set in partial equilibrium as they assume the existence of a riskless technology in infinitely elastic supply.

and [Heston, Loewenstein, and Willard \(2007\)](#) study bubbles on the price of derivatives, and [Jarrow, Protter, and Shimbo \(2010\)](#) analyze bubbles in models with incomplete markets. An important difference between these partial equilibrium studies and the present paper lies in the fact that they assume the existence of a risk neutral measure and, therefore, rule out the existence of a bubble on the riskless asset. By contrast, we show that such a pricing measure cannot exist in our model because the existence of a bubble on the riskless asset is a necessary condition for equilibrium.

The remainder of the paper is organized as follows. In [Section 2](#) we present our assumptions and provide a basic discussion of bubbles. In [Section 3](#) we solve the individual optimization problems, derive the equilibrium and discuss its main properties. In [Section 4](#) we analyze the equilibrium trading strategies and present the implications of the model for excess volatility and the leverage effect. In [Appendix A](#) we show that our results remain qualitatively unchanged under an alternative, less cyclical specification of the credit facility that puts a constant bound on the discounted losses that the arbitrageurs is allowed incur. All proofs are gathered in [Appendix B](#).

2 The model

2.1 Securities markets

We consider a continuous-time economy on an infinite horizon.⁸ Uncertainty in the economy is represented by a probability space carrying a Brownian motion Z_t and in what follows we assume that all random processes are adapted with respect to the usual augmentation of the filtration generated by this Brownian motion.

Agents trade in two securities: A riskless asset in zero net supply and a stock in positive supply of one unit. The price of the riskless asset evolves according to

$$S_{0t} = 1 + \int_0^t S_{0u} r_u du \tag{1}$$

for some short rate process r_t that is to be determined in equilibrium. On the other hand, the stock is a claim to a dividend process δ_t that evolves according to a geometric Brownian motion with constant drift μ_δ and volatility $\sigma_\delta > 0$. The stock price is denoted

⁸We assume an infinite horizon to avoid having to keep track of time as a state variable, but this assumption is not needed for the validity of our conclusions. In particular, bubbles also arise in a finite horizon version of our model and the arbitrage activity that they attract results in a price decrease and a countercyclical excess volatility component in equilibrium stock returns.

by S_t and evolves according to

$$S_t + \int_0^t \delta_u du = S_0 + \int_0^t S_u (\mu_u du + \sigma_u dZ_u) \quad (2)$$

for some initial value $S_0 > 0$, some drift μ_t and some volatility σ_t that are to be determined endogenously in equilibrium.

2.2 Trading strategies

A trading strategy is a pair $(\pi_t; \phi_t)$ where π_t represents the amount invested in the stock while ϕ_t represents the amount invested in the riskless asset. A trading strategy is said to be self-financing given initial wealth w and consumption rate $c_t \geq 0$ if the corresponding wealth process satisfies

$$W_t \equiv \pi_t + \phi_t = w + \int_0^t (\phi_u r_u + \pi_u \mu_u - c_u) du + \int_0^t \pi_u \sigma_u dZ_u. \quad (3)$$

Implicit in the definition is the requirement that the trading strategy and consumption plan be such that the above stochastic integrals are well-defined.

2.3 Agents

The economy is populated by three agents indexed by $k \in \{1, 2, 3\}$. Agent k is endowed with $n_k \in [0, 1]$ units of the stock and his preferences are represented by

$$U_k(c) \equiv E \left[\int_0^\infty e^{-\rho t} \log(c_t) dt \right]$$

for some subjective discount rate $\rho > 0$. In what follows we let $w_k \equiv n_k S_0$ denote the initial wealth of agent k computed at equilibrium prices.

The three agents have homogenous preferences and beliefs but differ in their trading opportunities. Specifically, agent 1 is free to choose any strategy whose wealth remains nonnegative at all times, and we will refer to him as the unconstrained agent. Agent 2, to whom we will refer as the constrained agent, is subject to the same nonnegative wealth requirement as agent 1 but is also required to choose a strategy that satisfies

$$\pi_t \in \mathcal{C}_t \equiv \{ \pi \in \mathbb{R} : |\sigma_t \pi| \leq (1 - \varepsilon) \sigma_\delta W_t \},$$

for some given $\varepsilon \in [0, 1]$. This portfolio constraint can be thought of as limiting the amount of risk the agent is allowed to take. In particular, if $\sigma_t \geq \sigma_\delta$ (which we will show is the case in equilibrium) then it forces agent 2 to keep a strictly positive fraction of his wealth in the riskless asset at all times and, thereby, introduces an imbalance that ultimately generates bubbles. This constraint is also a special case of the general risk constraints in [Cuoco, He, and Isaenko \(2008\)](#) and is studied in both [Gârleanu and Pedersen \(2007\)](#) and [Prieto \(2012\)](#) as a constraint on conditional value-at-risk.

Agent 3 is free to choose any self-financing strategy but, in contrast to the two other agents, he is not required to maintain nonnegative wealth. Instead, we assume that this agent has access to a credit facility that allows him to withstand short term deficits provided that his wealth satisfies the lower bound

$$W_{3t} + \psi S_t \geq 0, \quad t \geq 0, \tag{4}$$

and the transversality condition

$$\liminf_{T \rightarrow \infty} E[\xi_T W_T] \geq 0 \tag{5}$$

for some exogenously fixed $\psi \geq 0$ where ξ_t is the state price density process defined in (6) below. This agent should be thought of as an arbitrageur whose funding liquidity conditions are determined by the magnitude of ψ . The fact that the amount of credit available to this arbitrageur increases with the size of the market is meant to capture in a simple way the observation that capital availability improves in times where the stock market is high, see for example [Ben-David, Franzoni, and Moussawi \(2012\)](#) and [Ang, Gorovyy, and Van Inwegen \(2011\)](#) for evidence on hedge fund trading.

Since agent 3 can continue trading in states of negative wealth, the wealth constraint of the arbitrageur in (4) allows for excess borrowing. Trades in these states may be considered uncollateralized as agent 3 does not have enough assets to cover his liabilities in case of instantaneous liquidation. Note however that, due to his preferences, agent 3 will never willingly stop servicing debt or risk a rollover freeze of short-term debt. In other words, agent 3 is balance sheet insolvent but never cash-flow insolvent in states of negative wealth and, as a result, his debt is riskless at all times.

To emphasize the interpretation of agent 3 as an arbitrageur, we assume from now on that $n_3 = 0$ so that his initial wealth is zero.⁹ This in turn implies that the endowments

⁹For simplicity we assume that the mass and initial wealth of arbitrageurs are exogenously fixed but these quantity can be easily endogenized by assuming that arbitrageurs are heterogenous in the size of

of the other agents can be summarized by the number $n = n_2 \in (0, 1]$ of shares of the stock initially held by the constrained agent.

2.4 Definition of equilibrium

The concept of equilibrium that we use is similar to that of equilibrium of plans, prices and expectations introduced by Radner (1972):

Definition 1. *An equilibrium is a pair of security price processes (S_{0t}, S_t) and an array $\{c_{kt}, (\pi_{kt}, \phi_{kt})\}_{k=1}^3$ of consumption plans and trading strategies such that:*

1. *Given (S_{0t}, S_t) the plan c_{kt} maximizes U_k over the feasible set of agent k and is financed by the trading strategy (π_{kt}, ϕ_{kt}) .*
2. *Markets clear: $\sum_{k=1}^3 \phi_{kt} = 0$, $\sum_{k=1}^3 \pi_{kt} = S_t$ and $\sum_{k=1}^3 c_{kt} = \delta_t$.*

An equilibrium is said to have arbitrage activity if the consumption plan of the arbitrageur is not identically zero.

Since the arbitrageur starts from zero wealth it might be that the set of consumption plans he can finance is empty. In such cases, his consumption and optimal portfolio are set to zero and the equilibrium only involves the two other agents. To determine conditions under which the arbitrageur participates it is necessary to characterize his feasible set. This is the issue to which we now turn.

2.5 Feasible sets, bubbles, and limited arbitrages

Let (S_{0t}, S_t) denote the prices in a given equilibrium and assume that there are no riskless arbitrages for otherwise the market could not be in equilibrium. As is well-known (see e.g., Duffie (2001)), this implies that $\mu_t = r_t + \sigma_t \theta_t$ for some process θ_t . This process is referred to as the market price of risk and is uniquely defined on the set where volatility is non zero. Now consider the state price density defined by

$$\xi_t = \frac{1}{S_{0t}} \exp \left(- \int_0^t \theta_u dZ_u - \frac{1}{2} \int_0^t |\theta_u|^2 du \right). \quad (6)$$

their credit facility and have to pay a cost to enter the market. In such a setting an arbitrageur will enter only if the profits that his credit facility allows him to generate exceed the entry cost, and the aggregation of these decisions gives rise to a representative arbitrageur similar to the one we use but whose credit facility reflects the entry costs and the cross-sectional distribution of individual credit facilities.

The following proposition shows that the ratio $\xi_{t,u} = \xi_u/\xi_t$ can be used as a pricing kernel to characterize the feasible sets of agents 1 and 3 and provides conditions under which the arbitrageur participates in the market.

Proposition 1. *Assume that $\lim_{T \rightarrow \infty} E[\xi_T S_T] = 0$.¹⁰ A consumption plan c_t is feasible for agent $k \in \{1, 3\}$ if and only if*

$$E \left[\int_0^\infty \xi_t c_t dt \right] \leq w_k + 1_{\{k=3\}} \psi(S_0 - F_0)$$

where the nonnegative process

$$F_t \equiv E_t \left[\int_t^\infty \xi_{t,u} \delta_u du \right] \tag{7}$$

gives the minimal amount that agent 1 needs to hold at time $t \geq 0$ to replicate the dividends of the stock while maintaining nonnegative wealth. In particular, the feasible set of agent 3 is non empty if and only if $\psi(S_0 - F_0) > 0$.

Following the literature on rational bubbles (see for example Santos and Woodford (1997), Loewenstein and Willard (2000), and Hugonnier (2012)) we refer to F_t in (7) as the *fundamental value* of the stock; and to

$$B_t \equiv S_t - F_t = S_t - E_t \left[\int_t^\infty \xi_{t,u} \delta_u du \right] \geq 0$$

as the *bubble* on its price. Using this terminology, Proposition 1 shows that the feasible set of the arbitrageur is empty unless two conditions are satisfied: There needs to be a strictly positive bubble on the stock, and the agent must have access to the credit facility. The intuition behind this result is clear: Since the arbitrageur does not hold any initial wealth he can only consume in the future if there are arbitrage opportunities in the market that he is able to exploit, at least to some extent.

