Relationship Capital and Financing Decisions*

Thomas Geelen† Erwan Morellec‡ Natalia Rostova§

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Abstract

Lending relationships matter for firm financing. In a model of debt dynamics, we study how lending relationships are formed and how they impact leverage and debt maturity choices, thereby rationalizing recent empirical findings and generating new testable predictions. In the model, lending relationships evolve through repeated interactions between firms and debt investors. Stronger lending relationships lead firms to adopt higher leverage ratios, issue longer term debt, and raise funds from non-relationship lenders when relationship quality is sufficiently high. Debt contracts involving non-relationship investors, such as syndicated loans or bonds, have longer maturity than those exclusively issued to relationship investors.

Keywords: relationship lending, capital structure, debt maturity, default.
JEL Classification: G20, G32, G33.

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†Copenhagen Business School and Danish Finance Institute. tag.fi@cbs.dk.
‡Ecole Polytechnique Fédérale de Lausanne, Swiss Finance Institute, and CEPR. erwan.morellec@epfl.ch.
§Ecole Polytechnique Fédérale de Lausanne and Swiss Finance Institute. natalia.rostova@epfl.ch

Over the past 20 years, outstanding U.S. corporate debt has nearly tripled from \$2.5 trillion in 2000 to \$7.2 trillion in 2020. Corporate debt is often closely held by banks or large institutional investors. Detragiache, Garella, and Guiso (2000) document for instance that 44.5% of firms have a relationship with only one bank and that the median number of relationships is two. Chen, Schuerhoff, and Seppi (2020) find that on average more than half of a new bond issuance gets placed with issuers' incumbent institutional bondholders. Strong relationships with debt investors improve financing terms and affect financing choices. Bharath, Dahiya, Saunders, and Srinivasan (2011) and Karolyi (2018) for example show that firms with existing banking relationships are able to obtain larger loans at lower interest rates (see also Engelberg, Gao, and Parsons (2012) or Herpfer (2021)). Zhu (2021) finds that bond mutual funds that hold a firm's existing bonds are five times more likely to provide capital in future bond issues and do so at lower yields (see also Kubitza (2021)).[1]

Even though there is mounting evidence that lending relationships matter for firm financing, most existing capital structure theories implicitly assume that credit supply is perfectly elastic, so that financing decisions depend solely on firm characteristics. That is, although the Modigliani and Miller irrelevance does not hold on the demand side of the market in these models, it is assumed to hold on the supply side. Our objective in this paper is to relax this assumption and examine how lending relationships are built over time and how they shape firms' leverage and debt maturity choices.

To do so, we develop a dynamic model of financing in the spirit of Fischer, Heinkel, and Zechner (1989) and Strebulaev (2007) and consider a firm with assets that generate a continuous cash flow stream. The firm pays taxes on corporate income and, thus, has incentives to issue debt. Debt financing however increases the likelihood of costly financial distress and is subject to financing frictions, the severity of which depends on the quality of the firm's lending relationships. As in these classic models, firms in our setting have repeated

[1]Recent evidence from the COVID-19 crisis also highlights the importance of lending relationships in times of market stress. Halling, Yu, and Zechner (2020) and Goel and Serena (2020) show that due to well-established relationships with underwriters, borrowers with more bond issuance experience before the pandemic were able to issue bonds at lower spreads during the crisis. Relationship borrowers also received larger loans and faster approvals (Amiram and Rabetti (2020)).

interactions with lenders. An important innovation of our model is that a debt investor's willingness to invest in the firm's debt depends on the quality of its relationship with the firm while the cost of issuing debt depends on the endogenous composition of the pool of debt investors (i.e. relationship vs non-relationship investors). Our analysis encompasses both bank loans and bond issues. The "relationship investor" is the firm's bank in case of a loan issue and existing bondholders in case of a bond issue. "Non-relationship investors" are the other banks (bond investors) involved in a syndicated loan (bond issue) if the firm intends to issue more debt than the "relationship investor" is willing or able to purchase.

In the model, management acts in the best interests of shareholders and maximizes shareholder value by selecting the amount of debt to issue with relationship and non-relationship investors, the maturity of corporate debt, and the firm's default policy. The firm repeatedly interacts with debt investors who differ in their ability to supply credit (or in their appetite for the firm's debt). Stronger lending relationships lead to an increase in the financing provided by the relationship investor and thus make it more likely that the relationship investor will meet the firm's demand for debt. In addition, because raising funds from non-relationship investors is associated with higher issuance costs, stronger lending relationship are also associated with lower frictional costs of debt.[2] The quality of the relationship between the firm and the relationship investor evolves through time and depends on the number of past financing rounds, i.e. the number of loans made by the bank to the issuer or the number of bond issues in which incumbent bondholders invested, in line with the evidence in Bharath et al. (2011)). The quality of the lending relationship is therefore positively related to the duration of the relationship—the proxy used for the quality of the relationship in the empirical studies of Petersen and Rajan (1994) or Berger and Udell (1995).

How do lending relationships affect debt dynamics and maturity choices? When lending

[2] Empirically, Yasuda (2005) finds that banks charge lower fees to those firms with which they have relationships. The additional cost incurred by the firm when raising debt from multiple investors may be due to search costs incurred by the underwriter for a bond issue (see Chen et al. (2020)) or to costs incurred by the lead bank when securing the funds and syndicating the loan (see e.g. Ivashina (2009)). Graham and Harvey (2001) and Graham (2022) highlight the central role played by transaction costs and fees in the decision to issue debt. Their survey of corporate CEOs shows that transaction costs and fees come much before bankruptcy costs or personal taxes as a determinant of capital structure.

relationships are weak, the creditor's willingness to invest in the firm's debt (or to lend) is low, so that firms typically have a debt level that is below first best. While firms could in principle borrow from multiple lenders, this is (more) costly so that they will most of the time choose to stay away from their preferred leverage ratio and follow conservative financing policies. Stronger lending relationships increase the creditor's willingness to invest in the firm's debt. As a result, they lead to an increase in firm value and to a decrease in default risk, for any given debt level. The decrease in the frictional (issuance) and non-frictional (default) costs of debt leads to an increase in optimal leverage and pushes the firm to issue more debt as lending relationships improve, which is now possible due to the better lending relationships. In our base case environment, the effect of lending relationship on debt supply is large and implies that optimal leverage increases from 22% to 33% (a 50% increase) as the relationship between the firm and its creditors improves.

A striking result of the model is that stronger lending relationships allow the firm not only to issue more debt from relationship lenders but also to raise additional debt from outside (i.e. non-relationship) investors. That is, the sharp increase in leverage associated with better lending relationships is not solely due to the additional financing provided by relationship lenders. In effect, by decreasing the non-frictional costs of debt, stronger lending relationships make it optimal for the firm to also raise debt with non-relationship investors at a higher issuance cost, e.g. through loan syndication. That is, we find that stronger lending relationships lead to a higher likelihood of issuing syndicated loans or bonds.

In the model, the firm chooses not only how much debt to issue but also the maturity of corporate debt. When the quality of the lending relationship is low, the amount of debt purchased by the relationship investor is low. This leads the firm to issue short maturity debt, allowing it to refinance and adjust leverage sooner, possibly at better terms. As the relationship quality improves due to further interactions (or financing rounds), the relationship investor purchases larger amounts of debt, allowing the firm to get closer to its target leverage and leading to an increase in debt maturity. That is, a unique prediction of our model is that debt maturity should increase with the quality of lending relationships. As the lending relationship improves, default risk decreases for any given amount of debt. This decrease in the non-frictional cost of debt makes debt issuance to outside investors more attractive. When the relationship becomes strong enough, the firm issues debt to both relationship and non-relationship investors and the optimal debt maturity further increases, reflecting the higher costs of issuing debt to non-relationship investors. That is, the model futher predicts that the maturity of debt contracts involving non-relationship lenders is higher than that of relationship investors, in line with the evidence in Bharath et al. (2011). With our baseline parameters, the model implies that optimal debt maturities increase from 3 years to 9 years as the quality of the lending relationship improves.

Our analysis also shows that the wedge between the costs of debt issuance with relationship and outside investors is an important driver of leverage, debt composition, and debt maturity choice. This wedge can be related for instance to the cost of attracting loan participants and structuring and originating a syndicated loan (Berg, Saunders, and Steffen (2016)) or to the severity of search frictions in the bond market (Chen, Schuerhoff, and Seppi (2020)). When the wedge is large (reflecting, e.g., a lower supply of funds in credit markets), debt issuance with outside investors is relatively more costly, and firms issue debt mainly to the relationship investor, maintain a low leverage ratio, and issue short term debt. In this case, firms increase both optimal leverage and debt maturity as the lending relationship improves. When the wedge is small, firms issue debt to both relationship and outside investors. As the relationship quality improves, the share of debt financing coming from the relationship investor increases. As a result, average costs of issuance decrease, the firm increases its leverage ratio, and the loan structure becomes more concentrated.

In an important extension of our baseline model, we study how idiosyncratic shocks to the relationship lender affect corporate financing decisions. An example of such a shock is the lending cut by Commerzbank due to losses on its international trading book during the financial crisis. We show that such shocks decrease leverage but that the magnitude of the effect is highly dependent on the quality of the lending relationship. Specifically, firms with intermediate relationship quality are most affected as they optimally choose to maintain the lending relationship, leading to a significant drop in leverage and to a sharp shortening of debt maturity. Firms with weak lending relationships terminate their current relationship and borrow from a new lender, leading to a lower leverage ratio and to a shortening of debt maturity. Firms with strong lending relationships are almost unaffected.

Our paper contributes to several strands of the literature. First, we contribute to the literature on dynamic capital structure choice; see, e.g., Fischer et al. (1989), Leland (1998), Hackbarth, Miao, and Morellec (2006), Strebulaev (2007), Morellec, Nikolov, and Schurhoff (2012), Danis, Rettl, and Whited (2014), Dangl and Zechner (2021), or DeMarzo and He (2021). In that literature, our paper is most closely related to Hugonnier, Malamud, and Morellec (2015), in which firms search for debt investors when seeking to raise new debt and can only be matched once with a given debt investor. We instead allow firms to build lending relationships and shows how these impact capital structure and debt maturity choices. We find that optimal leverage is increasing while credit spreads are decreasing in the quality of lending relationships, in line with the empirical results in Karolyi (2018) and Zhu (2021).

Second, we advance the literature on dynamic debt maturity choice; see e.g. He and Xiong (2012), Cheng and Milbradt (2012), He and Milbradt (2016), Huang, Oehmke, and Zhong (2019), Geelen (2020), or Chen, Xu, and Yang (2021) for recent contributions. We advance this literature by allowing firms to build relationships with debt investors, which impacts optimal leverage, the debt maturity choice, and default risk. We show that firms initially issue short maturity debt and increase debt maturity as the quality of the lending relationship improves. We also show that the maturity of debt contracts involving non-relationship investors is higher than that of contracts issued solely to relationship investors.

Third, we add to the literature on relationship lending (see, e.g., Diamond (1991), Petersen and Rajan (1994), or Boot and Thakor (1994)) by showing how firms' debt maturity choice impacts the relationships building process and how these relationships in turn affect the joint choice of leverage and debt maturity as well as the decision to default. We also show that lending relationship have large quantitative effects on leverage ratios and are tightly associated with the debt maturity choice.

Lastly, there exists a large empirical literature documenting the role of debt investors in shaping many aspects of firm financing (see e.g. Lemmon and Roberts (2010), Faulkender and Petersen (2006), Leary (2009) for early contributions). In this literature, recent studies show that firms with stronger relationships with debt investors can raise more debt and benefit from better financing terms (see e.g. Bharath et al. (2011), Engelberg et al. (2012), Karolyi (2018), or Herpfer (2021)). Most of the early literature on supply-side frictions in corporate debt markets focuses on bank loans. Recent studies find that capital supply conditions and relationships are also important in primary bond markets (see, e.g., Chen et al. (2020), Zhu (2021), Kubitza (2021), Siani (2022), or Coppola (2022)). Our model captures some key features of primary debt markets and demonstrates how lending relationships shape leverage and debt maturity choices.

Section I presents the model. Section II analyzes the model implications for optimal leverage and debt maturity. Section III extends the model to allow for idiosyncratic shocks to a relationship investor's credit supply. Section IV concludes. Technical developments including the numerical implementation of the model solution are gathered in the Appendix.

I Model

A Assumptions

Throughout the paper, agents are risk-neutral and discount cash flows at the constant rate $r > 0$. Time is continuous and uncertainty is modeled by a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where the filtration $\{\mathcal{F}_t : t \geq 0\}$ satisfies the usual conditions.

Our objective is to characterize the effects of lending relationships on firm financing in a model that captures the dynamics of corporate financing behavior. To do so, we build on Fischer, Heinkel, and Zechner (1989) and Strebulaev (2007) and consider a firm with assets in place that generate a cash flow X_t at time $t \geq 0$ as long as the firm is in operation. This operating cash flow is independent of financing choices and governed by the process:

$$dX_t = \mu X_t dt + \sigma X_t dB_t,$$

where $\mu < r$ and $\sigma > 0$ are constant parameters and $(B_t)_{t\geq 0}$ is a Brownian motion.

Operating cash flows are taxed at the constant rate $\gamma \in (0,1)$, providing the firm with an incentive to issue debt. Debt contracts are characterized by a principal ρ and a coupon c and mature with Poisson intensity $\frac{1}{m}$, all of which are endogenously chosen. We model debt maturity as lumpy, in that all outstanding debt matures simultaneously, as in Geelen (2016) and Chen et al. (2021). This assumption is consistent with the finding in Choi, Hackbarth, and Zechner (2018) that lumpiness in maturity structure is a prevalent feature in the data. Denoting the next time that debt matures by τ_m, we have that (barring default) expected debt maturity is given by $\mathbb{E}(\tau_m) = m$. We allow the firm to re-optimize its capital structure when debt matures.[3] This feature differs from the assumption used, e.g., in Leland and Toft (1996), He and Milbradt (2014), or Della Seta, Morellec, and Zucchi (2020), that firms are committed to roll over any retired debt and continuously issue debt. We also allow the firm to default on its debt, which can occur when the debt matures at time τ_m or before maturity at the endogenous time τ_D. In default, creditors recover a fraction $(1-\alpha)$ of the unlevered asset value, where $\alpha \in (0,1)$ is a frictional default cost.