At first glance, it might seem that a stock bubble should be inconsistent with optimal choice, and therefore also with the existence of an equilibrium, since it implies that two assets with the same cash flows (the stock and the portfolio that replicates its dividends) are traded at different prices. The reason why this is not so is that, due to wealth constraints, bubbles only constitute limited arbitrage opportunities. To see this, consider

¹⁰This transversality condition guarantees that the deflated price of the stock converges to zero at infinity and allows for a simple characterization of the feasible set of agent 3. This condition can be relaxed at the cost of a more involved characterization (see Lemma B.1 in the Appendix) but is without loss of generality since we show that it necessarily holds in equilibrium.

the textbook strategy that sells short $x > 0$ shares, buys the portfolio that replicates the corresponding cash flows over a given finite time interval $[0, T]$ and invests the remainder in the riskless asset. The value process of this trading strategy is

$$A_t(x; T) = x(S_{0t}B_0(T) - B_t(T))$$

where

$$B_t(T) \equiv S_t - F_t(T) = S_t - E_t \left[\int_t^T \xi_{t,u} \delta_u du + \xi_{t,T} S_T \right] \geq 0 \quad (8)$$

gives the bubble on the price of the stock over the interval $[t, T]$. This trade requires no initial investment and if the stock price includes a strictly positive bubble then its terminal value $A_T(x; T) = xS_{0T}B_0(T)$ is strictly positive so it does constitute an arbitrage opportunity in the usual sense. But this opportunity is risky because it entails the possibility of interim losses and, therefore, cannot be implemented to an arbitrary scale by the agents in the economy. In particular, the arbitrageur can only implement this trade up to size ψ because otherwise the solvency constraint (4) would not be satisfied. Similarly, the unconstrained agent can only implement this arbitrage if he holds sufficient collateral in the form of cash or securities.

The discussion has so far focused on the stock but bubbles may be defined on any security, including the riskless asset. Indeed, over an interval $[0, T]$ the riskless asset can be viewed as a derivative security that pays a single terminal dividend equal to S_{0T} . The fundamental value of such a security is $F_{0t}(T) = E_t [\xi_{t,T} S_{0T}]$ whereas its market value is simply given by S_{0t} , and this naturally leads to defining the finite horizon bubble on the riskless asset as

$$B_{0t}(T) \equiv S_{0t} - F_{0t}(T) = S_{0t} \left(1 - E_t \left[\xi_{t,T} \frac{S_{0T}}{S_{0t}} \right] \right). \quad (9)$$

As in the case of the stock, a bubble on the riskless asset is consistent with both optimal choice and the existence of an equilibrium in our economy. In fact, we show below that due to the presence of constrained agents bubbles on both the stock and the riskless asset are necessary for markets to clear.

Remark 1. Equation (9) shows that the riskless asset has a bubble over $[0, T]$ if and only if the process $M_t \equiv S_{0t} \xi_t$ satisfies $E[M_T] < M_0 = 1$. Since the economy is driven by a single source of risk this process is the unique candidate for the density of the risk-neutral

measure and it follows that the presence of a bubble on the riskless asset is equivalent to the non existence of the risk-neutral measure, see [Loewenstein and Willard \(2000\)](#), and [Heston, Loewenstein, and Willard \(2007\)](#) for derivatives pricing implications.

Remark 2. Combining (2) and (3) reveals that under the assumption of Proposition 1 a given consumption plan is feasible for the arbitrageur if and only there exists a trading strategy $(\pi_t^*; \phi_t^*)$ such that the process

$$W_t^* = W_t + \psi S_t = \psi S_0 + \int_0^t (r_u \phi_u^* + \pi_u^* \mu_u - c_u - \psi \delta_u) du + \int_0^t \pi_u^* \sigma_u dZ_u \quad (10)$$

is nonnegative and satisfies the transversality condition (5). This shows that the feasible set of the arbitrageur coincides with that of an auxiliary agent who has initial capital ψS_0 , receives income at rate $-\psi \delta_t$ and is subject to a liquidity constraint that requires him to maintain nonnegative wealth at all times as in [He and Pagès \(1993\)](#), [El Karoui and Jeanblanc-Picqué \(1998\)](#) and [Detemple and Serrat \(2003\)](#).

Given this observation it might seem surprising that the static characterization of the arbitrageur's feasible set involves only the unconstrained state price density ξ_t rather than a family of shadow state price densities. This is due to the fact that, because the implicit income rate $-\psi \delta_t$ in (10) is negative, the liquidity constraint of the auxiliary agent is non binding. The intuition is clear: In order to consume at a nonnegative rate while simultaneously receiving negative income over time this agent must necessarily maintain nonnegative wealth at all times. Mathematically, it follows from [El Karoui and Jeanblanc-Picqué \(1998\)](#) and the above observation that a consumption plan is feasible for the arbitrageur if and only if it satisfies

$$\sup_{\Lambda \in \mathcal{L}} E \left[\int_0^\infty \Lambda_u \xi_u (c_u + \psi \delta_u) du \right] \leq \psi S_0$$

where \mathcal{L} denotes the set of nonnegative, decreasing processes with initial value smaller than one and, since $\psi \delta_t \geq 0$, we have that the supremum is attained by $\Lambda_u^* \equiv 1$. This important simplification implies that the marginal utility of the arbitrageur is a function of the unconstrained state price density and allows us to characterize the equilibrium in terms of an endogenous state variable that follows a standard diffusion process rather than a reflected diffusion process, see Proposition 3.

3 Equilibrium

3.1 Individual optimality

Combining Proposition 1 with well-known results on logarithmic utility maximization leads to the following characterization of optimal policies.

Proposition 2. *Assume that $\lim_{T \rightarrow \infty} E[\xi_T S_T] = 0$. Then the optimal consumption and trading strategies of the three agents are given by*

$$c_{kt} = \rho (W_{kt} + 1_{\{k=3\}} \psi B_t) \quad (11)$$

and

$$\begin{aligned} \pi_{1t} &= (\theta_t / \sigma_t) W_{1t} \\ \pi_{2t} &= \kappa_t (\theta_t / \sigma_t) W_{2t} \end{aligned} \quad (12)$$

$$\pi_{3t} = (\theta_t / \sigma_t) (W_{3t} + \psi B_t) - \psi (\Sigma_t^B / \sigma_t) \quad (13)$$

where

$$\kappa_t = \min \left(1; \frac{(1 - \varepsilon) \sigma_\delta}{|\theta_t|} \right) \in [0, 1], \quad (14)$$

and the process Σ_t^B denotes the diffusion coefficient of the process B_t .

The solution for the unconstrained agent 1 is standard given logarithmic preferences. Indeed, this agent invests in an instantaneously mean-variance efficient portfolio and has a constant marginal propensity to consume equal to his discount rate. The solution for the constrained agent 2 follows from Cvitanić and Karatzas (1992) and shows that the constraint binds in states where the market price of risk is high. This is intuitive: Since agent 2 has logarithmic preferences we know that absent the portfolio constraint he would invest in proportion to the market price of risk and the result follows by noting that the constraint limits the amount of risk he is allowed to take.

The solution for the arbitrageur is novel and illustrates how this agent is able to reap arbitrage profits, and thereby consume, despite the fact that he holds no initial capital. Specifically, (13) shows that the optimal strategy for this agent is to short ψ shares of the stock, buy ψ units of the portfolio that replicates the stock dividends, and invest the strictly positive net proceeds of these transactions into the same mean

variance efficient portfolio as agent 1. This strategy is only admissible because of the credit facility and allows the arbitrageur to increase his consumption basis from W_{3t} to $W_{3t} + \psi B_t = e^{-\rho t} \psi B_0 / \xi_t$. The optimal consumption in (11) then follows by noting that, since the arbitrageur has logarithmic preferences, his marginal propensity to consume is constant and equal to his subjective discount rate.

Remark 3. The optimal policy of the arbitrageur bears a close resemblance to that of an hypothetical agent with logarithmic utility and no initial wealth who receives labor income at rate e_t in a complete market with state price density ξ_t . Indeed, the optimal consumption of such an agent is

$$c_t = \rho(W_t + H_t) = \rho e^{-\rho t} (H_0 / \xi_t)$$

where H_t gives the fundamental value of the agent's future income, and an application of Itô's lemma shows that his optimal trading strategy is

$$\pi_t = (\theta_t / \sigma_t)(W_t + H_t) - (\Sigma_t^H / \sigma_t)$$

where Σ_t^H denotes the diffusion coefficient of the process H_t . This solution is isomorphic to that given in Proposition 2 with one important caveat: Instead of arising exogenously from the agent's income, the process $H_t = \psi B_t$ in this paper is endogenously generated by the profits that arbitrageurs are able to reap from the market.

Remark 4. Since $W_{3t} + \psi S_t = \psi(F_t + e^{-\rho t} B_0 / \xi_t) > 0$ the arbitrageur never exhausts his credit limit. Although of a different nature, this result is reminiscent of [Liu and Longstaff \(2004\)](#) who study the portfolio decisions of an arbitrageur facing an exogenous arbitrage opportunity modeled as a Brownian bridge and a margin constraint.

3.2 Equilibrium allocations and risk-sharing

To characterize the equilibrium we use a representative agent with stochastic weights that allows to easily clear markets despite the imperfect risk sharing induced by the presence of the constrained agent (see [Cuoco and He \(1994\)](#)). The utility function of this representative agent is defined by

$$u(c, \gamma, \lambda_t) \equiv \max_{c_1 + c_2 + c_3 = c} (\log(c_1) + \lambda_t \log(c_2) + \gamma \log(c_3))$$

where $\lambda_t > 0$ is an endogenously determined weighting process that encapsulates the differences across the agents and $\gamma \geq 0$ is an endogenous constant that determines the relative weight of arbitrageurs in the economy.

By Proposition 2, we have that the first order condition for agents 1 and 3 can be written as $\xi_t = e^{-\rho t}(c_{k0}/c_{kt})$ for $k = 1, 3$. Comparing these to the first order condition of the representative agent's problem shows that the equilibrium state price density and the equilibrium consumption are given by

$$\xi_t = e^{-\rho t} \frac{u_c(\delta_t, \gamma, \lambda_t)}{u_c(\delta_0, \gamma, \lambda_0)} = e^{-\rho t} \frac{\delta_0(1-s_0)}{\delta_t(1-s_t)}, \quad (15)$$

and

$$c_{2t} = \rho W_{2t} = s_t \delta_t, \quad (16)$$

$$c_{1t} = \rho W_{1t} = \frac{1}{1+\gamma}(1-s_t)\delta_t, \quad (17)$$

$$c_{3t} = \rho(W_{3t} + \psi B_t) = \delta_t - c_{1t} - c_{2t} = \frac{\gamma}{1+\gamma}(1-s_t)\delta_t,$$

where the endogenous state variable

$$s_t \equiv \frac{c_{2t}}{\delta_t} = \frac{\lambda_t}{1+\gamma+\lambda_t} \in (0, 1) \quad (18)$$

tracks the consumption share of the constrained agent. Combining (6), (12), (15) and (17) then allows to determine the equilibrium drift and volatility of this state variable, and delivers the following explicit characterization of equilibrium.

Proposition 3. *In equilibrium, the riskless rate of interest and the market price of risk are explicitly given by*

$$\theta_t = \sigma_\delta \left(1 + \frac{\varepsilon s_t}{1-s_t} \right), \quad (19)$$

$$r_t = \rho + \mu_\delta - \sigma_\delta \theta_t = \rho + \mu_\delta - \sigma_\delta^2 \left(1 + \frac{\varepsilon s_t}{1-s_t} \right), \quad (20)$$

and the consumption share of the constrained agent evolves according to the stochastic differential equation

$$ds_t = -s_t \varepsilon \sigma_\delta \left(dZ_t + \frac{s_t}{1-s_t} \varepsilon \sigma_\delta dt \right) \quad (21)$$

with initial condition $s_0 = \rho w_2 / \delta_0$.

The above characterization of equilibrium is notable for two reasons. First, it follows from (12), (14) and (19) that the portfolio of agent 2 satisfies

$$\sigma_t \pi_{2t} = W_{2t}(1 - \varepsilon)\sigma_\delta < W_{2t}\theta_t. \quad (22)$$

This shows that the portfolio constraint binds at all times in equilibrium and it follows that agent 2 constantly has a positive demand for the riskless asset. This in turn implies that prices should adjust to entice agents 1 and 3 to borrow and explains why, as shown by (19) and (20), the market price of risk increases and the interest rate decreases compared to an unconstrained economy ($\varepsilon = 0$). Second, (21) shows that the consumption share of the constrained agent is negatively correlated with dividends and therefore tends to decrease (increase) following sequences of positive (negative) cash flow shocks. The intuition for this result is clear: by limiting the amount of risk that agent 2 can take, the portfolio constraint implies that his consumption is less sensitive to bad shocks but also limits the extent to which it benefits from sequences of good shocks.

3.3 Equilibrium prices

To compute the equilibrium stock price, we rely on the market clearing conditions which require that $S_t = \sum_{k=1}^3 W_{kt}$. Combining this identity with (11), (15) and the clearing of the consumption good market gives

$$S_t = P_t - \psi B_t = P_t - \psi(S_t - F_t) \quad (23)$$

where $P_t = \delta_t / \rho$ is the stock price that would prevail in equilibrium if arbitrageurs were absent from the economy and

$$F_t = E_t \left[\int_t^\infty \xi_{t,u} \delta_u du \right] = \delta_t (1 - s_t) E_t \left[\int_t^\infty e^{-\rho(u-t)} \frac{du}{1 - s_u} \right]$$

gives the fundamental value of the stock. Setting $\alpha = \psi / (1 + \psi) \in [0, 1]$ and solving (23) gives

$$S_t = \alpha F_t + (1 - \alpha) P_t.$$

To complete the characterization of the equilibrium, it remains to determine whether the price includes a bubble. Using the above expression together with (18) shows that

$$B_t = (1 - \alpha)(P_t - F_t) = \frac{(1 - \alpha)\delta_t}{1 + \gamma + \lambda_t} E_t \left[\int_t^\infty e^{-\rho(u-t)} (\lambda_t - \lambda_u) du \right] \quad (24)$$

and it follows that the stock price is bubble free if and only if the weighting process is a martingale. An application of Itô's lemma gives

$$d\lambda_t = (1 + \gamma)d\left(\frac{s_t}{1 - s_t}\right) = -\lambda_t(1 + \gamma + \lambda_t)\frac{\varepsilon\sigma_\delta}{1 + \gamma}dZ_t,$$

so that the weighting process is a local martingale. However, the following proposition shows that this local martingale fails to be a true martingale and, thereby, proves that any equilibrium includes a bubble on the stock and arbitrage activity.

Proposition 4. *The weighting process is a strict local martingale. In particular, the stock price includes a strictly positive bubble in any equilibrium.*

Combining the above proposition with (7), (24) and Itô's product rule reveals that in equilibrium the discounted gains process

$$\xi_t S_t + \int_0^t \xi_u \delta_u du = E_t \left[\int_0^\infty \xi_u \delta_u du \right] + \xi_t B_t$$

is a local martingale, as required to rule out riskless arbitrage opportunities, but not a true martingale. This result shows that the distinction between local and true martingales, which is usually perceived as a technicality, actually captures an important economic phenomenon, namely the presence of an asset pricing bubble. It also clearly indicates that continuous-time bubbles are of a different nature than those that may arise in discrete-time models. In particular, since a discrete-time local martingale is a true martingale over any finite horizon (see Meyer (1972)), the same arguments as in Santos and Woodford (1997) imply that a stock bubble cannot arise in a discrete-time version of our model. By contrast, Proposition 4 shows that in continuous-time a bubble arises as soon as there are constrained agents in the economy and cannot be eradicated by arbitrage activity unless arbitrageurs have access to unbounded credit.

Our next result establishes the existence and uniqueness of the equilibrium and derives closed form expressions for the stock price and its bubble.

Theorem 1. *There exists a unique equilibrium. In this equilibrium the stock price and its bubble are explicitly given by*

$$S_t = (1 - \alpha s_t^\eta) P_t \quad (25)$$

$$B_t = S_t - F_t = (1 - \alpha) s_t^\eta P_t \quad (26)$$

with the constant

$$\eta \equiv \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2\rho}{(\varepsilon\sigma_\delta)^2}} > 1,$$

and the consumption share of the constrained agent evolves according to (21) with initial condition given by the unique $s_0 \in (0, 1)$ such that $s_0 = n(1 - \alpha s_0^\eta)$.

The above theorem offers several important conclusions. First, it shows that the unique equilibrium includes a strictly positive bubble on the stock and, therefore, generates arbitrage activity as soon as agent 3 has access to a credit facility in that $\alpha > 0$.

Second, it shows that in stark contrast to existing equilibrium models with frictions and logarithmic agents (see e.g. [Detemple and Murthy \(1997\)](#), [Basak and Cuoco \(1998\)](#) and [Basak and Croitoru \(2000\)](#)) the combination of portfolio constraints and arbitrage activity generates a price/dividend ratio that is both time-varying and lower than that which would have prevailed in the absence of arbitrageurs. Importantly, (25) shows that the price/dividend ratio is a decreasing function of the consumption share of constrained agents and, since the latter is negatively correlated with fundamental shocks, we have that arbitrage activity generates both excess volatility and the leverage effect despite the fact that all agents have logarithmic preferences. We will come back to this important property in Section 4.2 where we will be able to interpret it in light of the equilibrium portfolio strategies that we derive in Section 4.1.

Third, the expression for the stock price shows that arbitrage activity has a negative impact on the equilibrium price level. One way to understand this result is to observe that arbitrage activity makes the stock more volatile and thereby reduces the value of the collateral services it provides. Another way to understand this result is to view the price decrease as rents to the arbitrage technology. Comparing the consumption of the agent 1 to that of agent 3 and using (26) shows that these rents accrue to the arbitrageur in the form of an infinitely-lived stream of consumption at rate $\alpha \delta_t s_t^\eta$. This additional consumption reduces the amount of dividend available to stockholders to $\delta_t(1 - \alpha s_t^\eta)$ and

the equilibrium stock price is simply given by the usual logarithmic valuation formula applied to this reduced dividend process.

Our next result shows that, in addition to a stock bubble, the equilibrium price system also includes a bubble on the riskless asset over any investment horizon. Importantly, combining this result with Remark 1 shows that in the unique equilibrium of our economy a risk-neutral probability measure does not exist.

Proposition 5. *In the unique equilibrium the price of the riskless asset is*

$$S_{0t} = e^{qt} \frac{P_t}{P_0} \left(\frac{s_t}{s_0} \right)^{1/\varepsilon} \quad (27)$$

with the constant $q = \rho - \frac{1}{2}(1 - \varepsilon)\sigma_\delta^2$. Over the time interval $[t, T]$ the stock and the riskless asset include bubble components that satisfy

$$\frac{B_t(T)}{S_t} = H(T - t, s_t; 2\eta - 1) \frac{B_t}{S_t} \leq H(T - t, s_t; 2/\varepsilon - 1) = \frac{B_{0t}(T)}{S_{0t}} \quad (28)$$

where we have set

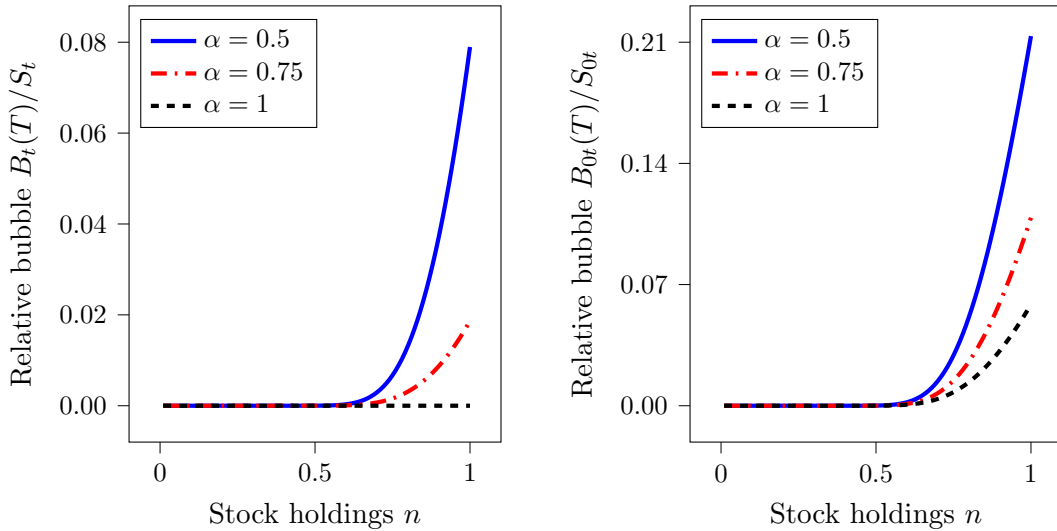
$$\begin{aligned} d_\pm(\tau, s; a) &\equiv \frac{\log(s)}{\varepsilon\sigma_\delta\sqrt{\tau}} \pm \frac{a}{2}\varepsilon\sigma_\delta\sqrt{\tau}, \\ H(\tau, s; a) &\equiv N(d_+(\tau, s; a)) + s^{-a}N(d_-(\tau, s; a)), \end{aligned} \quad (29)$$

and the function $N(x)$ denotes the cdf of a standard normal random variable.