We are interested in building a model in which capital structure and debt maturity depend not only on firm characteristics but also on frictions in primary debt markets and the quality of issuer-debt investor (or lending) relationship. Indeed, as documented by Petersen and Rajan (1994), Berger and Udell (1995), Bharath et al. (2011), Karolyi (2018), Zhu (2021), and Kubitza (2021), issuer-investor relationships are first-order determinants of debt issuance decisions, leverage ratios, and credit spreads. In addition, as discussed for example in Chen, Schuerhoff, and Seppi (2020), the primary market for corporate debt is subject to significant frictions that are reflected in the cost of debt. A typical example is the U.S. corporate bond market, an over-the-counter market which is illiquid and subject to search frictions. Chen, Schuerhoff, and Seppi (2020) find that, as a result of search frictions, on average more than half of a new bond issuance gets placed with issuers' incumbent bondholders and that firms with reduced search frictions face lower costs of issuing new bonds. Relatedly, Zhu (2021)

[3]We can also extend our model to allow for debt restructuring as in Fischer et al. (1989). Geelen (2016) shows that having the ability to increase leverage by buying back outstanding debt and issuing new debt lengthens optimal debt maturity. The reason is that the ability to restructure debt lowers the value of the option to adjust capital structure at maturity, which gives firms an incentive to lengthen their debt maturity.

finds that bond mutual funds that hold a firm's existing bonds are five times more likely to provide capital in future bond issues and do so at a lower cost.

To capture these features of primary debt markets, we consider that debt issuance at time $t > 0$ works as follows. The firm initially starts a relationship with a single debt investor or a single pool of investors in the case of bond issues. Throughout the paper, we call this (pool of) investor(s) the relationship investor. When seeking to refinance existing debt and potentially change its leverage, the firm contacts its relationship investor—directly for a loan or via its lead underwriter for a bond issue—for a debt issue of endogenous size $\hat{\rho}$ and maturity $m \in \mathcal{M}$, where $\mathcal{M} \subseteq (\underline{m}, \infty]$ with $\underline{m} > 0$ the set of available maturities. There is uncertainty regarding the relationship investor's ability to provide financing (or appetite for debt). We denote by $\beta_t \in [0, 1]$ the fraction of the new issue that the relationship investor purchases, where β_t is drawn from a distribution that reflects the relationship quality and where the highest (respectively lowest) quality relationship investor H (L) has a higher (lower) chance of filling the firm's demand for debt. In this specification, β_t also captures the relationship investor's ability to supply capital, that depends for instance on inflows and outflows to and from the relationship investor and on its investment (risk) strategy.

We assume that the quality of the relationship between the firm and the relationship investor at time t depends on the number $n_t \in \mathbb{N}_+$ of financing rounds in which the investor participated before time t (the proxy used in the empirical study of Bharath et al. (2011)),[4] i.e. the number of loans made by the bank to the issuer or the number of bond issues in which incumbent bondholders invested.[5] Notably, we consider that if the investor participated in n_t financing rounds by time t, then β_t is drawn from a distribution $\beta \sim \mathbb{P}(\beta|n_t)$ given by

$$\mathbb{P}(\beta|n) = q(n)\mathbb{P}^H(\beta) + (1 - q(n))\,\mathbb{P}^L(\beta)$$

[4] This implies that n_t is right-continuous with left limits. Therefore, if the firm issues debt at time t then $\lim_{s\uparrow t} n_s = n_t = \lim_{s\downarrow t} n_s - 1$.

[5] In our model, the quality of the issuer-investor relationship is therefore positively related to the duration of the relationship—the proxy used for the quality of the relationship in the empirical studies of Petersen and Rajan (1994) or Berger and Udell (1995)). We can easily extend the model to allow the relationship quality to also improve due to monitoring or other interactions that the firm has with its relationship investor.

where $q(n) \in [0,1]$ is non-decreasing and invertible (for $q(n) < 1$). In this specification, $q(n)$ measures the quality of the lending relationship and $\mathbb{P}^H(\beta)$ and $\mathbb{P}^L(\beta)$ respectively govern the credit supply associated with the highest and lowest relationship qualities. In our framework, the number of previous debt issues n_t and the relationship quality $q_t = q(n_t)$ are both sufficient statistics for the lending relationship. In the remainder of the paper, we work with the relationship quality q_t, which is a non-decreasing process.

Debt issuance with the relationship investor incurs a proportional issuance (underwriting) cost $\psi_R \geq 0$. After the relationship investor has announced the amount of debt it purchases $\beta_t \hat{\rho}$, the firm decides whether it issues additional debt to outside investors $\rho - \beta_t \hat{\rho} \geq 0$ at a proportional issuance cost ψ_O. Better issuer-investor relationships reduce financing frictions in that $\psi_O \geq \psi_R$.[6] This assumption is consistent with the finding in Yasuda (2005) that banks charge lower fees to those firms with which they have relationships. The additional cost incurred by the firm when raising debt from multiple investors may be due to additional search costs incurred by the underwriter for a bond issue (see Chen et al. (2020)) or to costs incurred by the lead bank when securing the funds and syndicating the loan (see e.g. Ivashina (2009)). When issuing new debt, the firm therefore issues a contract (loan or bond) that is purchased either solely by the relationship lender (at a low frictional cost) or by both the relationship lender and outside investors (at a higher frictional cost). The firm is always able to sell its full debt issue, but the average cost of issuance decreases with the fraction of the issue purchased by the relationship investor.

After the size ρ and the maturity m of the debt issue are chosen and the composition of the pool of investors is determined, the coupon rate is set such that debt is issued at par. Finally, debt investors impose the following restriction on the size of the debt issue, which guarantees that firm value is finite.[7]

Assumption 1. *When the firm issues debt at time τ_m, debt investors require the interest coverage ratio $\frac{X_{\tau_m}}{c}$ to be above some strictly positive constant, which can be arbitrarily small.*

[6] When $\psi_O = \psi_R$, raising funds from relationship and non-relationship investors is (counter-factually) subject to the same financing costs and the firm can issue any amount of debt it wants at the issuance cost $\psi_O = \psi_R$. As a result, lending relationships play no role in financing decisions.

[7] This constraint does not bind in equilibrium and plays no role in the analysis.

The timeline for debt issuance, summarized in Figure 1, is therefore as follows:

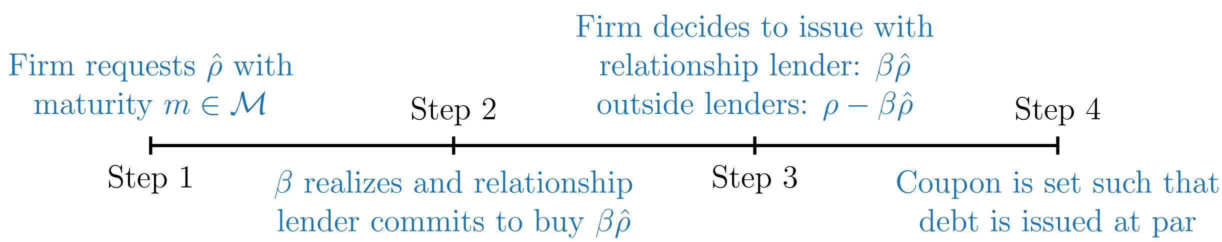

Figure 1: **Debt issuance process.**

1. The firm contacts its relationship investor (lender) for a debt issue of endogenous size $\hat{\rho}$ and maturity $m \in \mathcal{M}$.

2. The relationship investor commits to purchase $\beta\hat{\rho}$ of debt with maturity m.

3. The firm chooses chooses how much debt to issue to non-relationship investors $\rho - \beta\hat{\rho} \geq 0$, determining the total face value $\rho \in [\beta\hat{\rho}, \hat{\rho}]$ of the debt issue, total debt issuance costs $\psi_R\rho + (\psi_O - \psi_R)(\rho - \beta\hat{\rho})$, and the proceeds from the debt issue

$$\underbrace{(1 - \psi_R)\beta\hat{\rho}}_{\textit{Relationship investor}} + \underbrace{(1 - \psi_O)(\rho - \beta\hat{\rho})}_{\textit{Outside investors}}.$$

4. The coupon rate is set such that debt is issued at par, given the face value and the maturity of the debt contract, the level of cash flows, and the quality of the issuer-debt investor relationship at the time of issuance.

Figure 2 illustrates the firm's financing choices (top panel) and the dynamics of the relationship quality and their effects on credit supply (bottom panel) for a given path of operating cash flows. The stopping times $(\tau_m^n)_{n=1}^{+\infty}$ indicate the dates at which debt matures and new debt is issued and therefore the dates when the coupon rate, the face value of debt, debt maturity, and the relationship quality change. The characteristics of the debt issue (face value, maturity, number of debt investors) at any time τ_m depend on the two state

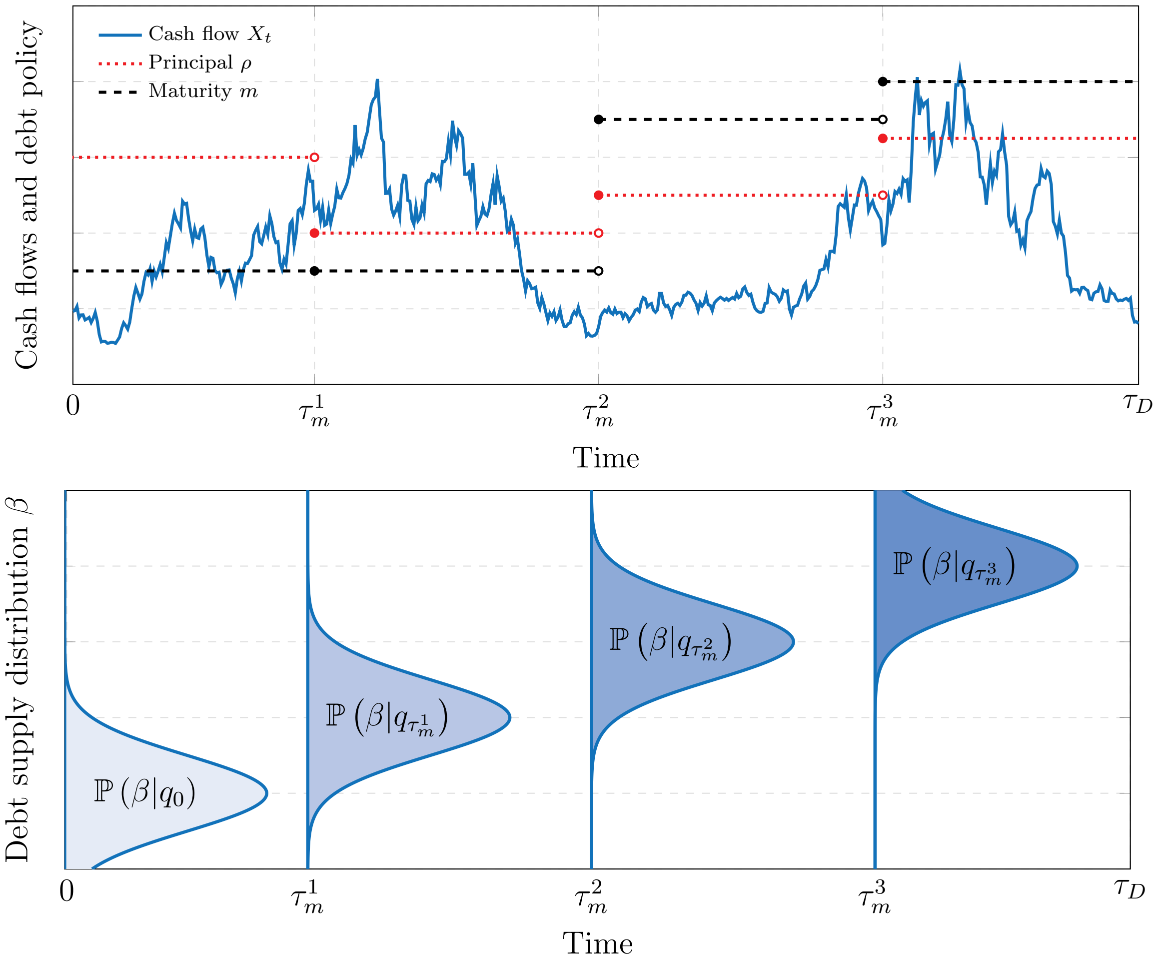

Figure 2: **Financing and lending relationship dynamics**. Jumps in endogenous quantities (principal, coupon, maturity) occur on maturity dates $(\tau_m^n)_{n=1}^{+\infty}$. Default occurs at time τ_D.

variables $(X_{\tau_m^n}, q_{\tau_m^n})_{n=1}^{+\infty}$. Notably, the processes describing the coupon, face value, and debt maturity are piece-wise constant and only change when new debt is issued. At that time, the debt supply distribution shifts up, reflecting an improvement in the relationship quality.

B Optimal Financing and Default Policies

To determine the effects of issuer-debt investor relationships on leverage and debt maturity choices, we need to determine the prices of corporate debt and equity. To aid in the understanding of the pricing formulas, Figure 3 shows the cash flows to shareholders and creditors (relationship and outside debt investors) at different points in time. The top row in the blue boxes indicates the cash flow to shareholders while the bottom row indicates the cash flow to creditors. The gray area describes the decisions made by shareholders at maturity/issuance and their effects on cash flows. On the maturity date τ_m, shareholders decide whether to default on maturing debt or not. If there is no default, maturing debt is repaid and new debt with face value ρ' and maturity m' is issued.

Debt value is given by the present value of the cash flows that creditors expect to receive and depends on the firm's current cash flow x, debt coupon c, the relationship quality q, debt principal ρ, and debt maturity m. Specifically, the value of outstanding debt is given by

$$
\begin{aligned}
&D(x, c, q, \rho, m) \\
&= \mathbb{E}_x \left[\int_0^{\tau_m \wedge \tau_D} e^{-rt} c dt + \mathbb{I}_{\{\tau_D \leq \tau_m\}} e^{-r\tau_D} \frac{(1-\alpha)(1-\gamma) X_{\tau_D}}{r-\mu} \right] \\
&+ \mathbb{E}_x \left[\mathbb{I}_{\{\tau_m < \tau_D\}} e^{-r\tau_m} \left(\mathbb{I}_{\{F(X_{\tau_m}, q) \geq \rho\}} \rho + \mathbb{I}_{\{F(X_{\tau_m}, q) < \rho\}} \frac{(1-\alpha)(1-\gamma) X_{\tau_m}}{r-\mu} \right) \right]
\end{aligned}
$$

where $F(x, q)$ is the continuation value of shareholders defined in equation (1) below, $\mathbb{I}_{\{x \leq y\}}$ is the indicator function of the event $x \leq y$, and $\tau_m \wedge \tau_D \equiv \inf\{\tau_m, \tau_D\}$ is the first time that debt matures or the firm defaults. As shown by this equation, creditors receive coupon payments until the debt matures or the firm defaults. The firm can default either if cash flows deteriorate sufficiently before maturity (in which case $\tau_D \leq \tau_m$) or when the debt matures if the continuation value of shareholders is less than the debt principal (i.e. if $F(x, q) < \rho$). If the firm defaults before debt maturity, creditors recover a fraction $(1 - \alpha)$ of the unlevered asset value. If debt matures and the firm does not default, creditors receive the principal ρ. Otherwise, they get the recovery value. As we show below, the quality of the lending relationship feeds back not only in financing decisions (i.e. on the choice of (ρ, c, m)) but also in the decision to default for given (ρ, c, m) by affecting the continuation value of equity.