The above proposition shows that in relative terms, i.e. for each dollar of investment, the bubble on the riskless asset is larger than that on the stock over any investment horizon. Since an agent subject to a nonnegative wealth constraint cannot short both assets, we naturally expect that agent 1 will choose a strategy that exploits the bubble on the riskless asset because it requires less collateral per unit of initial profit. This intuition will be confirmed in the Section 4.1 below where we show that the equilibrium strategy of the unconstrained agent can be seen as the combination of an all equity portfolio and a short position in the riskless asset bubble.

Comparing the results of Theorem 1 and Proposition 5 reveals that arbitrage activity has a different effect on the stock and riskless asset bubbles. Indeed, (26) shows that arbitrage activity impacts the stock bubble both directly through the constant $1 - \alpha$ and indirectly through the initial value of the constrained agent's consumption share process, while (28) shows that only the later channel is at work for the riskless asset bubble.

Figure 1: Bubbles and credit conditions



Notes. This figure plots the relative bubbles at horizon $T = 100$ on the stock (left panel) and the riskless asset (right panel) as functions of the initial stock holdings of the constrained agent for different value of the parameter governing the size of the credit facility. To construct this figure we set $\sigma_\delta = 0.0357$, $\rho = 0.001$ and $\varepsilon = 0.5$.

An immediate consequence of this observation is that while the stock bubble disappears when arbitrageurs have access to unlimited credit ($\alpha \rightarrow 1 \Leftrightarrow \psi \rightarrow \infty$), the riskless asset bubble is always strictly positive. The reason for this difference is that our formulation of the wealth constraint gives the arbitrageur a comparative advantage over agent 1 in exploiting the stock bubble, but not the riskless asset bubble.

Since the right hand side of (18) is increasing and concave in the weighting process, it follows from Jensen's inequality and Proposition 4 that s_t is a supermartingale. This implies that the consumption share of constrained agents is expected to decline over time and a direct calculation provided in the appendix shows that we have $s_\infty = 0$ so that constrained agents, and the bubbles that their presence generates, actually disappear in the long run. A natural way to correct this behavior and thereby ensure that bubbles subsist even in the long run is to allow for birth and death of agents of the various kinds in such a way as to continuously repopulate the group of constrained agents. See [Gârleanu and Panageas \(2014\)](#) for a recent contribution along these lines.

To illustrate the magnitude of the bubbles and the impact of arbitrage activity we plot in Figure 1 the relative bubbles on the stock and the riskless asset as functions of the initial stock holdings of the constrained agent in an economy where agents have

discount rate $\rho = 0.001$, the constraint is set at $\varepsilon = 0.5$, and the volatility of the aggregate consumption is taken to be $\sigma_\delta = 0.0357$ as proposed by [Basak and Cuoco \(1998\)](#) to match the estimates of [Mehra and Prescott \(1985\)](#). As shown by the figure the bubbles can account for a significant fraction of the equilibrium prices and this fraction decreases both as the fraction of the market held by constrained agents decreases and as credit conditions improves. To gain more insight into these properties it is necessary to determine how arbitrage activity affects the path of the constrained agent's consumption share and this is the issue we turn to next.

3.4 Comparative statics

In equilibrium the consumption share of the constrained agent plays the role of an endogenous state variable. Therefore, to understand the impact of arbitrage activity we need to understand how the credit facility parameter α influences the path of this state variable. To state the result, let $s_t(\alpha)$ denote the constrained agent's consumption share seen as a function of time and the credit facility parameter.

Proposition 6. *We have*

$$-\frac{ns_0(\alpha)^{1+\eta}}{s_0(\alpha) + n\eta s_0(\alpha)^\eta} \left(\frac{s_t(\alpha)}{s_0(\alpha)} \right) \leq \frac{\partial s_t(\alpha)}{\partial \alpha} \leq 0. \quad (30)$$

In particular, the consumption share of the constrained agent, the stock price and the stock bubble are decreasing in arbitrage activity at all dates.

The above result shows that arbitrage activity reduces the consumption share of the constrained agent at every point in time and, hence, also his welfare. Furthermore, a direct calculation based on (30) and the definition of the initial value $s_0(\alpha)$ shows that the consumption share of the arbitrageur

$$s_{3t}(\alpha) = \frac{\gamma(\alpha)}{1 + \gamma(\alpha)} (1 - s_t(\alpha)) = \alpha s_0(\alpha)^\eta \frac{1 - s_t(\alpha)}{1 - s_0(\alpha)}$$

is increasing in the credit facility parameter α at all dates, while the consumption share of the unconstrained agent

$$s_{1t}(\alpha) = \frac{1}{1 + \gamma(\alpha)} (1 - s_t(\alpha)) = (1 - \alpha s_0(\alpha)^\eta - s_0(\alpha)) \frac{1 - s_t(\alpha)}{1 - s_0(\alpha)}$$

is decreasing in α at the initial time but non monotone at subsequent dates. The intuition for these findings is the following. As α increases the consumption share of the constrained agent decreases so that more consumption becomes available for agents 1 and 3, but the repartition of that consumption simultaneously tilts towards the arbitrageur as a higher α improves his comparative advantage over the unconstrained agent. This generates an initial decrease in the consumption share of the unconstrained agent but this effect may reverse itself over time depending on the path of the economy.

4 Analysis

4.1 Equilibrium portfolio strategies

Proposition 2 and Theorem 1 allow us to derive in closed-form the trading strategies employed in equilibrium by each of the three groups of agents.

Proposition 7. *Define a strictly positive function by setting*

$$v(s) = v(s; \alpha) \equiv \varepsilon \eta \left(\frac{\alpha s^\eta}{1 - \alpha s^\eta} \right). \quad (31)$$

Then the equilibrium trading strategies of the three agents, and their respective signs, are explicitly given by

$$\begin{aligned} (\pi_{1t}; \phi_{1t}) &= \left(\frac{1 - (1 - \varepsilon)s_t}{(1 + \gamma)(1 + v(s_t))}; \frac{v(s_t) - (\varepsilon + v(s_t))s_t}{(1 + \gamma)(1 + v(s_t))} \right) P_t \in \mathbb{R}_+ \times \mathbb{R}_- \\ (\pi_{2t}; \phi_{2t}) &= \left(\frac{1 - \varepsilon}{1 + v(s_t)}; \frac{\varepsilon + v(s_t)}{1 + v(s_t)} \right) s_t P_t \in \mathbb{R}_+^2 \end{aligned} \quad (32)$$

and

$$(\pi_{3t}; \phi_{3t}) = \gamma(\pi_{1t}, \phi_{1t}) + \left(\frac{\varepsilon \eta - 1}{1 + v(s_t)}; -\frac{\varepsilon \eta + v(s_t)}{1 + v(s_t)} \right) \alpha s_t^\eta P_t \in \mathbb{R}_\pm \times \mathbb{R}_-$$

with the strictly positive constant $\gamma = c_{30}/c_{10}$.

Since the portfolio constraint binds at all times in equilibrium (see (22)), agent 2 constantly keeps a strictly positive fraction of his wealth in the riskless asset and the above proposition shows that this long position in the riskless asset is offset by the borrowing positions held by the two other agents.

As can be seen from (32) the unconstrained agent 1 uses this borrowing to invest larger amounts in the stock despite the fact that its price includes a strictly positive bubble.

To understand this finding recall that agent 1 is required to maintain nonnegative wealth and, therefore, cannot simultaneously exploit both bubbles. Since by Proposition 5 the riskless asset bubble requires less collateral per dollar of initial profit we expect this agent to short the riskless asset bubble and use the stock as collateral. Our next result confirms this intuition by showing that the wealth of agent 1 is the outcome of a dynamic strategy that buys the stock and shorts the riskless asset bubble.

Proposition 8. *The wealth of agent 1 expressed as a self-financing strategy in the stock and the riskless asset bubble over the interval $(t, T]$ is given by*

$$W_{1t} = \phi_{1t}^S(T) + \phi_{1t}^{B_0}(T)$$

where

$$\begin{aligned}\phi_{1t}^S(T) &= \frac{\varepsilon s_t + (1 - \Sigma_{0t}(T))(1 - s_t)}{(1 + \gamma)(1 + v(s_t) - \Sigma_{0t}(T))} P_t \geq 0 \\ \phi_{1t}^{B_0}(T) &= W_{1t} - \phi_{1t}^S(T) = \frac{v(s_t) - (\varepsilon + v(s_t))s_t}{(1 + \gamma)(1 + v(s_t) - \Sigma_{0t}(T))} P_t \leq 0\end{aligned}$$

and the process $\Sigma_{0t}(T)$ is the diffusion coefficient of $(1/\sigma_\delta) \log B_{0t}(T)$.

Turning to the last group of agents, Proposition 7 shows that the equilibrium trading strategy of the arbitrageur can be decomposed into two parts: A short position of size ψ in the stock bubble, and a long position in the same strategy as the unconstrained agent. The short position in the stock bubble is worth $W_{at} \equiv W_{3t} - \gamma W_{1t} = -\alpha s_t^\eta P_t$ and an application of Itô's lemma shows that this part of the arbitrageur's equilibrium portfolio corresponds to a self-financing trading strategy that holds

$$n_{at} \equiv \frac{(\eta\varepsilon - 1)\alpha s_t^\eta}{1 - \alpha s_t^\eta(1 - \eta\varepsilon)}$$

units of the stock, and invests

$$\phi_{at} \equiv W_{at} - n_{at} S_t = -\frac{\eta\varepsilon\alpha s_t^\eta P_t}{1 - \alpha s_t^\eta(1 - \eta\varepsilon)} \leq 0$$

in the riskless asset. While the stock position can be either positive or negative, we have that the position in the riskless asset is negative and decreasing in $\delta_t = \rho P_t$. This implies that, compared to the unconstrained agent, the arbitrageur levers up in good times and delevers in bad times. This feature of the equilibrium is in line with the evidence in Ang,

Gorovyy, and Van Inwegen (2011) and leads to an amplification of fundamental shocks that generates excess volatility and the leverage effect as we now explain.

4.2 Equilibrium volatility

Combining our explicit formula for the equilibrium stock price with the comparative statics result of Proposition 6 directly leads to the following characterization of the equilibrium volatility of the stock.

Proposition 9. *The equilibrium volatility of the stock is given by*

$$\sigma_t = (1 + v(s_t; \alpha))\sigma_\delta \quad (33)$$

where the nonnegative function $v(s; \alpha)$ is defined as in (31). In particular, we have that the equilibrium volatility of the stock increases with arbitrage activity.

Proposition 9 shows that contrary to existing models with frictions and logarithmic agents, our model with constrained agents and risky arbitrage activity generates excess volatility. Furthermore, this excess volatility is self-generated within the system and therefore provides a new example of the endogenous risks advocated by Danielsson and Shin (2003), He and Krishnamurthy (2012) and Brunnermeier and Sannikov (In Press) among others. In our model, excess volatility is explicitly given by

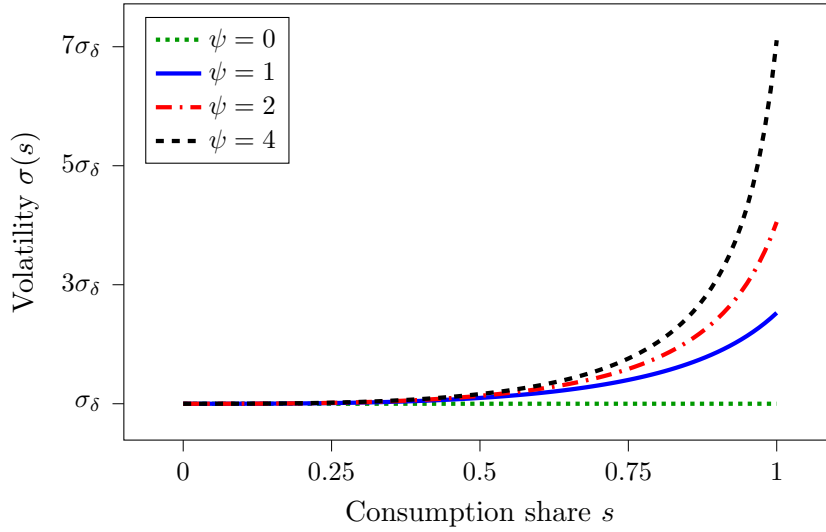
$$\sigma_t - \sigma_\delta = v(s_t; \alpha)\sigma_\delta = \eta\varepsilon \left(\frac{\alpha s_t^\eta}{1 - \alpha s_t^\eta} \right) \sigma_\delta \quad (34)$$

and increases with both the constrained agent's consumption share and the amount α of arbitrage activity. Since shocks to the consumption share process are negatively correlated with fundamental shocks this implies that the stock volatility is negatively correlated with fundamental shocks and it follows that our model is consistent with the leverage effect (see Black (1976), Schwert (1989), Mele (2007) and Ait Sahalia, Fan, and Li (2013) for recent evidence) according to which stock volatility tends to increase when prices fall. In addition, since the equilibrium market price of risk in (19) is also negatively correlated with fundamental shocks we have that the equity premium

$$\sigma_t \theta_t = \sigma_\delta^2 (1 + v(s_t; \alpha)) \left(1 + \frac{\varepsilon s_t}{1 - s_t} \right) \geq \sigma_\delta^2$$

is countercyclical which is consistent with the evidence presented by Mehra and Prescott (1985, 2003) and Fama and French (1989) among others.

Figure 2: Volatility and credit conditions



Notes. This figure plots the equilibrium stock volatility as a function of the consumption share of the constrained agent for various values of the parameter that governs the size of the credit facility. To construct the figure we set $\sigma_\delta = 0.0357$, $\rho = 0.001$ and $\varepsilon = 0.5$.

To illustrate the amount of excess volatility that our model generates we plot in Figure 2 the volatility of the stock as a function of the consumption share of constrained agents for different values of the parameter $\psi = \alpha/(1 - \alpha)$ that governs the size of the credit facility. As shown by the figure the amplification of fundamental shocks induced by arbitrage activity is sizable and increases with both the consumption share of constrained agents and the amount of arbitrage activity. For example, with 80% of constrained agents the stock volatility varies between 1.5 and 2.03 times that of the underlying dividend depending on the size of the credit facility.

As can be seen from (34) the model generates excess volatility if and only if there is arbitrage activity in that $\alpha > 0$. Therefore, intuition suggests that the source of the excess volatility lies in what arbitrageurs do in response to fundamental shocks, and more precisely in the short stock bubble position that their preferential access to credit allows them to implement. To confirm this intuition recall from the previous section that this position requires the arbitrageur to borrow

$$|\phi_{at}| = |W_{at} - n_{at}S_t| = \frac{\eta\varepsilon\alpha s_t^\eta \delta_t}{\rho(1 - \alpha s_t^\eta(1 - \eta\varepsilon))}$$

from the constrained agent, and assume that the economy suffers a negative fundamental shock so that the dividend decreases. As in the model without arbitrageurs (see for example [Hugonnier \(2012\)](#)) this triggers a relative decrease of the same magnitude in the price of the stock. However, since ϕ_{at} is negative and decreasing in $\delta_t = \rho P_t$ we have that this decrease in δ_t prompts the arbitrageur to delever. This puts additional downward pressure on the stock compared to the case without arbitrage activity and leads to a larger total decrease of the stock price. Symmetrically, the arbitrageur tends to lever up his position in reaction to positive fundamental shocks. This puts additional upward pressure on the price and, therefore, amplifies the effect of the shock compared to the case without arbitrage activity. Since

$$\frac{\partial^2 |\phi_{at}|}{\partial \delta_t \partial s_t} = \frac{\varepsilon \alpha \eta^2 s_t^{1+\eta}}{\rho(1 - \alpha s_t^\eta (1 - \eta \varepsilon))^2} \geq 0$$

we have that the magnitude of the amplification increases with the consumption share of the constrained agent. This property explains the convexity of the stock volatility function that is apparent from [\(33\)](#) and is quite intuitive as we know from [Section 3.3](#) that the size of the stock bubble, and hence also the influence of the arbitrageur, increases with the consumption share of the constrained agent.

5 Conclusion

In this paper we derive a novel and analytically tractable equilibrium model of dynamic arbitrage. Specifically, we consider an economy populated by three groups of agents: Constrained agents who are subject to a portfolio constraint that tilts their demand toward the riskless asset, unconstrained agents who are only subject to a nonnegativity constraint on wealth, and arbitrageurs who have no initial wealth but have access to a credit facility that allows them to weather transitory losses.

We show that the presence of constrained agents in the economy gives rise to risky arbitrage opportunities in the form of asset pricing bubbles, and that these bubbles make the credit facility valuable by allowing arbitrageurs to consume despite the fact that they initially hold no wealth. We solve for the equilibrium in closed-form and show that it is characterized by bubbles on both traded assets, a time varying price-dividend ratio and a sizable countercyclical excess volatility component.

A Alternative credit facility

In this appendix we show that our results remain qualitatively unchanged if we replace the stock by the riskless asset in the wealth constraint (4) of the arbitrageur. Specifically, we fix a constant $\ell \geq 0$ and assume that the arbitrageur is subject to

$$W_{3t} + \ell S_{0t} \geq 0, \quad t \geq 0. \quad (\text{A.1})$$

Proceeding as in the baseline model, we obtain the following characterization of the feasible consumption set for the arbitrageur.

Proposition A.1. *A consumption plan is feasible for agent 3 if and only if*

$$E \left[\int_0^\infty \xi_t c_{3t} dt \right] \leq \ell \left(1 - \lim_{T \rightarrow \infty} E[\xi_T S_{0T}] \right).$$

In particular, the feasible set of agent 3 is non empty if and only if the price of the riskless asset includes a bubble over some horizon.

Going through the same steps as in Section 3 we deduce that the state price density and the optimal consumptions of agents 1 and 2 are given by (15), (16) and (17) where the consumption share evolves according to (21). Itô's lemma then shows that the market price of risk and the interest rate are given by (19) and (20), and it follows that over an interval $[t, T]$ the price of the riskless asset includes a bubble that is given by

$$B_{0t}(T) = S_{0t} \left(1 - E_t \left[\frac{\xi_T S_{0T}}{\xi_t S_{0t}} \right] \right) = S_{0t} H(T - t, s_t, 2/\varepsilon - 1) > 0. \quad (\text{A.2})$$

In particular we have that $\lim_{T \rightarrow \infty} E[\xi_T S_{0T}] = 0$ and it thus follows from the same arguments as in the proof of Proposition 2 that the optimal consumption of the arbitrageur is given by $c_{3t} = \rho(W_{3t} + \ell S_{0t})$. Combining this identity with Proposition 5 and the market clearing conditions then shows that

$$S_t = \frac{\delta_t}{\rho} - \ell S_{0t} = P_t \left[1 - e^{qt} \frac{\ell}{P_0} \left(\frac{s_t}{s_0} \right)^{1/\varepsilon} \right] \quad (\text{A.3})$$

and it now remains to prove that, under appropriate parametric assumptions, the initial value of the consumption share process can be chosen in such a way that the agents consumption and the stock price are positive at all dates.

Theorem A.2. *Denote by $\hat{x} \in (0, 1)$ the unique solution to $\hat{x}^\varepsilon = n(1 - \hat{x})$ and let q be defined as in Proposition 5.*

- a) *If $q > 0$ and $\ell \geq \hat{x} P_0$ then no equilibrium exists.*

- b) If $q \leq 0$ and $0 \leq \ell < \hat{x}P_0$ then there exists a unique equilibrium in which the price of the traded securities are given by (27) and (A.3) where the consumption share process evolves according to (21) with initial value $s_0 = n(1 - \ell/P_0)$.