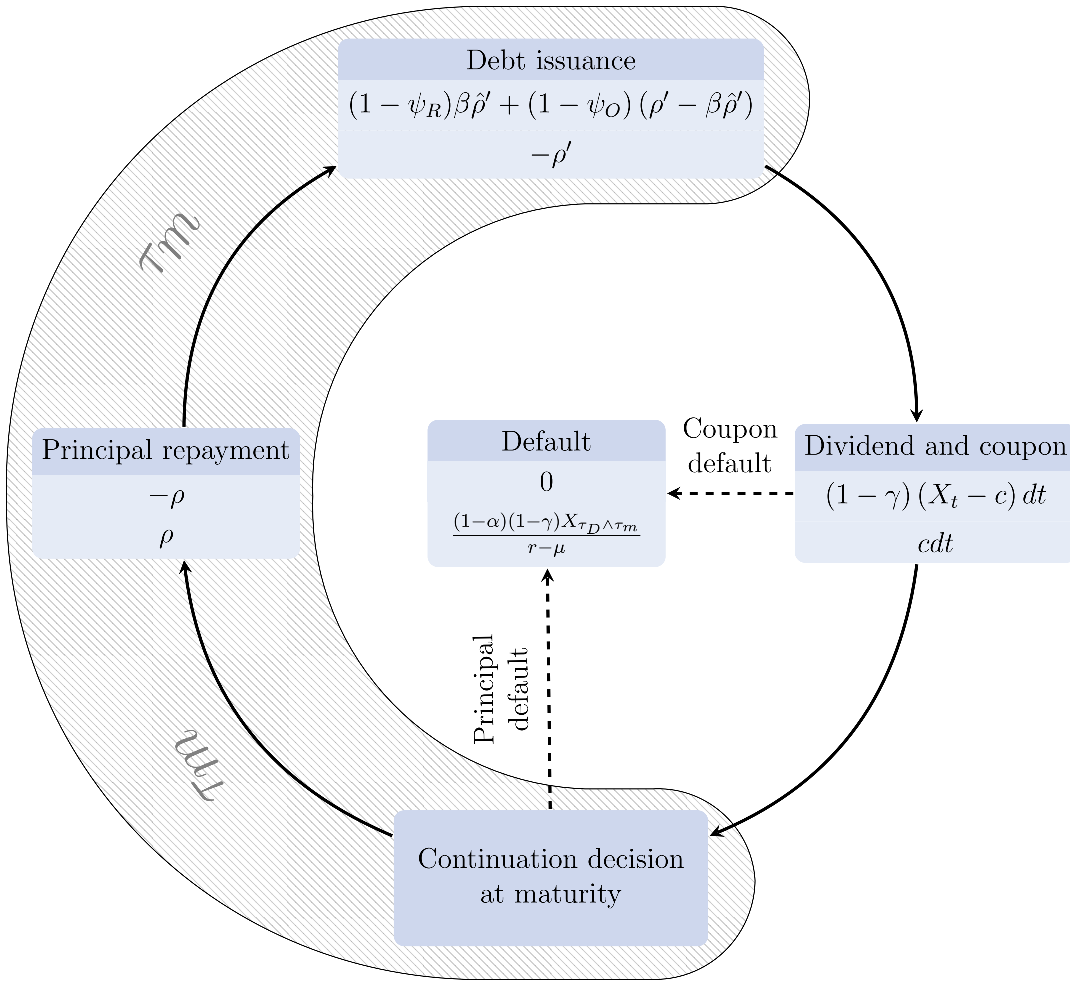

Figure 3: **Cash flows to and from shareholders and creditors (relationship + outside investors).** The middle row in the blue boxes indicates the cash flow to shareholders while the bottom row indicates the cash flow to creditors. The gray area describes the decisions made by shareholders at maturity/issuance and their effects on cash flows. In this figure, $\hat{\rho}'$ and ρ' respectively indicate the quantity of debt requested and issued at time τ_m.

Shareholders' levered equity value, denoted by $E(x, c, q, \rho, m)$, is in turn given by:

$$E(x, c, q, \rho, m) = \sup_{\tau_D} \mathbb{E}_x \left[\int_0^{\tau_m \wedge \tau_D} e^{-rt}(1-\gamma)(X_t - c)\,dt + \mathbb{I}_{\{\tau_m < \tau_D\}} e^{-r\tau_m} \left(F(X_{\tau_m}, q) - \rho \right)^+ \right]$$

where $x^+ = \max\{0, x\}$. As shown by this equation, shareholders receive the firm's cash flows minus coupon payments net of taxes until either the debt matures at τ_m or the firm defaults at τ_D. If the firm defaults before maturity (i.e. $\tau_D \leq \tau_m$), absolute priority is enforced and shareholders receive zero. When the debt matures, shareholders decide whether to repay the principal ρ. If debt is repaid, shareholders get the continuation value defined in equation (1) net of the debt principal ρ. Otherwise, they get zero.

Lastly, firm value at issuance is given by

$$F(x,q) = \tag{1}$$
$$\sup_{(\hat{\rho},m)\in\mathbb{R}_+\times\mathcal{M}} \mathbb{E}_q \left[\sup_{\rho\in[\beta\hat{\rho},\hat{\rho}]} \left\{ E(x,c,q',\rho,m) + (1-\psi_R)\beta\hat{\rho} + (1-\psi_O)\left(\rho - \beta\hat{\rho}\right) \right\} \right]$$

such that

$$c = \left\{ c' \middle| D(x,c',q',\rho,m) = \rho \right\} \tag{2}$$

where q' is the relationship quality after an additional financing round given a current relationship quality q. Equation (1) shows that shareholders first decide on the amount of debt to request from their relationship investor $\hat{\rho}$ and on the maturity m of this debt. The inner maximization operator shows that shareholders decide on how much debt to issue $\rho \in [\beta\hat{\rho}, \hat{\rho}]$ after observing the relationship investor's credit supply β. Finally, equation (2) indicates that the coupon is set such that debt is issued at par.

Given the functional forms of issuance costs, default costs, and taxes, shareholder's optimization problem is homogeneous of degree one in x. Notably, we can establish the following result (see the Appendix for a proof):

Proposition 1 (Firm value). *Firm value exists, is finite, and satisfies $F(x,q) = xf(q)$.*

The homogeneity of the firm value function in x works through the levered equity and debt values, which can be written as

$$E(x,c,q,\rho,m) = xe\left(\frac{c}{x}, q, \frac{\rho}{c}, m\right) \quad \text{and} \quad D(x,c,q,\rho,m) = xd\left(\frac{c}{x}, q, \frac{\rho}{c}, m\right),$$

where the functions e and d are defined in the Appendix (see Lemma 1).

Proposition 1 implies that in our model, all claims to cash flows scale with the level of operating cash flows as in Leland (1998), Strebulaev (2007), or Morellec et al. (2012). Using this scaling property, we can also establish the following result:

Proposition 2 (Optimal default). *The optimal strategy for shareholders is to default:*

1. ***Coupon default**: Before maturity if the ratio of the coupon payment to the firm cash flow $z = \frac{c}{x}$ rises above an endogenous threshold $z_D(q, \frac{\rho}{c}, m)$, that is determined by the equity value's smooth pasting condition.*

2. ***Principal default**: On the maturity date of the debt contract if the debt principal ρ exceeds firm value $F(x, q)$.*

Proposition 2 shows that there are two types of default in our model: *i*) The firm can default when its current cash flow drops sufficiently and shareholders are unwilling to cover additional losses, which we call a *coupon default*, and *ii*) At maturity the principal needs to be repaid and shareholders are unwilling to do so, which we call a *principal default*.

II Model Analysis

A Parameters

This section examines the effects of the quality of lending relationships on leverage, debt maturity choices, and default risk. To do so, we calibrate the model parameters to reflect a typical U.S. public firm. Parameter values are reported in Table 1.

In the analysis, firms are characterized by a set of parameters that determine their cash flow growth rate μ and volatility σ, corporate taxes γ, default costs α, the costs of issuing debt with the relationship investor ψ_R and outside investors ψ_O, and the risk-free rate r. We split the parameters in two sets: parameters that we take from other studies $(r, \gamma, \alpha, \mu, \alpha)$ and parameters that we calibrate by matching empirical moments

Parameter	Symbol	Value
Interest rate	r	4.2%
Tax rate	γ	15%
Default costs	α	40%
Issuance costs (relationship investor)	ψ_R	0.6%
Issuance costs (outside investors)	ψ_O	2.7%
Cash flow drift	μ	3%
Cash flow volatility	σ	25%
Maturity set	$\mathcal{M}$	$[1, \infty]$
Relationship investor debt appetite	β	$\{0.4, 1\}$
Probabilities high-quality relationship investor	$\mathbb{P}(\beta = 1 \vert \theta = H)$	0.9
Probabilities low-quality relationship investor	$\mathbb{P}(\beta = 1 \vert \theta = L)$	0.2
Relationship quality function	$q(n)$	$\frac{exp^{0.25(n-7)}}{exp^{0.25(n-7)}+1}$

Table 1: **Baseline parameters.**

$(\psi_R, \psi_O, \beta, \mathbb{P}(\beta = 1|\theta = H), \mathbb{P}(\beta = 1|\theta = L))$, such as the fraction of debt held by relationship lenders, average refinancing costs, average leverage, or average debt maturity (determining the average frequency of debt rebalancings), and average maturity of bonds/syndicated loans.

The risk-free rate is set equal to $r = 4.2\%$ as in Morellec et al. (2012). The tax benefits of debt are set equal to $\tau = 15\%$. As in Graham (1999), this estimate reflects both corporate and personal taxes. We set α to 40%, which is similar to Glover (2016)'s estimate of the default cost. In our base case parametrization, we set the costs of debt issuance with the relationship investor ψ_R and outside investors ψ_O to 0.6% and 2.7%, respectively. This produces an average cost of debt issuance between 0.6% and 0.8% of the issue size under the optimal financing policy, which is in the range reported by Altınkılıç and Hansen (2000). We additionally set the growth rate and the volatility of the cash flow process to $\mu = 3\%$ and $\sigma = 25\%$ as in, e.g., Morellec and Schürhoff (2010). The homogeneity property of the model implies that we can set the initial level of the cash flow to $X_0 = 1$ without loss of generality. We assume the firm can issue debt with a maturity of one year or longer $\mathcal{M} = [1, \infty]$.

We set the distribution for β such that the relationship investor acquires all the firm's debt or only 40% of it. The highest-quality (respectively lowest-quality) relationship investor has a 90% (20%) probability of purchasing the entire debt issue. Given these parameters, if the firm deals with the highest-quality investor and issues debt, on average, every 4 years, then the investor does not provide the full amount requested once every 40 years. A large part of our analysis is dedicated to studying the effects of varying these parameters on outcome variables. The function describing the quality of the lending relationship is given by $q(n) = \frac{exp^{0.25(n-7)}}{exp^{0.25(n-7)}+1}$, where $q_0 = q(0) \approx 0.15$ captures factors, other than past interactions, that affect the relationship quality (such as geographic proximity or industry specialization). When $n = 0$, the relationship is of the lowest quality. As n increases the quality of the relationship improves, allowing the firm to raise more funds from the relationship investor.[8] The numerical implementation used to solve our model is described in Appendix C.

With these baseline parameters, the model predicts an optimal debt maturity between 3.38 and 8.54 years and leverage ratios (at issuance) between 22.0% and 32.5% depending on the quality of issuer-investor relationship, in line with empirical estimates. For instance, Choi et al. (2018) show that the average of firms' debt maturities in the Compustat database is 5.15 years. Colla, Ippolito, and Li (2020) report that mean market leverage ratios for firms covered by Compustat from 2002 to 2018 are between 18.1% and 31.6%.

B Fixed Debt Maturity

To aid in the intuition of the model, we start by analyzing corporate policies assuming that debt maturity m is exogenously given. Under this assumption, shareholders only have to decide on how much debt to issue and when to default. An improvement in the lending relationship leads to an increase in the willingness of the relationship lender to provide financing at a low cost. As a result, net benefits of debt and optimal leverage increase with the quality of the lending relationship for all maturities, as illustrated by Figure 4. For example, when issuing 5-year debt, optimal leverage increases from 22.5% to 30.2%, as q

[8]This functional form is motivated by a learning model where for intermediate beliefs information moves priors the most. Observe that the inverse of the log-likelihood ratio is $q = exp(z)/(exp(z)+1)$. In an earlier version of the paper, we assumed that the quality of the relationship between the firm and the relationship investor was unknown ex ante but both the firm and the investor learned over time from past debt purchases by the relationship investor. This setup leads to very similar implications as our current setup.

increases from its minimum to its maximum value.

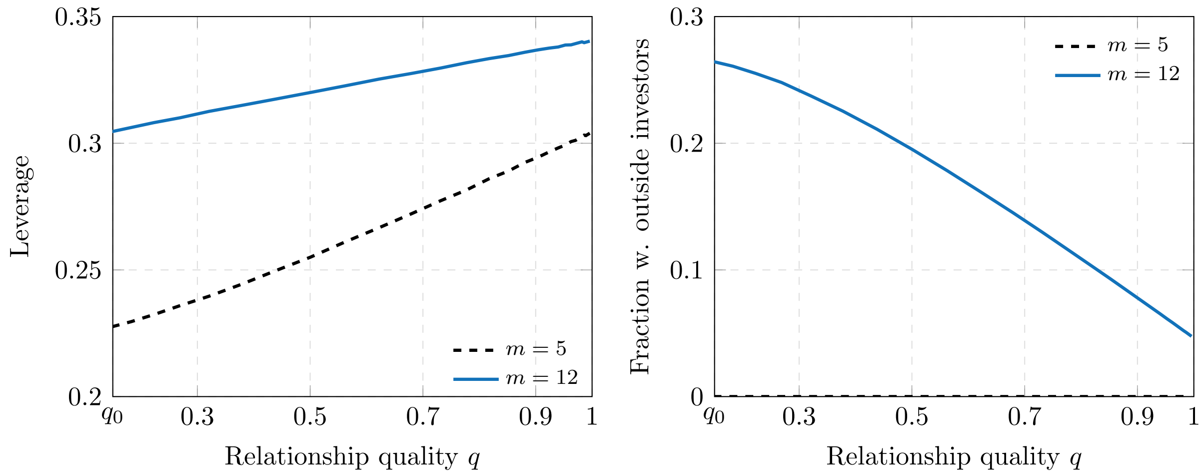

Figure 4: **Lending relationships, leverage, and the fraction of debt issued to outside investors.** Maturity is fixed at either 5 years (black dashed line) or 12 years (blue solid line). Parameters are as in Table 1. Leverage is defined as $\frac{\rho}{\rho+E(x,c,q,\rho,m)}$. Thhe fraction of debt issued with outside investors is given by $\frac{\rho-\beta\hat{\rho}}{\rho}$.

Figure 4 also shows that an increase in average debt maturity increases the target leverage ratio. When debt has a longer maturity and the firm is expected to grow, shareholders need to wait longer to adjust capital structure and therefore issue more debt ex-ante. Interestingly, when issuing long-maturity debt, the firm decides to borrow both from relationship and non-relationship lenders as shown by the right panel of Figure 4. The reason is that debt issuance to non-relationship lenders becomes profitable because the issuance cost can be spread over the longer duration of the debt. As the relationship quality improves, the relationship lender invests more in the firm's debt so that the fraction of debt held by non-relationship investors decreases. For shorter maturities (and hence higher refinancing costs), optimal leverage is lower. Furthermore, the firm finds it too costly to raise funds from outside investors and only borrows from relationship lenders.

To illustrate the value effects of lending relationships, Figure 5 plots firm value as a function of q. The better the relationship quality, the greater the incentives of the relationship investor to provide debt financing. Therefore, the firm is more certain that it can issue the amount of debt it wants at a lower cost, leading to an increase in firm value. With the baseline parameters, lending relationships can improve total firm value by more than 1.0%. This effect is large compared to the total net benefits of debt estimated between 3.5% and 5.5% of firm value by Korteweg (2010) and Van Binsbergen, Graham, and Yang (2010).

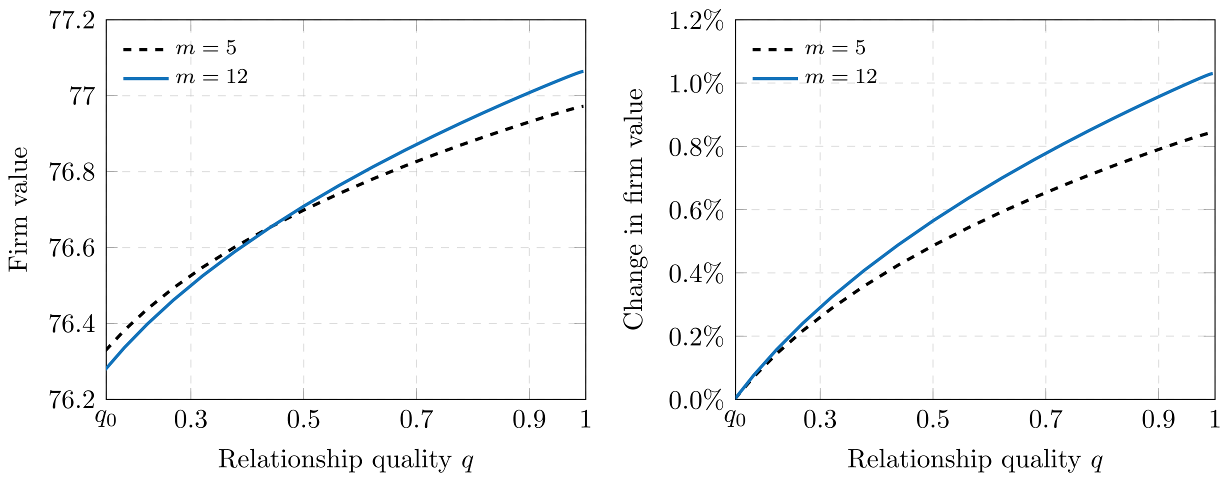

Figure 5: **Impact of the quality of lending relationships on firm value for different debt maturities.** Maturity is fixed at either 5 years (black dashed line) or 12 years (blue solid line). Parameters are as in Table 1. Change in firm value is defined as $\frac{f(q|m)-f(q_0|m)}{f(q_0|m)}$.

Figure 5 additionally shows that when the relationship quality is low, issuing shorter-term debt leads to a greater improvement in firm value. By contrast, when the relationship quality is high, issuing longer-term debt is preferable. The reason is that with a low-relationship quality, the firm benefits from the repeated interactions with the relationship investor, that only occur when issuing new debt. For a high relationship quality, these benefits of short-term debt are smaller and the firm prefers to chose a debt maturity that is closer to the optimal maturity, which is driven by the trade-off between financial flexibility and debt issuance costs. Figures 4 and 5 therefore suggest that as lending relationships improve, firms issue more debt and longer-maturity debt. Longer-maturity debt is associated not only with much higher leverage (and thus much larger debt issues) but also characterized by the participation of non-relationship lenders in loan syndicates or bond issues.

C Financing with Endogenous Debt Maturity

I Lending Relationships, Debt Supply, and Firm Value

We start by examining the effects of lending relationships on the size of debt issues. To do so, we plot in Figure 6 the amount of debt purchased by relationship and non-relationship investors as a function of the relationship quality q. The figure shows that stronger lending relationships lead to an increase in the financing provided by the relationship investor. Higher availability of debt financing at a lower cost leads a firm with better lending relationships to issue more debt, in line with empirical findings. For instance, Bharath et al. (2011) find that relationship borrowers receive larger loans. Figure 6 therefore shows that debt issuance is driven not only by a firm's demand for debt but also by credit supply, in line with the evidence in Lemmon and Roberts (2010), Leary (2009), Zhu (2021), and Kubitza (2021).

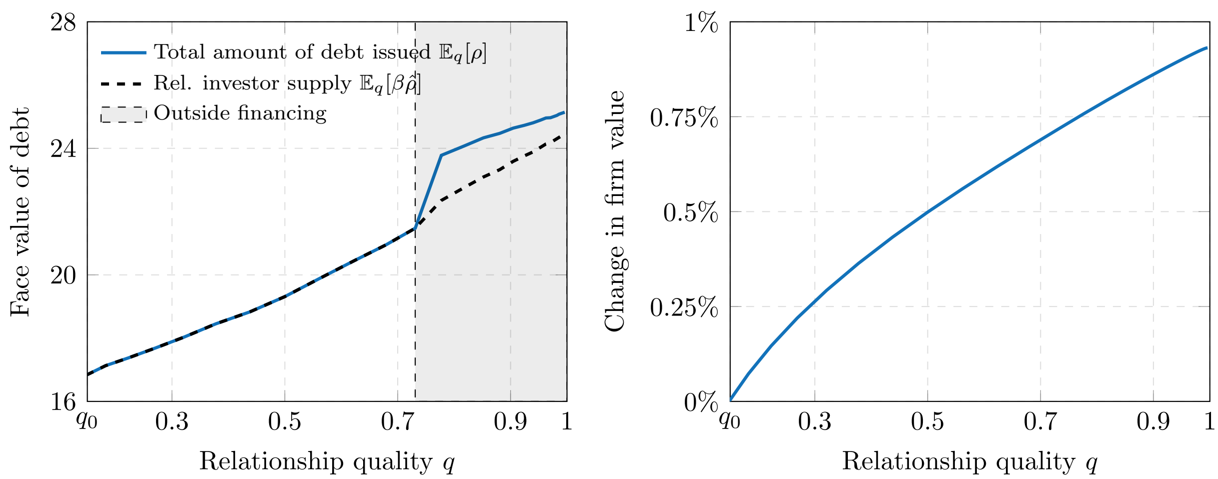

Figure 6: **Lending relationship, debt supply, and firm value.** The grey area indicates the region in which the firm also raises debt from outside investors. Parameters are as in Table 1. We take the average over credit supply β realizations.

A striking result in Figure 6 is that stronger lending relationships allow the firm not only to issue more debt from relationship lenders but also to raise additional debt from outside investors, due to the associated decrease in the non-frictional cost of debt. In our base case environment, this occurs for $q > 0.73$. In the context of our model, debt issuance with both the relationship and outside investors can be interpreted as the issuance of a syndicated loan or a bond. Among the firms that issue debt to outside investors, the fraction of the debt issue acquired by the relationship investor is lower for firms with weaker relationship quality. Therefore, the model predicts that stronger lending relationships lead to a higher likelihood of issuing syndicated loans. It also predicts that, conditional on issuing a syndicated loan, the loan structure becomes more concentrated as the lending relationship improves.

Stronger lending relationships allow the firm to borrow more at better terms. As a result, they are associated with a higher firm value, as illustrated by Figure 6. In our base case environment, a firm with a high-quality relationship investor ($q = 1$) has a value that is almost 1% higher than the value of a firm that issues debt to a new relationship investor ($q = 0.15$). The reason is that the better the quality of the lending relationship is, the higher is the likelihood that the relationship investor provides the entire amount of debt requested. As a result, firms with a strong lending relationship are more likely to issue debt at a lower cost. In addition, firms with better lending relationships may also borrow more from outside investors, allowing the firm to sustain a higher leverage ratio. With our baseline parameters, the leverage ratio increases from 22.0% to 32.5% as the lending relationship improves when firms can choose debt maturity (see Figure 8 below).

II Optimal Debt Maturity

In the model, the firm chooses not only how much debt to issue but also the maturity of this debt. By issuing shorter maturity debt, the firm can change its capital structure more frequently by repaying existing debt and optimally adjust its leverage ratio and debt maturity. On the other hand, shorter debt maturities also imply that the firm incurs debt issuance costs more frequently. Optimal debt maturity balances these different effects.

Figure 7 illustrates the effects of lending relationships on optimal debt maturity. We note several results. First, at the beginning of the relationship (when q is low), the firm issues shorter maturity debt. Indeed, when the quality of the relationship is low, the firm can raise little debt from the relationship investor. As a result, it abstains from issuing longer maturity

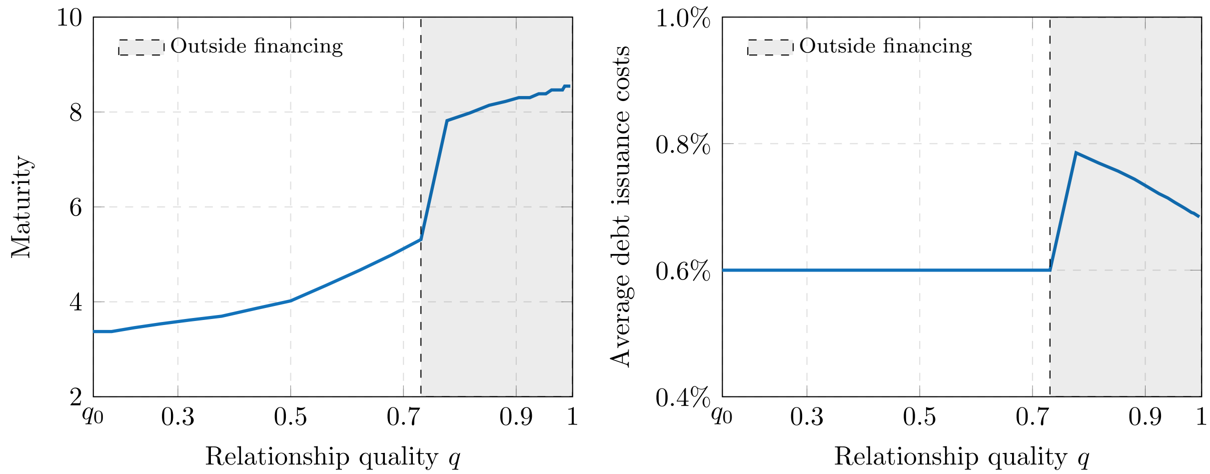

Figure 7: **Lending relationships, debt maturity, and debt issuance costs.** In the grey area, the firm raises debt from both relationship and outside investors. Parameters are as in Table 1. Average debt issuance costs are defined as $\psi_R \frac{\beta\hat{\rho}}{\rho} + \psi_O \frac{\rho-\beta\hat{\rho}}{\rho}$. We take the average over credit supply β realizations.

debt. Debt contracts with shorter maturity allow the firm to refinance debt at better terms sooner. Once the relationship quality improves, more debt financing is available from the relationship investor, allowing the firm to get closer to its target leverage ratio when issuing new debt and leading the firm to issue longer maturity debt.

Second, firms that issue shorter maturity debt incur debt issuance costs more frequently. Since the firm only receives tax benefits over the interest payments but pays issuance costs over the full face value of debt, debt issuance costs are relatively larger for shorter maturity debt. This effect makes debt issuance with outside investors relatively less attractive for firms with weak lending relationships that issue shorter maturity debt. As a result, these firms abstain from raising funds from outside investors (see Figure 6).

As the relationship quality improves, default risk and the non-frictional cost of debt decrease and the firm issues longer maturity debt. When the cost of debt decreases sufficiently, the marginal cost of issuing debt to outside investors falls below the marginal benefit of issuing additional debt and it becomes optimal for the firm to issue debt with outside investors. At that point, the average issuance cost jumps and, as a result, so does the optimal maturity as illustrated by Figure 7. The model therefore predicts that the maturity of debt contracts issued to non-relationship investors is higher than that of relationship investors, in line with the evidence in Bharath, Dahiya, Saunders, and Srinivasan (2011). Overall, Figures 6 and 7 highlight the central role played by transaction costs and fees in leverage and maturity choices. This is consistent with the survey evidence in Graham and Harvey (2001) and Graham (2022) where transaction costs and fees come before bankruptcy costs or personal taxes as a determinant of capital structure choice.

III Comparative Statics

How do changes in a firm's environment affect the relation between lending relationships and leverage and debt maturity choices? To answer this question, Figure 8 shows the optimal maturity, leverage ratio, and the fraction of the debt issued to outside investors as functions of the relationship quality for different values of the corporate tax rate γ, default costs α, and cash flow volatility σ.

Optimal leverage is determined by the trade-off between the costs and benefits of debt. As a result, an increase in the corporate tax rate leads i) to an increase in leverage and ii) to a decrease of the threshold for the relationship quality q above which firms raise debt with outside investors. With the base case issuance costs and low tax benefits of debt (a tax rate of 10%), firms abstain from issuing debt with non-relationship investors at all levels of relationship quality. As the corporate tax rate γ increases to 15%, tax benefits become larger relative to the cost of debt. As a result, outside debt issuance becomes attractive for firms with high relationship quality that pay lower average debt issuance costs. A further increase in tax benefit of debt ($\gamma = 20\%$) makes outside debt issuance attractive for firms with even weaker lending relationships.