The first part shows that the equilibrium fails to exist whenever agents are sufficiently impatient and the credit limit exceeds a threshold that depends positively on the initial endowment of the constrained agent and the initial size of the economy as measured by P_0 . The intuition is clear: If ℓ is too large relative to P_0 then the consumption $c_{3t} - \rho W_{3t}$ generated by the arbitrageur's profits exceeds the available stock of the consumption good which leads the stock price in (A.3) to take negative values that are incompatible with limited liability.

Our next result shows that in equilibrium the stock price includes a bubble that is dominated in relative terms by the riskless asset bubble over any horizon.

Corollary A.3. *Under the conditions of Theorem A.2.b) the equilibrium price of the stock includes a bubble component that satisfies*

$$0 \leq B_t(T) = s_t^\eta P_t H(T - t; s_t; 2\eta - 1) - \ell B_{0t}(T) \leq \frac{S_t}{S_{0t}} B_{0t}(T)$$

with $\eta > 1$ and the function $H(\tau; s; a)$ as in Proposition 5.

Comparing Theorem A.2 and Corollary A.3 to the results of Sections 3 and 4 shows that the asset pricing implications of (A.1) are qualitatively similar to those of our main model. First, the equilibrium includes bubbles that allow arbitrageurs to consume despite the fact that they hold no initial capital. Second, arbitrage activity has a negative impact on the stock price and the bubbles on both assets. It follows that arbitrage activity reduces mispricing but we can no longer show that arbitrageurs are eradicate the stock bubble in the limit of infinite credit since equilibrium fails to exist for large values of ℓ . Third, the price/dividend ratio is time varying and the equilibrium stock volatility

$$\sigma(s_t(\ell)) = \sigma_\delta \left[1 - e^{qt} \frac{\ell}{P_0} \left(\frac{s_t(\ell)}{s_0(\ell)} \right)^{1/\varepsilon} \right]^{-1} \geq \sigma_\delta$$

is increasing in the size of the credit limit, decreasing in the consumption share of constrained agents and negatively correlated with fundamental shocks so that arbitrage activity generates both excess volatility and the leverage effect.

B Proofs

Proof of Proposition 1. The static budget constraint for agent 1 is well-known and follows from Lemma B.1 below by letting $b_t = \Pi_t = 0$. On the other hand, the static budget constraint

for the arbitrageur is novel and follows from Lemma B.1 below by letting $b_t = \psi\delta_t$, $\Pi_t = \psi S_t$ and using the assumption that $\lim_{T \rightarrow \infty} E[\xi_T \Pi_T] = 0$. \blacksquare

Consider an asset with dividend rate $b_t \geq 0$ and price process

$$\Pi_t = \Pi_0 + \int_0^t (r_s \Pi_s + \Sigma_s^\Pi \theta_s - b_s) ds + \int_0^t \Sigma_s^\Pi dZ_s \geq 0$$

for some diffusion coefficient Σ_t^Π such that the above integrals are well-defined. In order to simultaneously cover the models of Section 2 and Appendix A consider an agent who is subject to the transversality condition (5) and the wealth constraint

$$W_0 = 0 \leq W_t + \Pi_t, \quad t \geq 0. \quad (\text{B.1})$$

The following lemma provides a static characterization of the feasible set associated with these constraints and implies the results of Propositions 1 and A.1.

Lemma B.1. *A consumption plan c_t is feasible if and only if*

$$E \left[\int_0^\infty \xi_s c_s ds \right] \leq \Pi_0 - \lim_{T \rightarrow \infty} E \left[\xi_T \Pi_T + \int_0^T \xi_s b_s ds \right]. \quad (\text{B.2})$$

In particular, the feasible set of the arbitrageur is non empty if and only if Π_t includes a strictly positive bubble component.

Proof. Assume that c_t is feasible and let W_t satisfying (5) and (B.1) denote the corresponding wealth process. An application of Itô's lemma shows that

$$\xi_t (W_t + \Pi_t) + \int_0^t \xi_s (c_s + b_s) ds$$

is a nonnegative local martingale and therefore a supermartingale. Using this property together with (B.1) the monotone convergence theorem then gives

$$\begin{aligned} E \left[\int_0^\infty \xi_s c_s ds \right] &= \lim_{T \rightarrow \infty} E \left[\int_0^T \xi_s c_s ds \right] \\ &\leq \lim_{T \rightarrow \infty} \left(\Pi_0 - E \left[\xi_T (W_T + \Pi_T) + \int_0^T \xi_s b_s ds \right] \right) \\ &\leq \Pi_0 - \liminf_{T \rightarrow \infty} E [\xi_T W_T] - \liminf_{T \rightarrow \infty} E \left[\xi_T \Pi_T + \int_0^T \xi_s b_s ds \right] \\ &\leq \Pi_0 - \lim_{T \rightarrow \infty} E \left[\xi_T \Pi_T + \int_0^T \xi_s b_s ds \right] \end{aligned}$$

where the last inequality follows from the fact that since

$$Y_t = \xi_t \Pi_t + \int_0^t \xi_s b_s ds = \Pi_0 + \int_0^t \xi_s (\Sigma_s^\Pi - \Pi_s \theta_s) dZ_s$$

is a nonnegative local martingale the function $T \mapsto E[Y_T] \leq Y_0$ is decreasing and hence admits a well-defined limit. Conversely, assume that the consumption plan c_t satisfies (B.2) with an equality. Since $Y_t \geq 0$ is a local martingale it follows from Fatou's lemma and Doob's supermartingale convergence theorem that the limit

$$L = \lim_{T \rightarrow \infty} \xi_T \Pi_T = \lim_{T \rightarrow \infty} Y_T - \int_0^\infty \xi_s b_s ds$$

is well-defined and nonnegative. Now consider the process

$$W_t = -\Pi_t + \frac{1}{\xi_t} E_t \left[\int_t^\infty \xi_s (c_s + b_s) ds + aL \right]$$

where the nonnegative constant a is defined by $a = 0$ if the random variable L is almost surely equal to zero and by

$$a = \lim_{T \rightarrow \infty} \frac{E[\xi_T \Pi_T]}{E[L]} \in (1, \infty)$$

otherwise. The martingale representation theorem, Itô's lemma and the definition of a imply that W_t is the wealth process of a self-financing strategy that starts from no initial capital and consumes at rate $c_t + b_t$. On the other hand, it is clear from the definition that (B.1) holds and using the definition of a and the monotone convergence theorem we deduce that

$$\liminf_{T \rightarrow \infty} E[\xi_T W_T] = - \lim_{T \rightarrow \infty} E[\xi_T \Pi_T] + aE[L] = 0$$

and the proof is complete. ■

Proof of Proposition 2. The optimal strategy of the unconstrained agent follows from well known results, see for example Duffie (2001, Chapter 9.E) and Karatzas and Shreve (1998, Chapter 3). Letting $p_t = \pi_t/W_{2t}$ denote the proportion of wealth that agent 2 invests in the stock we have that the portfolio constraint can be written as

$$p_t \in \{p \in \mathbb{R} : |\sigma_t p| \leq (1 - \varepsilon)\sigma_\delta\}$$

and the optimal strategy of the constrained agent now follows from Cvitanić and Karatzas (1992, Section 11). Let us now turn to the optimal strategy of the arbitrageur. Using Proposition 1

we have that his optimization problem can be formulated as

$$\max_{c \geq 0} E \left[\int_0^\infty e^{-\rho s} \log c_s ds \right] \quad \text{subject to} \quad E \left[\int_0^\infty \xi_s c_s ds \right] \leq \psi B_0.$$

Using the concavity of the utility function together with the same arguments as in the second part of the proof of Lemma B.1 we have that the solution to this problem and the corresponding wealth process are explicitly given by $c_{3t} = (y_3 e^{\rho t} \xi_t)^{-1}$ and

$$W_{3t} + \psi B_t = \frac{1}{\xi_t} E_t \left[\int_t^\infty \xi_s c_{3s} ds \right] = \frac{c_{3t}}{\rho} = e^{-\rho t} \frac{\psi B_0}{\xi_t}.$$

for some Lagrange multiplier $y_3 > 0$ that is determined in such a way that $W_{30} = 0$. This shows that (11) holds for the arbitrageur. On the other hand, applying Itô's lemma to the right hand side of the above expression and using the fact that

$$B_t = B_0 + \int_0^t (r_s B_s ds + \Sigma_s^B (dZ_s + \theta_s ds))$$

we deduce that the optimal wealth process evolves according to

$$W_{3t} = \int_0^t ((r_s W_{3s} - c_{3s}) ds + ((W_{3s} + \psi B_s) \theta_s - \psi \Sigma_s^B) (dZ_s + \theta_s ds))$$

and the desired result now follows by comparing this expression to (3). ■

Proof of Proposition 3. To determine the dynamics of the consumption share, assume that $ds_t = m_t dt + k_t dZ_t$ for some drift m_t and volatility k_t . Applying Itô's lemma to (6) and (15) and comparing the results shows that

$$\theta_t = \sigma_\delta - \frac{k_t}{1 - s_t}, \tag{B.3}$$

$$r_t = \rho + \mu_\delta - \sigma_\delta^2 + \frac{\sigma_\delta k_t - m_t}{1 - s_t} - \left(\frac{k_t}{1 - s_t} \right)^2. \tag{B.4}$$

On the other hand, Proposition 2 shows that along the optimal path $W_{2t} = s_t \delta_t / \rho$. Applying Itô's lemma to this expression, and matching terms with the dynamic budget constraint (3), shows that the drift and volatility of the consumption share process are related by

$$m_t + \frac{k_t^2}{1 - s_t} = 0, \tag{B.5}$$

$$\sigma_t \pi_{2t} = W_{2t} \left(\sigma_\delta + \frac{k_t}{s_t} \right). \tag{B.6}$$

Substituting (B.3) risk into (12) and comparing the result with (B.6) then shows that

$$\sigma_\delta = \frac{k_t}{1-s_t} + \left(\sigma_\delta + \frac{k_t}{s_t} \right) \max \left\{ 1; \frac{|\sigma_\delta(1-s_t) - k_t|}{(1-\varepsilon)(1-s_t)\sigma_\delta} \right\}. \quad (\text{B.7})$$

Solving (B.5) and (B.7) gives the drift and volatility of the consumption share in (21) and the remaining claims follow by substituting these coefficients into (B.3) and (B.4). ■

Lemma B.2. *We have*

$$q_t(T) = \int_t^T \rho e^{-\rho(u-t)} \frac{\lambda_t - E_t[\lambda_u]}{1 + \gamma + \lambda_t} du = s_t^\eta H(\tau, s_t; 2\eta - 1) - e^{-\rho\tau} s_t H(\tau, s_t, 1) \quad (\text{B.8})$$

where $\tau = T - t$ and the function $H(\tau, s, a)$ is defined as in (29).