Corporate taxes also have an effect on the debt maturity choice. Notably, firms that would benefit the most from issuing debt (i.e. with high γ) but with weak lending relationships are further from their target leverage and issue shorter term debt to be able to re-optimize financing as lending relationships improve. These firms are also those that raise debt from

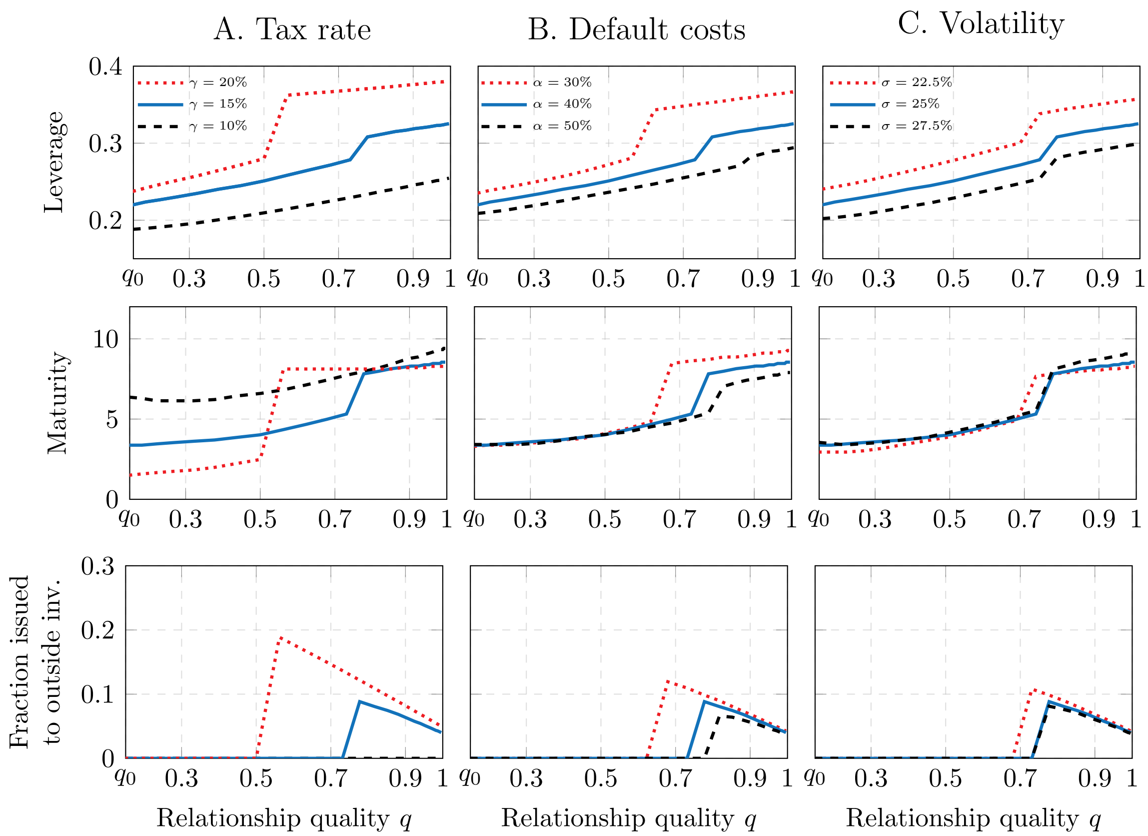

Figure 8: **The effects of lending relationships on financing decisions for varying levels of the corporate tax rate, default costs, and cash flow volatility.** The base case (Table 1) is depicted by the blue solid line. Leverage is defined as $\frac{\rho}{\rho+E(x,c,q,\rho,m)}$ and the fraction of debt issued with outside investors as $\frac{\rho-\beta\hat{\rho}}{\rho}$. We take the average over credit supply β realizations.

non-relationship investors the fastest, leading to a large increase in debt maturity.

The effects of bankruptcy costs on financing choices follow the same logic. Firms with higher default costs optimally choose lower target leverage ratios at all levels of relationship quality. In addition, these firms issue less debt with non-relationship investors at all levels of the relationship quality. Lastly, lower volatility decreases the probability of default and, thus, expected bankruptcy costs. In addition, firms with lower cash flow volatility also benefit from reduced uncertainty regarding tax benefits. This encourages these firms to issue more debt. Panel C of Figure 8 also shows that lowering volatility decreases the threshold for the quality

q of the lending relationship above which the firm raises debt with outside investors. That is, lower volatility of cash flows σ implies a lower non-frictional cost of debt, thus making borrowing from outside investors more attractive.

Tax Rate		**Default costs**		**Cash Flow Volatility**	
γ	$\frac{f(1)-f(q_0)}{f(q_0)}$	α	$\frac{f(1)-f(q_0)}{f(q_0)}$	σ	$\frac{f(1)-f(q_0)}{f(q_0)}$
20%	1.33%	30%	1.10%	22.5%	1.11%
15%	0.95%	40%	0.95%	25%	0.95%
10%	0.51%	50%	0.81%	27.5%	0.81%

Table 2: **Effects of lending relationships on firm value.** Parameters are as in Table 1.

Figure 8 and Table 2 demonstrate that the benefits of stronger lending relationships are larger for firms facing a higher tax rate, subject to lower default costs, and with less volatile cash flows. Firms with these characteristics have higher target leverage ratios and, therefore, need to raise more debt from investors. As a result, stronger lending relationships are more valuable for these firms. The predictions related to default costs and volatility are opposite to those coming out of a mechanism based on informational asymmetries.

D Relationship Versus Outside Investors

The wedge between the costs of issuing debt with the relationship investor versus outside investors reflects the severity of frictions in primary debt markets. This wedge can arise because underwriters need to search for new investors within a limited time frame when placing a new bond issue (Chen et al. (2020)). In the case of syndicated loans, it can arise because of the upfront fees necessary to compensate lead arrangers for attracting loan participants and structuring and originating the syndicated loan (Berg et al. (2016)).

Figure 9 shows optimal maturity and leverage choices for different costs of debt issuance with outside investors. When issuing debt with outside investors is relatively more expensive (first column of Figure 9), firms optimally choose to issue debt only with the relationship investor or to raise a small fraction of the debt issue in the outside market. Because of this,

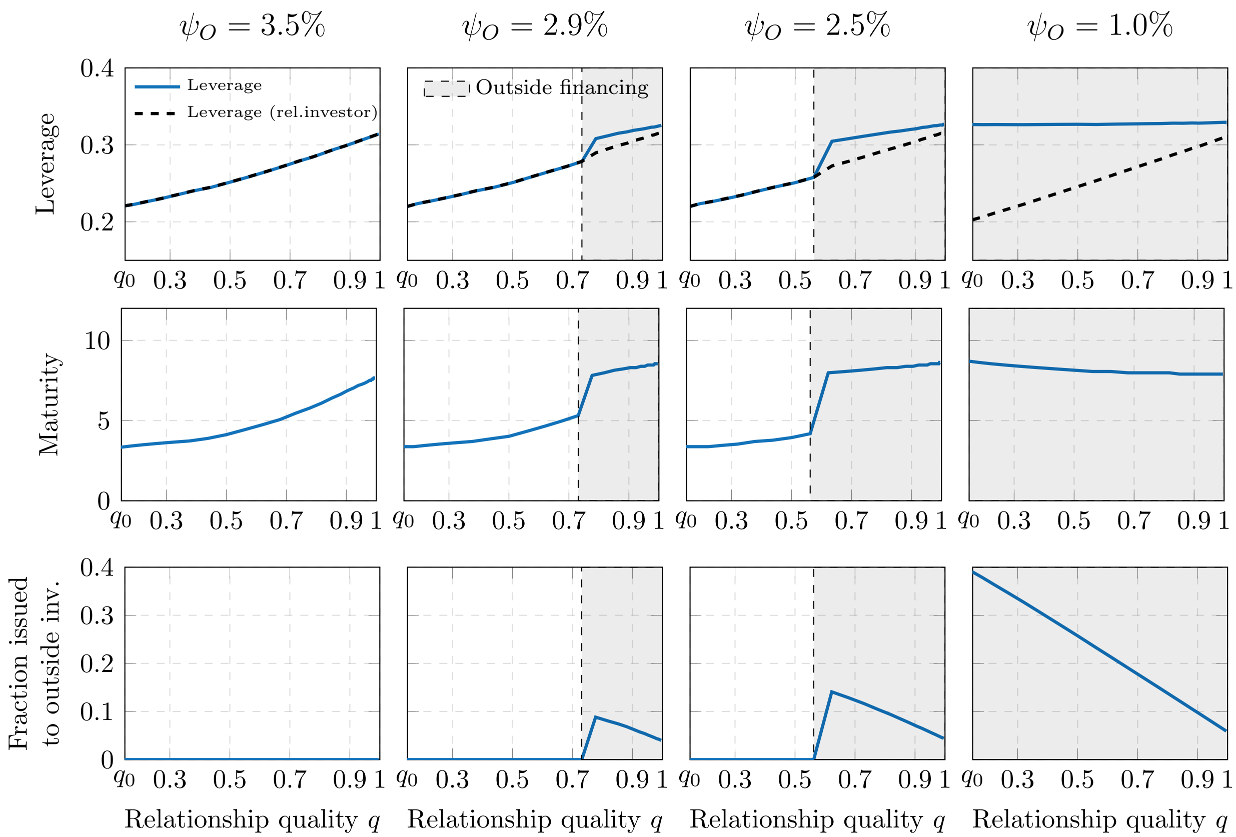

Figure 9: **The impact of lending relationships for various costs of debt issuance with outside investors.** The grey area indicates the region in which the firm raises debt from outside investors. Leverage is defined as $\frac{\rho}{\rho+E(x,c,q,\rho,m)}$ and the fraction of debt issued with outside investors as $\frac{\rho-\beta\hat{\rho}}{\rho}$. Parameters are as in the Table 1. We take the average over credit supply β realizations.

the average cost of debt issuance is similar for firms with different relationship qualities. As a result, the optimal maturity choice is driven mainly by the availability of debt financing from the relationship investor, and, thus, monotonically increases as quality improves.

As the cost of issuing debt with outside investors decreases, firms naturally choose to issue more debt with non-relationship investors. When the cost of issuing debt with outside investors decreases sufficiently, all firms issue debt in the outside market (the last column of Figure 9). As a result, all firms have approximately the same target leverage ratios at refinancing points.

Figure 9 also shows that the benefits of having better relationships with debt investors decrease as the cost of issuing debt with outside investors decreases. This is in agreement with the empirical evidence in Karolyi (2018) and recent evidence from the COVID-19 crisis (Halling et al. (2020) or Amiram and Rabetti (2020)) that relationships with debt investors are particularly important in times of economic downturns, i.e. when credit supply is weaker.

III Shocks to the Relationship Investor

A number of empirical studies have shown that relationship investors face shocks that may affect their ability to supply credit and therefore the financing choices of the firms they finance. For instance, Huber (2018) shows that Commerzbank—a major German commercial bank—suffered significant losses on its international trading book during the financial crisis, resulting in a reduction in the bank debt of companies that had a relationship with Commerzbank before the crisis.[9] Karceski, Ongena, and Smith (2005) and Di Patti and Gobbi (2007) show that bank consolidation—another type of exogenous shock to the lending relationship—negatively impacts firms with which the (target) bank has a relationship.

This section studies the effects of idiosyncratic shocks to the relationship investor's ability to supply credit on firm financing. To do so, we assume that the availability of credit from the relationship investor depends on the relationship investor's state s which can be either good, G, or bad, B. We assume that $\beta_t \in \{\beta_B, 1\}$ in state B and $\beta_t \in \{\beta_G, 1\}$ in state G, with $\beta_B < \beta_G$ so that the ability of the relationship investor to purchase decreases in state B. We also assume that $\mathbb{P}_B(\beta_B|\theta) = \mathbb{P}_G(\beta_G|\theta)$. As a result, the expected fraction of a debt issue that is purchased by the relationship investor is lower when she is in state B at all levels of relationship quality. The relationship investor transits from the good to the bad state with intensity κ_G and from the bad to the good state with intensity κ_B. The credit supply shock is transitory if $\kappa_B > 0$ and permanent if $\kappa_B = 0$. The state s is observable.

This setup implies that financing policy is a function of the firm's current cash flow x,

[9]Similarly, Fernando, May, and Megginson (2012) show that firms that had stronger security underwriting relationships with Lehman Brothers before the financial crisis were affected more severely by its collapse.

the quality of the relationship investor q, and, additionally, the relationship investor's state s. The model remains homogeneous of degree one in x so that firm value can be written as $F(x, q, s) = xf(q, s)$. Similarly, the debt and equity values scale linearly in x and additionally depend on the relationship investor's state s; see Lemma 3 in the Appendix.

Because shocks to the relationship investor affect the cost of debt, the firm may decide to request debt from a new relationship investor at the time of refinancing if the current relationship investor is in state B. The firm will do so if the value of starting a new relationship exceeds the value from staying with the current debt investor, i.e. if

$$f(q(0), G) > f(q, B).$$

Since $f(q, B)$ is monotonically increasing in q, there exists a replacement threshold q_R such that for $q < q_R$ the firm replaces its relationship investor if in state B at the time of refinancing. Using the scaling property of the model, it can be further shown that the optimal strategy for shareholders is to default $i)$ before maturity if the ratio of the coupon payment to the firm cash flow $z = \frac{c}{x}$ rises above an endogenous threshold $z_D(q, \frac{\rho}{c}, m, s)$ or $ii)$ on the maturity date of the debt contract if the debt principal ρ exceeds the continuation value of equity $F(x, q, s)$; see Proposition 4 in the Appendix.

Figure 10 plots optimal leverage, debt maturity, average issuance costs, and the fraction of debt issued to outside investors at issuance in the good and bad relationship investor states. The relationship investor's credit supply is set to $\beta_G = 0.4$ in the good state and to $\beta_B = 0.1$ in the bad state. The transition intensities are set to $\kappa_B = 0.1$ and $\kappa_G = 0.1$. Other parameters are as in Table 1.