Proof. See Hugonnier (2012, Lemma A.3). ■

Lemma B.3. *Let $a \in \mathbb{R}$ be a constant. Then the stochastic differential equation*

$$Y_t(a) = 1 - \int_0^t Y_u(a) \left(a + \frac{s_u}{1-s_u} \right) \varepsilon \sigma_\delta dZ_u \quad (\text{B.9})$$

admits a unique strictly positive solution which satisfies

$$E_t [Y_{t+T}(a)] = Y_t(a) (1 - H(T, s_t; 2a - 1))$$

where the function $H(\tau, s, a)$ is defined as in (29). In particular, the process $Y_t(a)$ is a strictly positive local martingale but not a martingale.

Proof. Itô's lemma and (21) show that the process $\Lambda_t = s_t/(1-s_t)$ evolves according to the driftless stochastic differential equation

$$d\Lambda_t = -\Lambda_t (1 + \Lambda_t) \varepsilon \sigma_\delta dZ_t$$

and the results follow from Hugonnier (2012, Lemma A.5). ■

Proof of Proposition 4. By application of Itô's lemma we have that the weighting process evolves according to

$$d\lambda_t = -\lambda_t (1 + \gamma + \lambda_t) \frac{\varepsilon \sigma_\delta}{1 + \gamma} dZ_t = -\lambda_t (1 + \Lambda_t) \varepsilon \sigma_\delta dZ_t.$$

Therefore the uniqueness of the solution to (B.9) implies that $\lambda_t = Y_t(1)$ and the desired result now follows from Lemma B.3. ■

Proof of Theorem 1. Combining (24) with the monotone convergence theorem shows that the bubble on the stock satisfies

$$\begin{aligned} \frac{B_t}{(1-\alpha)P_t} &= \frac{\rho}{1+\gamma+\lambda_t} E_t \left[\int_t^\infty e^{-\rho(u-t)} (\lambda_t - \lambda_u) du \right] \\ &= \lim_{T \rightarrow \infty} \frac{\rho}{1+\gamma+\lambda_t} E_t \left[\int_t^T e^{-\rho(u-t)} (\lambda_t - \lambda_u) du \right] = \lim_{T \rightarrow \infty} q_t(T) \end{aligned}$$

Taking the limit in (B.8) then gives (26) and substituting the result in (23) produces the formula for the stock price. Furthermore, the fact that $\alpha s_T^\eta \leq 1$ and the supermartingale property of the weighting process implies that we have

$$E[\xi_T S_T] = \frac{e^{-\rho T} P_0}{1+\gamma+\lambda_0} E[(1+\gamma+\lambda_T)(1-\alpha s_T^\eta)] \leq e^{-\rho T} P_0$$

and since $\rho > 0$ it follows that the assumption of Propositions 1 and 2 hold. To establish the existence and uniqueness of the equilibrium it suffices to show that whenever $n \in (0, 1]$ and $\alpha > 0$ there are unique constants $s_0 \in (0, 1)$ and $\gamma > 0$ such that

$$\begin{aligned} w_2 &= n(1 - \alpha s_0^\eta) P_0 = s_0 P_0, \\ w_1 &= (1 - n)(1 - \alpha s_0^\eta) P_0 = \frac{1}{1 + \gamma} (1 - s_0) P_0. \end{aligned}$$

A direct calculation shows that the pair $(s_0, \gamma) \in (0, 1) \times (0, \infty)$ is a solution to this system of equations if and only if

$$g(s_0) \equiv n(1 - \alpha s_0^\eta) - s_0 = 0 \text{ and } \gamma = \gamma(s_0) \equiv \frac{n - s_0}{s_0(1 - n)}.$$

Since the function $g(s)$ is continuous and strictly decreasing with $g(0) = n$ and $g(n) < 0$ we have that the nonlinear equation $g(s_0) = 0$ admits a unique solution s_0 that lies in the open interval $(0, n)$ and it follows that the constant $\gamma(s_0)$ is strictly positive. ■

Proof of Proposition 5. Applying Itô's lemma to the right hand side of (27) and using (21) we obtain that

$$e^{qt} \frac{P_t}{P_0} \left(\frac{s_t}{s_0} \right)^{1/\varepsilon} = 1 + \int_0^t e^{qu} \frac{P_u}{P_0} \left(\frac{s_u}{s_0} \right)^{1/\varepsilon} \left[\rho + \mu_\delta - \sigma_\delta^2 \left(1 + \frac{\varepsilon s_u}{1 - s_u} \right) \right] du$$

and the formula for the riskless asset price now follows from (1) and (20). Let us now turn to the finite horizon bubbles. Using (8) and the law of iterated expectations we obtain

$$\begin{aligned}
B_t(T) &= S_t - E_t \left[\int_t^T \xi_{t,u} \delta_u du + \xi_{t,T} S_T \right] \\
&= S_t - E_t \left[\int_t^\infty \xi_{t,u} \delta_u du - \int_T^\infty \xi_{t,u} \delta_u du + \xi_{t,T} S_T \right] \\
&= B_t - E_t \left[\xi_{t,T} \left(S_T - E_T \int_T^\infty \xi_{T,u} \delta_u du \right) \right] = B_t - E_t [\xi_{t,T} B_T].
\end{aligned}$$

To compute the term on the right hand side we start by observing that due to (B.8) and the monotone convergence theorem we have

$$s_u^\eta = \lim_{\Theta \rightarrow \infty} q_u(\Theta) = E_u \left[\int_u^\infty \rho e^{-\rho(k-u)} \frac{\lambda_u - \lambda_k}{1 + \gamma + \lambda_u} dk \right], \quad u \geq 0.$$

Using this identity in conjunction with Lemmas B.2 and B.3, the definition of the equilibrium state price density and (18) then gives

$$\begin{aligned}
\frac{B_t - E_t[\xi_{t,T} B_T]}{(1 - \alpha)P_t} &= s_t^\eta - e^{-\rho(T-t)} E_t \left[\frac{1 + \gamma + \lambda_T}{1 + \gamma + \lambda_t} s_T^\eta \right] \\
&= s_t^\eta - \rho E_t \left[\int_T^\infty e^{-\rho(u-t)} \frac{\lambda_T - \lambda_u}{1 + \gamma + \lambda_t} du \right] \\
&= s_t^\eta + q_t(T) - \lim_{\Theta \rightarrow \infty} q_t(\Theta) - e^{-\rho(T-t)} E_t \left[\frac{\lambda_T - \lambda_t}{1 + \gamma + \lambda_t} \right] \\
&= s_t^\eta H(T - t, s_t, 2\eta - 1)
\end{aligned}$$

and the formula of the statement now follows from (26). On the other hand, using (9) we have that the finite horizon bubble on the riskless asset is given by

$$B_{0t}(T) = S_{0t} \left(1 - E_t \left[\frac{\xi_T S_{0T}}{\xi_t S_{0t}} \right] \right).$$

Since the process $M_t = \xi_t S_{0t}$ evolves according to

$$M_t = 1 - \int_0^t M_s \theta_s dZ_s = 1 - \int_0^t M_s (1/\varepsilon + X_s) \epsilon \sigma_\delta dZ_s$$

we have that $M_t = Y_t(1/\varepsilon)$ by the uniqueness of the solution to (B.9) and the formula of the statement now follows from Lemma B.3. To complete the proof it remains to show that the relative bubble on the stock is dominated by the relative bubble on the riskless asset over any horizon. Consider the function defined by

$$G(\tau; s; a) = s^{\frac{1+a}{2}} H(\tau, s; a).$$

A direct calculation using (29) shows that

$$\begin{aligned}\frac{\partial}{\partial a}(G(\tau; s; a)) &= s^{\frac{1-a}{2}} \log(s) G_0(s) \\ \frac{\partial}{\partial a}(s^{-a} G(\tau; s; 2a-1)) &= 2s^{1-2a} \log(1/s) N(d_-(\tau; s; 2a-1)) \geq 0\end{aligned}$$

with the function defined by

$$G_0(s) = [s^a N(d_+(\tau; s; a)) - N(d_-(\tau; s; a))]/2,$$

and since

$$\begin{aligned}G_0(0) &= 0 < N(d_+(\tau; 1, a)) - 1/2 = G_0(1) \\ G_0'(s) &= (a/2) s^{a-1} N(d_+(\tau; s, a)) \geq 0\end{aligned}$$

we conclude that the functions $G(\tau; s, a)$ and $s^{-a} G(\tau; s, 2a-1)$ are respectively decreasing and increasing in a . Using these properties together with the first part of the proof and the facts that $s_t \in (0, 1)$, $\eta > 1$ and $\varepsilon \in [0, 1]$ we then deduce that

$$\begin{aligned}B_t(t+T)/S_t &= G(T, s_t, 2\eta-1)/(1+\psi(1-s_t^\eta)) \leq G(T, s_t, 2\eta-1) \\ &\leq G(T, s_t, 1) \leq s_t^{-1} G(T, s_t, 1) \leq s_t^{-1/\varepsilon} G(T, s_t, 2/\varepsilon-1) = B_{0t}(t+T)/S_{0t}\end{aligned}$$

and the proof is complete. ■

Asymptotic behaviour of the consumption share process. Since the consumption share process is a nonnegative supermartingale we have that it converges to a well-defined limit. On the other hand, an application of Itô's lemma to (21) shows that

$$0 \leq s_t = s_0 e^{-\int_0^t \frac{(\varepsilon\sigma_\delta)^2}{1-s_u} du - \frac{1}{2}(\varepsilon\sigma_\delta)^2 t - \varepsilon\sigma_\delta Z_t} \leq \bar{s}_t = s_0 e^{-\frac{1}{2}(\varepsilon\sigma_\delta)^2 t - \varepsilon\sigma_\delta Z_t}$$

and the desired result then follows from the fact that, by well-known results on geometric brownian motion, the process \bar{s}_t converges to zero. ■