The figure shows that when the relationship investor's ability to purchase debt decreases (i.e. when moving to state B), firms with lending relationships that are strong enough ($q > q_R$) do not switch to a new lender. When the relationship quality is below q_R, as shown by the blue area in Figure 10, the negative effects from a lending cut on firm value outweigh those from borrowing from a new relationship investor. As a result, firms terminate their current lending relationship and switch to a new lender, with little effect on leverage and

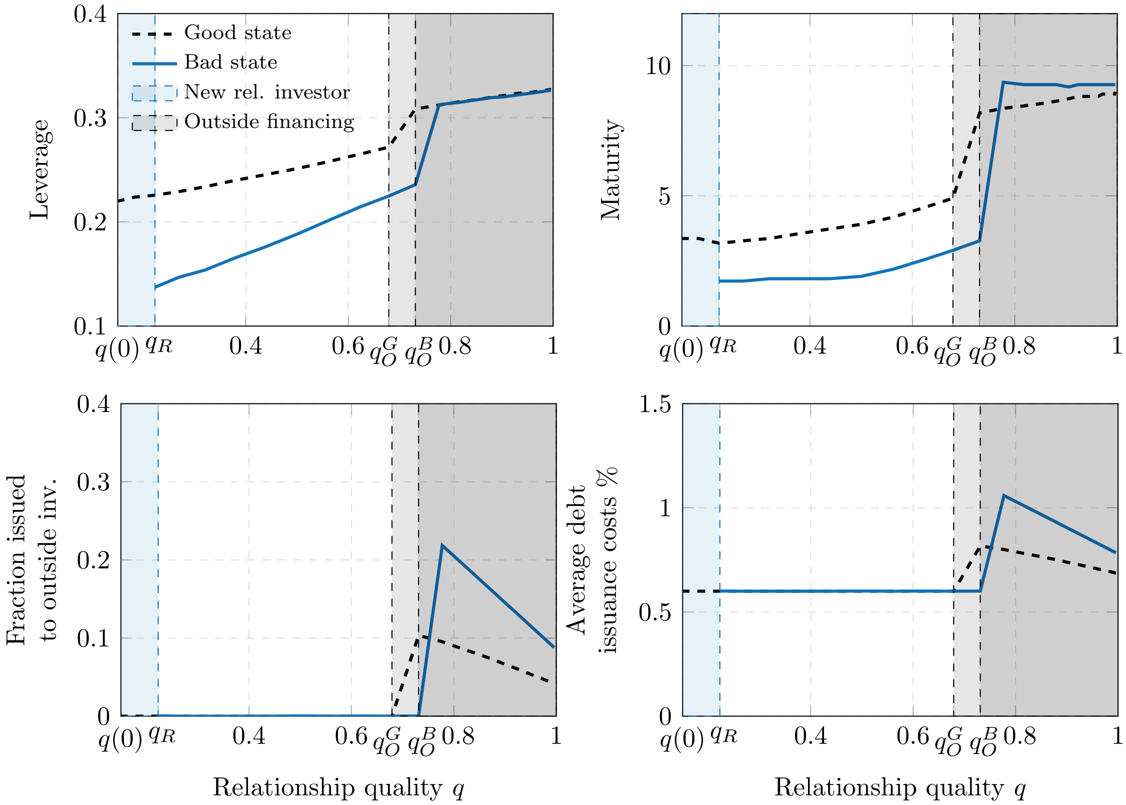

Figure 10: **Effects of idiosyncratic shocks to the relationship lender.** In the blue region, the firm is better off starting a new lending relationship. In the grey region, the firm raises debt from outside investors. Debt purchasing capacity is set to $\beta_G = 0.4$ and $\beta_B = 0.1$ in the good and bad states. Transition intensities are set to $\kappa_G = 0.1$ and $\kappa_B = 0.1$. Other parameters are as in Table 1. We take the average over credit supply β realizations.

debt maturity (as leverage and maturity are quantitatively close for $q = q_0$ and $q = q_R$).

Firms whose lending relationship ($q \in [q_R, q_O^B]$) are of intermediate quality are those that are the most affected by a shock to the relationship investor. These firms are better off maintaining their relationship with their current debt investor. As shown in Figure 10, they abstain from switching to a new relationship investor or issuing debt to outside investors and, as a result, experience a sharp drop in leverage (moving from the dashed black line to the solid blue line). These firms also significantly shorten the maturity of their debt. By doing so, they retain the possibility of refinancing at better terms with their existing relationship investor in case it moves back to a good state.

The figure also shows that when the quality of the lending relationship is sufficiently high (dark gray area in Figure 10), the shock to the relationship investor has much less impact on debt maturity and leverage choices as credit supply from the relationship investor is relatively unaffected. In response to any shortage in debt financing, firms sell to outside investors the debt that relationship investor did not buy. Given the higher cost of outside financing, these firms increase the maturity of their debt in the bad state.

IV Conclusion

In a model of debt dynamics, we study how lending relationships are formed and how they impact leverage and debt maturity choices. In the model, firms build lending relationships through repeated interactions with debt investors. Stronger lending relationships increase firm value by lowering financing costs and by improving access to credit. Financing risk therefore decreases as lending relationships improve, leading firms with stronger relationships to adopt higher leverage ratios.

Lending relationships are also an important driver of the debt maturity choice. We find that firms with weaker relationship quality issue shorter maturity debt, allowing them to refinance debt at better terms sooner if the relationship improves. Our model makes several predictions about optimal debt maturity that are consistent with the data. For instance, we find that the maturity of debt contracts issued to non-relationship investors is higher than that of debt issued to relationship investors.

Our analysis also shows that lending relationships are more valuable for firms that have higher target leverage ratios and thus need to raise more debt from investors. In the model, these are the firms with lower cash flow volatility, lower default costs, and higher tax benefits of debt. Finally, our model predicts that idiosyncratic shocks to debt investors that decrease availability of credit supply differently affect firms, depending on the strength of their lending relationships. We find that capital structures of firms with intermediate-quality lending relationships are affected the most. Overall, our results show that lending relationships potentially have large effects on leverage ratios and debt maturity.

References

Altınkılıç, Oya, and Robert S. Hansen, 2000, Are there economies of scale in underwriting fees? Evidence of rising external financing costs, *Review of Financial Studies* 13, 191–218.

Amiram, Dan, and Daniel Rabetti, 2020, The relevance of relationship lending in times of crisis, *Tel Aviv University Working Paper,* 1–46.

Berg, Tobias, Anthony Saunders, and Sasha Steffen, 2016, The total cost of corporate borrowing in the loan market: Don't ignore the fees, *Journal of Finance* 71, 1357–1392.

Berger, Allen, and Gregory Udell, 1995, Relationship lending and lines of credit in small firm finance, *Journal of Business* 68, 351–381.

Bharath, Sreedhar, Sandeep Dahiya, Anthony Saunders, and Anand Srinivasan, 2011, Lending relationships and loan contract terms, *Review of Financial Studies* 24, 1141–1203.

Boot, Arnoud, and Anjan Thakor, 1994, Moral hazard and secured lending in an infinitely repeated credit market game, *International Economic Review* 35, 899–920.

Chen, Hui, Yu Xu, and Jun Yang, 2021, Systematic risk, debt maturity, and the term structure of credit spreads, *Journal of Financial Economics* 139, 770–799.

Chen, Zhihua, Norman Schuerhoff, and Duane Seppi, 2020, Bondholder granularity, financing frictions, and corporate investment, *University of Lausanne Working Paper,* 1–29.

Cheng, Ing-Haw, and Konstantin Milbradt, 2012, The hazards of debt: Rollover freezes, incentives, and bailouts, *Review of Financial Studies* 25, 1070–1110.

Choi, Jaewon, Dirk Hackbarth, and Josef Zechner, 2018, Corporate debt maturity profiles, *Journal of Financial Economics* 130, 484–502.

Colla, Paolo, Filippo Ippolito, and Kai Li, 2020, Debt structure, *Annual Review of Financial Economics* 12, 193–215.

Coppola, Antonio, 2022, In safe hands: The financial and real impact of investor composition over the credit cycle, *Stanford Business School Working Paper,* 1–110.

Dangl, Thomas, and Josef Zechner, 2021, Debt maturity and the dynamics of leverage, *Review of Financial Studies* 34, 5796–5840.

Danis, Andras, Daniel Rettl, and Toni Whited, 2014, Refinancing, profitability, and capital structure, *Journal of Financial Economics* 114, 424–443.

Della Seta, Marco, Erwan Morellec, and Francesca Zucchi, 2020, Short-term debt and incentives for risk-taking, *Journal of Financial Economics* 137, 179–203.

DeMarzo, Peter M, and Zhiguo He, 2021, Leverage dynamics without commitment, *Journal of Finance* 76, 1195–1250.

Detragiache, Enrica, Paolo Garella, and Luigi Guiso, 2000, Multiple versus single banking relationships: Theory and evidence, *Journal of Finance* 55, 1133–1161.

Di Patti, Emilia Bonaccorsi, and Giorgio Gobbi, 2007, Winners or losers? The effects of banking consolidation on corporate borrowers, *Journal of Finance* 62, 669–695.

Diamond, Douglas, 1991, Monitoring and reputation: The choice between bank loans and directly placed debt, *Journal of Political Economy* 99, 689–721.

Engelberg, Joseph, Pengjie Gao, and Christopher Parsons, 2012, Friends with money, *Journal of Financial Economics* 103, 169–188.

Faulkender, Michael, and Mitchell Petersen, 2006, Does the source of capital affect capital structure?, *Review of Financial Studies* 19, 45–79.

Fernando, Chitru S, Anthony D May, and William L Megginson, 2012, The value of investment banking relationships: Evidence from the collapse of Lehman Brothers, *Journal of Finance* 67, 235–270.

Fischer, Edwin, Robert Heinkel, and Josef Zechner, 1989, Dynamic capital structure choice: Theory and tests, *Journal of Finance* 44, 19–40.

Geelen, Thomas, 2016, Debt maturity and lumpy debt, *Copenhagen Business School Working Paper,* 1–70.

Geelen, Thomas, 2020, Information dynamics and debt maturity, *Copenhagen Business School Working Paper,* 1–51.

Glover, Brent, 2016, The expected cost of default, *Journal of Financial Economics* 119, 284–299.

Goel, Tirupam, and José María Serena, 2020, Bonds and syndicated loans during the Covid-19 crisis: Decoupled again?, *The BIS Bulletin series, N°29* 1–7.

Graham, John, 1999, Do personal taxes affect corporate financing decisions?, *Journal of Public Economics* 73, 147–185.

Graham, John, and Campbell Harvey, 2001, The theory and practice of corporate finance: Evidence from the field, *Journal of Financial Economics* 60, 187–243.

Graham, John R., 2022, Corporate finance and reality, *Journal of Finance* 77, 1975–2049.

Hackbarth, Dirk, Jianjun Miao, and Erwan Morellec, 2006, Capital structure, credit risk, and macroeconomic conditions, *Journal of Financial Economics* 82, 519–550.

Halling, Michael, Jin Yu, and Josef Zechner, 2020, How did COVID-19 affect firms' access to public capital markets?, *Review of Corporate Finance Studies 9, 501–533* .

Harrison, Michael, 2013, *Brownian models of performance and control* (Cambridge University Press).

He, Zhiguo, and Konstantin Milbradt, 2014, Endogenous liquidity and defaultable bonds, *Econometrica* 82, 1443–1508.

He, Zhiguo, and Konstantin Milbradt, 2016, Dynamic debt maturity, *Review of Financial Studies* 29, 2677–2736.

He, Zhiguo, and Wei Xiong, 2012, Rollover risk and credit risk, *Journal of Finance* 67, 391–429.

Herpfer, Christoph, 2021, The role of bankers in the U.S. syndicated loan market, *Journal of Accounting and Economics* 71, 101383.

Huang, Chong, Martin Oehmke, and Hongda Zhong, 2019, A theory of multiperiod debt structure, *Review of Financial Studies* 32, 4447–4500.

Huber, Kilian, 2018, Disentangling the effects of a banking crisis: Evidence from german firms and counties, *American Economic Review* 108, 868–98.

Hugonnier, Julien, Semyon Malamud, and Erwan Morellec, 2015, Credit market frictions and capital structure dynamics, *Journal of Economic Theory* 157, 1130–1158.

Ivashina, Victoria, 2009, Asymmetric information effects on loan spreads, *Journal of Financial Economics* 92, 300–319.

Karceski, Jason, Steven Ongena, and David C Smith, 2005, The impact of bank consolidation on commercial borrower welfare, *Journal of Finance* 60, 2043–2082.

Karolyi, Stephen, 2018, Personal lending relationships, *Journal of Finance* 73, 5–49.

Korteweg, Arthur, 2010, The net benefits to leverage, *Journal of Finance* 65, 2137–2170.

Kubitza, Christian, 2021, Investor-driven corporate finance: Evidence from insurance markets, *European Central Bank, Working Paper* 1–96.

Leary, Mark, 2009, Bank loan supply, lender choice, and corporate capital structure, *Journal of Finance* 64, 1143–1185.

Leland, Hayne, 1998, Agency costs, risk management, and capital structure, *Journal of Finance* 53, 1213–1243.

Leland, Hayne, and Klaus Toft, 1996, Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads, *Journal of Finance* 51, 987–1019.

Lemmon, Michael, and Michael Roberts, 2010, The response of corporate financing and investment to changes in the supply of credit, *Journal of Financial and Quantitative Analysis* 45, 555–587.

Morellec, Erwan, Boris Nikolov, and Norman Schurhoff, 2012, Corporate governance and capital structure dynamics, *Journal of Finance* 67, 803–848.

Morellec, Erwan, and Norman Schürhoff, 2010, Dynamic investment and financing under personal taxation, *Review of Financial Studies* 23, 101–146.

Petersen, Mitchell, and Raghuram Rajan, 1994, The benefits of lending relationships: Evidence from small business data, *Journal of Finance* 49, 3–37.

Siani, Kerry, 2022, Raising bond capital in segmented markets, *Columbia Business School Working Paper,* 1–76.

Strebulaev, Ilya, 2007, Do tests of capital structure theory mean what they say?, *Journal of Finance* 62, 1747–1787.

Van Binsbergen, Jules H, John R Graham, and Jie Yang, 2010, The cost of debt, *Journal of Finance* 65, 2089–2136.

Yasuda, Ayako, 2005, Do bank relationships affect the firm's underwriter choice in the corporate-bond underwriting market?, *Journal of Finance* 60, 1259–1292.

Zhu, Qifei, 2021, Capital supply and corporate bond issuances: Evidence from mutual fund flows, *Journal of Financial Economics* 141, 551–572.

Appendix

This Appendix includes proofs of the results provided in Section I (the baseline model) and Section III (the two-state model). It also describes the numerical algorithm that we use to solve the baseline model.

A Baseline Model

This section consists of four parts. First, we show that the equity and debt value are homogeneous in X and c (Lemma 1). Second, we establish that firm value is finite (Lemma 2). Third, we show existence of the firm value (Proposition 1). Fourth, we prove the optimality of the default strategy (Proposition 2).

In the following, we assume that the firm can always issue debt when outstanding debt matures. In the proofs, we establish the results recursively, i.e. we first assume that the firm can issue debt n more times and we let n go to infinity. Using these recursive arguments simplifies the model solution. We denote by $f_n(q)$ the value of the firm if it can issue debt n more times, with $f_0(q) = \frac{1-\gamma}{r-\mu}$. The results presented in our paper are those for $f(q) = \lim_{n\to\infty} f_n(q)$. We abstain from explicitly writing down this recursive argument when it does not lead to confusion.