Proof of Proposition 6. By Protter (2004, Theorem 39 p. 305) we have that the first order derivative $\nabla_t(\alpha) = \frac{\partial s_t(\alpha)}{\partial s_0(\alpha)}$ of the consumption share process with respect to its initial value exists and satisfies

$$\nabla_t(\alpha) = 1 + \int_0^t \nabla_u(\alpha) \left[\frac{ds_u(\alpha)}{s_u(\alpha)} - \frac{s_u(\alpha)(\varepsilon\sigma_\delta)^2}{(1-s_u(\alpha))^2} du \right]$$

Solving this linear stochastic differential equation and using the fact that the consumption share is nonnegative then shows that

$$\nabla_t(\alpha) \leq \exp\left(-\int_0^t \frac{s_u(\alpha)(\varepsilon\sigma_\delta)^2}{(1-s_u(\alpha))^2} du\right) \frac{s_t(\alpha)}{s_0(\alpha)} \leq \frac{s_t(\alpha)}{s_0(\alpha)}.$$

On the other hand, using the chain rule we deduce that $\frac{\partial s_t(\alpha)}{\partial \alpha} = \nabla_t(\alpha) \frac{\partial s_0(\alpha)}{\partial \alpha}$ and the desired result now from the fact that

$$s_0(\alpha) = n(1 - \alpha s_0(\alpha)^\eta) \implies -\frac{\partial s_0(\alpha)}{\partial \alpha} = \frac{ns_0(\alpha)^{1+\eta}}{s_0(\alpha) + n\eta s_0(\alpha)^\eta}$$

by application of the implicit function theorem. ■

Proof of Proposition 7. Let σ_t denote the stock volatility. By Proposition 3 and Theorem 1 we have that along the equilibrium path agents' wealth are given by

$$W_{2t} = s_t P_t$$

$$W_{1t} = \frac{(1-s_t)P_t}{1+\gamma} \text{ and } W_{3t} = S_t - W_{1t} - W_{2t} = \gamma W_{1t} - \alpha s_t^\eta P_t$$

and the expression for the equilibrium trading strategies reported in the statement now follows by noting that we have

$$\pi_{it} = \frac{1}{\sigma_t} \left[\frac{\partial W_{it}}{\partial P_t} P_t \sigma_\delta - \frac{\partial W_{it}}{\partial s_t} s_t \varepsilon \sigma_\delta \right]$$

and $\phi_{it} = W_{it} - \pi_{it}$ as a result of (3), (21) and Itô's lemma. The signs of π_{1t} , π_{2t} and ϕ_{2t} follow from the expressions in the statement. To establish the sign of ϕ_{1t} we argue as follows. Using the fact that $v(s) \geq 0$ is nonnegative we deduce that

$$\text{sign}[\phi_{1t}] = \text{sign}[v(s) - s(\varepsilon + v(s))] = \text{sign}[-1 + \alpha s^{\eta-1}(s + \eta(1-s))].$$

Denote by $h(s)$ the continuous function inside the bracket. Since $\eta > 1$ we have that this function is increasing with $h(0) = -1$ and $h(1) = \alpha - 1$ and the result now follows by continuity. Finally, since $-\phi_{at} = \gamma\phi_{1t} - \phi_{3t} \geq 0$ we have that $\phi_{3t} \leq 0$ and the proof is complete. ■

Proof of Proposition 8. To establish the desired result we have to find a pair of adapted processes such that

$$W_{1t} = \frac{1}{1+\gamma}(1-s_t)P_t = \phi_{1t}^S(T) + \phi_{1t}^{B_0}(T)$$

and

$$dW_{1t} = \phi_{1t}^S(T) \left(\frac{dS_t}{S_t} + \frac{\delta_t}{S_t} dt \right) + \phi_{1t}^{B_0}(T) \frac{dB_{0t}(T)}{B_{0t}(T)} - \rho W_{1t} dt$$

where S_t denotes the equilibrium stock price and $B_{0t}(T)$ denotes the bubble on the riskless asset at horizon T . Expanding the dynamics of these two processes and using (3) shows that these equations are equivalent to

$$\begin{aligned} \phi_{1t}^{B_0}(T) + \phi_{1t}^S(T) &= W_{1t} \\ \phi_{1t}^{B_0}(T) \Sigma_{0t}(T) + \phi_{1t}^S(T) (1 + v(s_t)) &= \pi_{1t} (1 + v(s_t)) \end{aligned}$$

and solving this system gives the desired decomposition. The sign of the riskless asset bubble position follows by noting that $\text{sign } \phi_{1t}^{B_0}(T) = \text{sign } \phi_{1t}$. To establish the sign of the position in the stock and complete the proof it suffices to show that $\Sigma_{0t}(T) \leq 0$ or, equivalently, that the function $H(\tau, s; a)$ is increasing in s . Differentiating in (29) shows that

$$s^{1+a} \frac{\partial H}{\partial s}(\tau, s; a) = aN(d_-(\tau, s; a)) + \frac{2}{\varepsilon \sigma_\delta \sqrt{\tau}} N'(d_-(\tau, s; a))$$

and the required result now follows by noting that the function on the right hand side is increasing on $[0, 1]$ and equal to zero at zero. ■

Proof of Proposition 9. The expression for the stock volatility follows from (25) and Itô's lemma. To establish the second part we differentiate with respect to α . This gives

$$\frac{\partial \sigma_t(\alpha)}{\partial \alpha} = \frac{\varepsilon \eta s_t^\eta(\alpha)}{(1 - \alpha s_t^\eta(\alpha))^2} \left[1 + \frac{\alpha \eta}{s_t(\alpha)} \frac{\partial s_t(\alpha)}{\partial \alpha} \right]$$

and the desired result now follows from the derivative bound of Proposition 6. ■

Proof of Proposition A.1. The desired result follows from that of Lemma B.1 by taking $b_t = 0$ and $\Pi_t = \ell S_{0t}$. We omit the details. ■

Proof of Theorem A.2. Since individual optimality and market clearing are already taken into account we have that the existence of an equilibrium is equivalent to the existence of a pair $(s_0, \gamma) \in (0, 1) \times (0, \infty)$ such that

$$w_1 = n(P_0 - \ell) = s_0 P_0 \tag{B.10}$$

$$w_2 = (1 - n)(P_0 - \ell) = \frac{(1 - s_0)P_0}{1 + \gamma} \tag{B.11}$$

and $P[\{t : S_t > 0\}] = 1$ where the candidate equilibrium price process is defined as in (A.3). The unique solution to the system (B.10), (B.11) satisfies $s_0 = n(1 - \ell/P_0)$ and it follows

that $\ell < P_0$ is necessary for the existence of an equilibrium. Assume that this condition is satisfied and consider the non existence result. If $q > 0$ then we have

$$\frac{S_t}{P_t} = 1 - e^{qt} \frac{\ell}{P_0} \left(\frac{s_t}{s_0} \right)^{1/\varepsilon} \leq 1 - \frac{\ell}{P_0} \left(\frac{s_t}{s_0} \right)^{1/\varepsilon}$$

and it follows that

$$P[\{t : S_t \leq 0\}] \geq P \left[\left\{ t : s_t \geq b \equiv s_0 \left(\frac{\ell}{P_0} \right)^{-\varepsilon} \right\} \right]$$

If we have $\ell \geq \hat{x}P_0$ then $b \in (0, 1)$ and the proof will be complete once we show that the probability on the right hand side is strictly positive. A direct calculation using (21) shows that a scale function for the consumption share process is given by $p(x) = (2x - 1)/(4(1 - x))$ and the result now follows from Karatzas and Shreve (1991, Proposition 5.22). Let us now turn to the existence result. If $q \leq 0 < \hat{x}P_0 - \ell$ then the definition of \hat{x} implies that

$$\frac{S_t}{P_t} = 1 - e^{qt} \frac{\ell}{P_0} \left(\frac{s_t}{s_0} \right)^{1/\varepsilon} \geq 1 - \frac{\ell}{P_0} \left(\frac{1}{s_0} \right)^{1/\varepsilon} > 0 \quad (\text{B.12})$$

and the desired result now follows from the fact that the aggregate dividend process is strictly positive at all times. ■

Proof of Corollary A.3. Since the state price density is the same as in the basic model we have that the fundamental value of the stock is $F_t = (1 - s_t^\eta)P_t$ and it thus follows from (A.3) that the infinite horizon bubble on the stock is given by

$$B_t = S_t - (1 - s_t^\eta)P_t = s_t^\eta P_t - \ell S_{0t} = P_t \left[s_t^\eta - e^{qt} \frac{\ell}{P_0} \left(\frac{s_t}{s_0} \right)^{1/\varepsilon} \right].$$

Under the conditions of Theorem A.2.b) we have that $\eta \leq 1/\varepsilon$ and it thus follows from the second inequality in (B.12) that

$$B_t = s_t^{-\eta} P_t \left[1 - e^{qt} s_t^{1/\varepsilon - \eta} \frac{\ell}{P_0} \left(\frac{1}{s_0} \right)^{1/\varepsilon} \right] \geq s_t^{-\eta} P_t \left[1 - \frac{\ell}{P_0} \left(\frac{1}{s_0} \right)^{1/\varepsilon} \right] > 0.$$

Using the above identities in conjunction with (A.2), (A.3), the fact that $s_t \leq 1 \leq \eta \leq 1/\varepsilon$ and the monotonicity of $G(\tau; s; a)$ in the proof of Proposition 5 we obtain that

$$\begin{aligned}
B_t(t+T) &= B_t - E_t[\xi_{t,T} B_T] = s_t^\eta P_t - E_t[\xi_{t,T} s_T^\eta P_T] - \ell(S_{0t} - E_t[\xi_{t,T} S_{0T}]) \\
&= s_t^\eta P_t H(T-t; s_t, 2\eta-1) - \ell B_{0t}(T) \\
&= s_t^\eta P_t H(T-t; s_t, 2\eta-1) - \ell S_{0t} H(T-t; s_t, 2/\varepsilon-1) \\
&= S_t H(T-t, s_t, 2/\varepsilon-1) + P_t (s_t^\eta H(T-t; s_t, 2\eta-1) - H(T-t; s_t, 2/\varepsilon-1)) \\
&\leq S_t H(T-t, s_t, 2/\varepsilon-1) + P_t (G(T-t; s_t, 2\eta-1) - G(T-t; s_t, 2/\varepsilon-1)) \\
&\leq S_t H(T-t, s_t, 2/\varepsilon-1)
\end{aligned}$$

and the desired result now follows (A.2). ■

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