Lemma 1. *Assume that shareholders follow a Markovian default strategy in $z = c/x$. Then the equity and debt values satisfy*

$$\begin{aligned}
E(x, c, q, \rho, m) =& xe\left(\frac{c}{x}, q, \frac{\rho}{c}, m\right), \\
e(z, q, \rho, m) =& \sup_{\tau_D} \mathbb{E}_z^{\mathbb{Q}}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\left((1-\gamma)(1-Z_t) + \frac{1}{m}\left(f(q) - \rho Z_t\right)^+\right) dt\right] \qquad (3) \\
D(x, c, q, \rho, m) =& xd\left(\frac{c}{x}, q, \frac{\rho}{c}, m\right), \\
d(z, q, \rho, m) =& \mathbb{E}_z^{\mathbb{Q}}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\left(Z_t + \frac{1}{m}\left(\mathbb{I}_{\{f(q)\geq\rho Z_t\}}\rho Z_t + \mathbb{I}_{\{f(q)<\rho Z_t\}}(1-\alpha)\frac{(1-\gamma)}{r-\mu}\right)\right) dt\right] \\
&+ \mathbb{E}_z^{\mathbb{Q}}\left[e^{-\left(r-\mu+\frac{1}{m}\right)\tau_D}(1-\alpha)\frac{(1-\gamma)}{r-\mu}\right], \qquad (4) \\
F(x, q) =& xf(q),
\end{aligned}$$

where the dynamics of Z_t are given by

$$dZ_t = -\mu Z_t dt - \sigma Z_t dB_t^{\mathbb{Q}}$$

where $B_t^{\mathbb{Q}}$ is a standard Brownian motion under the probability measure $\mathbb{Q}$.

Proof. Observe that the equity value with the debt maturity date integrated out can be written as

$$E(x,c,q,\rho,m) = \sup_{\tau_D} \mathbb{E}_x \left[\int_0^{\tau_D} e^{-\left(r+\frac{1}{m}\right)t} X_t \left((1-\gamma)(1-Z_t) + \frac{1}{m}\left(f(q) - \frac{\rho}{c} Z_t \right)^+ \right) dt \right].$$

Using Girsanov's theorem, we can apply the following change of measure (see Harrison (2013) Theorem 1.17 on page 12)

$$\mathbb{Q}(A) = \mathbb{E}_0 \left[\mathbb{I}_{\{A\}} e^{-\frac{\sigma^2}{2}+\sigma B_t} \right] = \mathbb{E}_0 \left[\mathbb{I}_{\{A\}} e^{-\mu t} \frac{X_t}{X_0} \right] \ \forall A \subseteq \mathcal{F}_t,$$

which yields

$$\begin{aligned} &E(x,c,q,\rho,m) \\ &= \sup_{\tau_D} x \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t} \left((1-\gamma)(1-Z_t) + \frac{1}{m}\left(f(q) - \frac{\rho}{c} Z_t \right)^+ \right) dt \middle| Z_0 = \frac{c}{x} \right] \\ &= x e\left(\frac{c}{x}, q, \frac{\rho}{c}, m \right) \end{aligned}$$

where

$$dZ_t = -\mu Z_t dt - \sigma Z_t dB_t + \sigma^2 Z_t dt = -\mu Z_t dt - \sigma Z_t (dB_t - \sigma dt) = -\mu Z_t dt - \sigma Z_t dB_t^{\mathbb{Q}}.$$

The same change of measure can be applied to the debt value. □

Remark 1: For ease of exposition, we drop $\mathbb{Q}$ from the expectations.

Remark 2: We normalize $c = 1$ in the rest of the proofs without loss of generality.

Lemma 2. *The firm value, if it exists, is finite.*

Proof. The lower bound for the firm value is the unlevered value of assets. In addition, firm value is bounded from above by

$$\begin{aligned} &e(z,q,\rho,m) + d(z,q,\rho,m) \\ &\leq \mathbb{E}_z \left[\int_0^{\infty} e^{-\left(r-\mu+\frac{1}{m}\right)t} \left((1-\gamma) + \gamma Z_t + \frac{1}{m}\left(f(q) + \frac{(1-\alpha)(1-\gamma)}{r-\mu} \right) \right) dt \right] \\ &\leq \frac{(1-\gamma) + \frac{1}{m}\left(\sup_q f(q) + \frac{(1-\alpha)(1-\gamma)}{r-\mu} \right)}{r-\mu+\frac{1}{m}} + \frac{\gamma z}{r+\frac{1}{m}}, \end{aligned} \tag{5}$$

i.e. the firm value is smaller than the present value of all cash flows and tax benefits until maturity plus the payoff at maturity when there is default and when there is no default.

Given Assumption 1 and the fact that $\inf\{\mathcal{M}\} \geq \underline{m} > 0$, we have that[10]

$$\sup_q f(q) \leq \sup_{m \in \mathcal{M}} \frac{(1-\gamma) + \frac{1}{m}\left(\sup_q f(q) + \frac{(1-\alpha)(1-\gamma)}{r-\mu}\right)}{r-\mu+\frac{1}{m}} + \frac{\gamma \bar{z}}{r+\frac{1}{m}},$$

$$\sup_q f(q) \leq \sup_{m \in \mathcal{M}} \frac{(1-\gamma) + \frac{1}{m}\frac{(1-\alpha)(1-\gamma)}{r-\mu} + \frac{\left(r-\mu+\frac{1}{m}\right)\gamma \bar{z}}{r+\frac{1}{m}}}{r-\mu} < \infty.$$

□

Proof of Proposition 1. Let $f_0(q) = \frac{(1-\gamma)}{r-\mu}$ be the unlevered firm value assuming the firm cannot issue debt. Furthermore, let $f_n(q)$ be the unlevered firm value assuming the firm can issue debt n times. Given that we know $f_0(q)$, we can construct any $f_n(q)$ recursively since the face value and maturity of the debt requested, the prices, and default decisions are made sequentially. Furthermore, by construction we have that

$$f_{n+1}(q) \geq f_n(q).$$

Since we know from Lemma 2 that the firm value is finite, the monotone convergence theorem implies that $f(q) = \lim_{n\to\infty} f_n(q)$ exists. □

The final step is deriving the optimal default strategy.

Proof of Proposition 2. Optimality of the default strategy at maturity follows from the fact that it is a static choice. Given the default decision at maturity, equity value can be written as

$$e(z,q,\rho,m|z_D) = \mathbb{E}_z\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\left((1-\gamma)(1-Z_t) + \frac{1}{m}(f(q)-\rho Z_t)^+\right)dt\right]$$

where $\tau_D = \inf\{t>0|Z_t > z_D\}$. The goal is to show that the equity value (where z_D satisfies the smooth pasting condition) exists and that this equity value solves the Hamilton-Jacobi-Bellman equation for our optimal stopping problem.

We need to show that a solution z_D to the following equation exists

$$e_z(z_D,q,\rho,m|z_D) = 0.$$

[10] In the equation below we implicitly assume that $\sup_q f(q)$ is finite but this argument works because we establish our results recursively (i.e. assuming the firm can only issue debt n more times) and $f_0(q) = \frac{1-\gamma}{r-\mu} < \infty$. We just want to show that $f_n(q)$ cannot out grow a bound.

Define $\hat{z}$ as the solution to

$$\left((1-\gamma)(1-\hat{z}) + \frac{1}{m}(f(q) - \rho \hat{z})^+ \right) = 0$$

For $z < z_D < \hat{z}$, we must have that $e(z, q, \rho, m|z_D) > 0$ since the cash flow is always strictly positive. This result directly implies that $e_z(z_D, q, \rho, m|z_D) \leq 0$ for $z_D < \hat{z}$.

Furthermore, as $z_D \to \infty$ we must have that stopping is optimal since stopping is optimal for

$$e(z, q, \rho, m|z_D) \leq \sup_{\tau} \mathbb{E}_z \left[\int_0^{\tau} e^{-\left(r-\mu+\frac{1}{m}\right)t} \left((1-\gamma)(1-Z_t) + \frac{1}{m} f(q) \right) dt \right],$$

which is a standard optimal stopping problem, which has a threshold solution (see Harrison (2013) Chapter 5). Let $\hat{z}_D$ be the threshold solution to this auxiliary problem. Then for $z_D > \hat{z}_D$ we must have that for z close enough to z_D

$$e(z, q, \rho, m|z_D) < 0$$

and therefore $e_z(z_D, q, \rho, m|z_D) \geq 0$. Continuity in $e_z(z_D, q, \rho, m|z_D)$ with respect to z_D (see Lemma A.6 in Hugonnier et al. (2015)) then implies that a solution to

$$e_z(z_D, q, \rho, m|z_D) = 0$$

exists.

The next step is to show that this equity value satisfies the Hamilton-Jacobi-Bellman equation. For $z > z_D$, we have that $e(z|z_D) = 0$. For $z < z_D$ it solves the Feynman-Kac ordinary differential equation. Furthermore, at z_D (approaching it from the right) we have that

$$0 = (1-\gamma)(1-z_D) + \frac{1}{m}\left(f(q) - \rho z_D\right)^+ + \frac{1}{2}\sigma^2 z_D^2 e_{zz}(z_D, q, \rho, m|z_D)$$

Assume $e_{zz}(z_D, q, \rho, m|z_D) < 0$, then $(1-\gamma)(1-z_D) + \frac{1}{m}\left(f(q) - \rho z_D\right)^+ > 0$. This would imply that $e(z, q, \rho, m|z_D) > 0$ for $z < z_D$ since the cash flow is always positive. This result contradicts the fact that $e_z(z, q, \rho, m|z_D) > 0$ in some left neighborhood of z_D (since $e_{zz}(z_D, q, \rho, m|z_D) < 0$ and $e_z(z_D, q, \rho, m|z_D) = 0$) and $e(z_D, q, \rho, m|z_D) = 0$. Therefore, we must have that $e_{zz}(z_D, q, \rho, m|z_D) \geq 0$ and, as a result, $(1-\gamma)(1-z_D) + \frac{1}{m}\left(f(q) - \rho z_D\right)^+ \leq 0$. Since the cash flow is decreasing in z, this proves that the equity value satisfies the Hamilton-Jacobi-Bellman equation for $z \geq z_D$.

In some left neighborhood of z_D it must be that $e(z|z_D) > 0$. Assume $e_{zz}(z_D, q, \rho, m|z_D) = 0$. Then $(1-\gamma)(1-z_D) + \frac{1}{m}\left(f(q) - \rho z_D\right)^+ = 0$ and, therefore, the cash flow is positive for any $z < z_D$ and the equity value is always positive. Assume $e_{zz}(z_D, q, \rho, m|z_D) > 0$, then $e_z(z|z_D) < 0$ in some left neighborhood of z_D and, therefore, $e(z, q, \rho, m|z_D) > 0$ in this neighborhood.

For $z \leq z_D$, we only need to show that $e(z|z_D) \geq 0$. Assume this is not the case. Then there exists a local minimum $\tilde{z} \in (0, z_D)$ such that[11]

$$
\begin{aligned}
e(\tilde{z}, q, \rho, m|z_D) &< 0, \\
e_z(\tilde{z}, q, \rho, m|z_D) &= 0, \\
e_{zz}(\tilde{z}, q, \rho, m|z_D) &\geq 0, \\
(1-\gamma)(1-\tilde{z}) + \frac{1}{m}\left(f(q) - \rho\tilde{z}\right)^+ &\geq 0.
\end{aligned}
$$

where the last inequality follows from the fact that for some $z \in [\tilde{z}, z_D]$ the equity value is positive and thus the cash flow must be positive for some $z \in [\tilde{z}, z_D]$ and, as a consequence, also at $\tilde{z}$. But these inequalities lead to a contradiction

$$
\begin{aligned}
0 &> \left(r - \mu + \frac{1}{m}\right) e(\tilde{z}, q, \rho, m|z_D) \\
&= (1-\gamma)(1-\tilde{z}) + \frac{1}{m}\left(f(q) - \rho\tilde{z}\right)^+ - \mu\tilde{z}e_z(\tilde{z}, q, \rho, m|z_D) + \frac{1}{2}\sigma^2\tilde{z}^2 e_{zz}(\tilde{z}, q, \rho, m|z_D) \\
&\geq 0.
\end{aligned}
$$

Therefore, the equity value must be non-negative for $z \leq z_D$. This proves that the equity value satisfies the Hamilton-Jacobi-Bellman equation for $z \leq z_D$

Finally, the equity value function is piecewise C^2. Therefore, using Theorem 5.1 of Harrison (2013), we conclude that the optimal default strategy is a threshold default strategy where the threshold follows from the smooth pasting condition.

□

B Two-State Model

This section contains proofs of propositions provided in Section III and is organized as follows. First, we show that the equity and debt value are homogeneous in X and c (Lemma 3). Second, we show existence of the firm value (Proposition 3). Third, we show optimality of the default strategy (Proposition 4).

Lets denote by $S_t \in \{G, B\}$ the state the firm's relationship investor is in at time t. We can establish that

[11] Observe that $\lim_{z \to 0} e(z|z_D) > 0$, which follows from the ordinary differential equation the equity value satisfies.

Lemma 3. *Assume a Markovian default strategy in $z = c/x$ and s is used then the equity and debt values satisfy*

$$
\begin{aligned}
E(x,c,q,\rho,m,s) =& xe\left(\frac{c}{x}, q, \frac{\rho}{c}, m, s\right),\\
e(z,q,\rho,m,s) =& \sup_{\tau_D}\Bigg\{\mathbb{E}^{\mathbb{Q}}_{z,s}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}(1-\gamma)(1-Z_t)dt\right]\\
&+\mathbb{E}^{\mathbb{Q}}_{z,s}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\frac{1}{m}\left(\max\{f(q(0),G), f(q,S_t)\}-\rho Z_t\right)^+ dt\right]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
D(x,c,q,\rho,m,s) =& xd\left(\frac{c}{x}, q, \frac{\rho}{c}, m, s\right),\\
d(z,q,\rho,m,s) =& \mathbb{E}^{\mathbb{Q}}_{z,s}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\left(Z_t + \frac{1}{m}\mathbb{I}_{\{\max\{f(q(0),G),f(q,S_t)\}\geq \rho Z_t\}}\rho Z_t\right)dt\right]\\
&+\mathbb{E}^{\mathbb{Q}}_{z,s}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\frac{1}{m}\mathbb{I}_{\{\max\{f(q(0),G),f(q,S_t)\}<\rho Z_t\}}(1-\alpha)\frac{(1-\gamma)}{r-\mu}dt\right]\\
&+\mathbb{E}^{\mathbb{Q}}_{z,s}\left[e^{-\left(r-\mu+\frac{1}{m}\right)\tau_D}(1-\alpha)\frac{(1-\gamma)}{r-\mu}\right],\\
F(x,q,s) =& xf(q,s),
\end{aligned}
$$

where the dynamics of Z_t are given by

$$dZ_t = -\mu Z_t dt - \sigma Z_t dB^{\mathbb{Q}}_t$$

where $B^{\mathbb{Q}}_t$ is a standard Brownian motion under the probability measure $\mathbb{Q}$.

Proof. The proof is the same as for the baseline model (see the proof of Lemma 1). □

Proposition 3 (Firm value)**.** *Firm value exists, is finite, and satisfies $F(x,q,s) = xf(q,s)$.*

Proof of Proposition 3. The same arguments as in the proof of Lemma 2 imply that $f(q,s)$ is bounded from above and below. In equation (5) we now take the supremum over both q and s.

Let $f_0(q,s) = \frac{(1-\gamma)}{r-\mu}$ be the unlevered firm value assuming it can no longer issue debt. Furthermore, let $f_n(q,s)$ be the unlevered firm value assuming the firm can issue debt n times and the firm's relationship investor is in state s. Given that we know $f_0(q,s)$, we can construct any $f_n(q,s)$ recursively since the face value and maturity of the debt requested, the prices, and default decisions are made sequentially. Furthermore, by construction

$$f_{n+1}(q,s) \geq f_n(q,s).$$

Since we know that the firm value is finite, the monotone convergence theorem then tells us that $f(q,s) = \lim_{n\to\infty} f_n(q,s)$ exists. $\square$

Proposition 4 (Optimal default)**.** *The optimal strategy for shareholders is to default:*

1. ***Coupon default**: Before maturity if the ratio of the coupon payment to the firm cash flow* $z = \frac{c}{x}$ *rises above an endogenous threshold* $z_D(q, \frac{\rho}{c}, m, s)$*, that is determined by the equity value's smooth pasting conditions.*
2. ***Principal default**: On the maturity date of the debt contract if the debt principal* ρ *exceeds the continuation value of equity* $F(x,q,s)$*.*

Proof of Proposition 4. Optimality of the default strategy at maturity follows from the fact that it's a static choice.

For the coupon default strategy, we establish optimality recursively. Furthermore, we will integrate out the switching of the relationship investor's state. Fixing $f(q,s)$, ρ, and m and normalizing $c=1$, we define

$$\begin{aligned}
\tau_D^{si} =& \inf\left\{t > 0 | Z_t \geq z_D^{si}\right\}, \\
e_{s0}(z|0) =& 0, \\
e_{si}\left(z|z_D^{si}\right) =& \mathbb{E}_z\left[\int_0^{\tau_D^{si}} e^{-\left(r-\mu+\frac{1}{m}+\kappa_s\right)t}\left((1-\gamma)(1-Z_t) + \frac{1}{m}\left(\max\{f(q(0),G), f(q,s)\} - \rho Z_t\right)^+\right) dt\right] \\
&+ \mathbb{E}_z\left[\int_0^{\tau_D^{si}} e^{-\left(r-\mu+\frac{1}{m}+\kappa_s\right)t}\kappa_s e_{s'(i-1)}\left(Z_t\middle|z_D^{s'(i-1)}\right) dt\right]
\end{aligned}$$

where s' is the opposite state of s. Observe that by construction $e_{B0}(z|0) = 0$ and therefore $e'_{B0}(z|0) \leq 0$.

Take an uneven i and assume that $e_{B(i-1)}\left(z\middle|z_D^{B(i-1)}\right) \geq 0$ and $e'_{B(i-1)}\left(z\middle|z_D^{B(i-1)}\right) \leq 0$. We first want to establish that there exists a threshold z_D^{Gi} such that

$$e'_{Gi}\left(z_D^{Gi}\middle|z_D^{Gi}\right) = 0.$$

Define $\hat{z}$ as the solution to

$$\left((1-\gamma)(1-\hat{z}) + \frac{1}{m}\left(\max\{f(q(0),G), f(q,s)\} - \rho\hat{z}\right)^+\right) = 0$$

For $z < z_D < \hat{z}$, we must have that $e_{Gi}(z|z_D) > 0$ since the cash flow is always strictly positive. This result directly implies that $e'_{Gi}(z_D|z_D) \leq 0$ for $z_D < \hat{z}$.

Furthermore, as $z_D \to \infty$ we must have that stopping is optimal since stopping is optimal for

$$
\begin{aligned}
&e_{Gi}(z|z_D) \\
&\leq \sup_{\tau} \Bigg\{ \mathbb{E}_z \left[\int_0^{\tau} e^{-\left(r-\mu+\frac{1}{m}+\kappa_G\right)t} \left((1-\gamma)\left(1-Z_t\right) + \frac{1}{m} \max\{f(q(0),G), f(q,G)\} \right) dt \right] \\
&\qquad + \mathbb{E}_z \left[\int_0^{\tau} e^{-\left(r-\mu+\frac{1}{m}+\kappa_G\right)t} \kappa_G e_{B(i-1)} \left(0 \middle| z_D^{B(i-1)}\right) dt \right] \Bigg\} \qquad (6)
\end{aligned}
$$

which is a standard optimal stopping problem, which has a threshold solution (see Harrison (2013) Chapter 5). Let $\hat{z}_D$ be the threshold solution to this auxiliary problem then for $z_D > \hat{z}_D$ we must have that for z close enough to z_D

$$
e_{Gi}(z|z_D) < 0
$$

and therefore $e'_{Gi}(z_D|z_D) \geq 0$. Continuity in $e'_{Gi}(z_D|z_D)$ with respect to z_D (see Lemma A.6 in Hugonnier et al. (2015)) then implies that a solution to

$$
e'_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) = 0
$$

exists.

The next step is showing that this equity value satisfies the Hamilton-Jacobi-Bellman equation. For $z > z_D^{Gi}$, we have that $e_{Gi}\left(z \middle| z_D^{Gi}\right) = 0$ while for $z < z_D^{Gi}$ it solves the Feynman-Kac ordinary differential equation. Furthermore, at z_D^{Gi} (approaching it from the right) we have that

$$
\begin{aligned}
0 =& (1-\gamma)\left(1 - z_D^{Gi}\right) + \frac{1}{m}\left(\max\{f(q(0),G), f(q,G)\} - \rho z_D^{Gi}\right)^+ + \kappa_G e_{B(i-1)}\left(z \middle| z_D^{B(i-1)}\right) \\
&+ \frac{1}{2}\sigma^2 \left(z_D^{Gi}\right)^2 e''_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right)
\end{aligned}
$$

Assume $e''_{Gi}\left(z_D^{Gi}|z_D^{Gi}\right) < 0$, then $(1-\gamma)\left(1 - z_D^{Gi}\right) + \frac{1}{m}\left(\max\{f(q(0),G), f(q,G)\} - \rho z_D^{Gi}\right)^+ + \kappa_G e_{B(i-1)}\left(z_D^{Gi} \middle| z_D^{B(i-1)}\right) > 0$. Since the cash flow is always positive, this would imply that $e_{Gi}\left(z \middle| z_D^{Gi}\right) > 0$ for $z < z_D^{Gi}$. This result contradicts the fact that $e'_{Gi}\left(z \middle| z_D^{Gi}\right) > 0$ in some left neighborhood of z_D^{Gi} (since $e''_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) < 0$ and $e'_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) = 0$) and $e_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) = 0$. Therefore, we must have that $e''_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) \geq 0$ and as a result $(1-\gamma)\left(1 - z_D^{Gi}\right) +$

$\frac{1}{m}\left(\max\{f(q(0),G),f(q,G)\}-\rho z_D^{Gi}\right)^+ + \kappa_G e_{B(i-1)}\left(z_D^{Gi}\middle|z_D^{B(i-1)}\right) \leq 0$. Since the cash flow is decreasing in z, this proves that the equity value satisfies the Hamilton-Jacobi-Bellman equation for $z \geq z_D^{Gi}$.

In some left neighborhood of z_D^{Gi} it must be that $e_{Gi}\left(z\middle|z_D^{Gi}\right) > 0$. Assume $e''_{Gi}\left(z_D^{Gi}\middle|z_D^{Gi}\right) = 0$ then $(1-\gamma)\left(1-z_D^{Gi}\right)+\frac{1}{m}\left(\max\{f(q(0),G),f(q,G)\}-\rho z_D^{Gi}\right)^+ + \kappa_G e_{B(i-1)}\left(z_D^{Gi}\middle|z_D^{B(i-1)}\right) = 0$ and therefore the cash flow is positive for any $z < z_D^{Gi}$ and the equity value is always positive. Assume $e''_{Gi}\left(z_D^{Gi}\middle|z_D^{Gi}\right) > 0$ then $e'_{Gi}\left(z\middle|z_D^{Gi}\right) < 0$ in some left neighborhood of z_D^{Gi} and therefore $e\left(z_D^{Gi}\middle|z_D^{Gi}\right) > 0$ in this same neighborhood.

For $z \leq z_D^{Gi}$, we only need to show that $e_{Gi}\left(z\middle|z_D^{Gi}\right) \geq 0$. We will more generally show that $e'_{Gi}\left(z\middle|z_D^{Gi}\right) \leq 0$. For $z \geq z_D^{Gi}$ this result trivially holds. Assume this is not the case for some $z \in \left[0, z_D^{Gi}\right]$. First, it must be the case that

$$\begin{aligned}
&e_{Gi}\left(0\middle|z_D^{Gi}\right)\\
&= \frac{(1-\gamma)+\frac{1}{m}\max\{f(q(0),G),f(q,G)\}+\kappa_G e_{B(i-1)}\left(0\middle|z_D^{B(i-1)}\right)}{r-\mu+\frac{1}{m}+\kappa_G}\\
&= \int_0^\infty e^{-\left(r-\mu+\frac{1}{m}+\kappa_G\right)t}\left((1-\gamma)+\frac{1}{m}\max\{f(q(0),G),f(q,G)\}+\kappa_G e_{B(i-1)}\left(0\middle|z_D^{B(i-1)}\right)\right)dt\\
&\geq e_{Gi}\left(z\middle|z_D^{Gi}\right).
\end{aligned}$$

Therefore, there must exist a local minimum $z_1 > 0$ and local maximum $z_2 (> z_1)$ such that

$$\begin{aligned}
&e_{Gi}\left(z_1\middle|z_D^{Gi}\right) < e_{Gi}\left(z_2\middle|z_D^{Gi}\right),\\
&e'_{Gi}\left(z_1\middle|z_D^{Gi}\right) = e'_{Gi}\left(z_2\middle|z_D^{Gi}\right) = 0,\\
&e''_{Gi}\left(z_1\middle|z_D^{Gi}\right) \geq 0 \geq e'_{Gi}\left(z_2|z_D^{Gi}\right),\\
&CF(z) = (1-\gamma)(1-z)+\frac{1}{m}\left(\max\{f(q(0),G),f(q,G)\}-\rho z\right)^+ + \kappa_G e_{B(i-1)}\left(z\middle|z_D^{B(i-1)}\right)\\
&CF(z_1) > CF(z_2).
\end{aligned}$$

But these inequalities lead to a contradiction,

$$\begin{aligned}
0 &> \left(r-\mu+\frac{1}{m}+\kappa_G\right)\left(e_{Gi}\left(z_1|z_D^{Gi}\right)-e_{Gi}\left(z_2|z_D^{Gi}\right)\right)\\
&= CF(z_1)-CF(z_2)+\frac{1}{2}\sigma^2 z_1^2 e''_{Gi}\left(z_1|z_D^{Gi}\right)-\frac{1}{2}\sigma^2 z_2^2 e''_{Gi}\left(z_2|z_D^{Gi}\right) > 0,
\end{aligned}$$

and as a result $e'_{Gi}\left(z\middle|z_D^{Gi}\right) \leq 0$. Therefore, the equity value must be non-negative for $z \leq z_D^{Gi}$. This proves that the equity value satisfies the Hamilton-Jacobi-Bellman equation for $z \leq z_D^{Gi}$

Finally, the equity value function is piecewise C^2. Using Theorem 5.1 (Harrison (2013)), we conclude that the optimal default strategy is a threshold default strategy where the threshold follows from the smooth pasting condition.

The same arguments as for the case $s = G$ and i allow us to establish the existence of a threshold $z_D^{B(i+1)}$ that solves $e'_{B(i+1)}\left(z_D^{B(i+1)}\middle|z_D^{B(i+1)}\right) = 0$ and which is the optimal stopping threshold with $e_{B(i+1)}\left(z\middle|z_D^{B(i+1)}\right) \geq 0$ and $e'_{B(i+1)}\left(z\middle|z_D^{B(i+1)}\right) \leq 0$. We can then iteratively obtain the default thresholds for all values i (with $s = G$ when i is uneven and $s = B$ when i is even).

There exists an upper bound on the default threshold (using equation (6) and the fact that $e_{si}\left(0\middle|z_D^{si}\right) \leq \sup_{q,s} f(q,s) < \infty$) $z_D^{si} \leq \bar{z}_D < \infty$). Furthermore, $e_{si}\left(z\middle|z_D^{si}\right)$ follows from the monotone mapping

$$\mathcal{T}_s(e) =$$

$$\sup_{\tau_D^s}\left\{\mathbb{E}_z\left[\int_0^{\tau_D^s} e^{-\left(r-\mu+\frac{1}{m}+\kappa_s\right)t}\left((1-\gamma)(1-Z_t)+\frac{1}{m}\left(\max\{f(q(0),G), f(q,s)\} - \rho Z_t\right)^+\right)dt\right]\right.$$

$$\left. + \mathbb{E}_z\left[\int_0^{\tau_D^s} e^{-\left(r-\mu+\frac{1}{m}+\kappa_s\right)t}\kappa_s e(Z_t)dt\right]\right\}.$$

As a result, $e_{B2}\left(z\middle|z_D^{B2}\right) \geq e_{B0}\left(z\middle|z_D^{B0}\right)$ and therefore $e_{G3}\left(z\middle|z_D^{G3}\right) \geq e_{G1}\left(z\middle|z_D^{G1}\right)$. This implies that the sequence $(z_D^{G(2i-1)}, z_D^{B(2i)})$ is increasing in i since $e_{G(2i-1)}\left(z\middle|z_D^{G(2i-1)}\right)$ and $e_{B(2i)}\left(z\middle|z_D^{B(2i)}\right)$ are increasing in i. The monotone convergence theorem then implies that $\lim_{i\to\infty}\left(z_D^{G(2i-1)}, z_D^{B(2i)}\right)$ converges. Call this limit (z_D^G, z_D^B). We know that both these default thresholds satisfy the smooth pasting condition (simultaneously) and that both stopping times are optimal (since each stopping time in the sequence is optimal given the stopping time in the other state). □

C Numerical Implementation Baseline Model

This appendix describes the numerical algorithm used to calculate the firm value at issuance $f(q)$ in the baseline model. The firm's capital structure choices are a byproduct of this calculation.

First, note that the firm value at issuance $f(q)$ is a fixed point of the mapping $\mathcal{I}$ defined as:

$$\mathcal{I}(q,f) = \sup_{(\hat{\rho},m)\in\mathbb{R}_+\times\mathcal{M}} \mathbb{E}_q\left[\sup_{\rho\in[\beta\hat{\rho},\hat{\rho}]}\{e(z,\rho,m|f(q')) + (1-\psi_R)\beta\hat{\rho} + (1-\psi_O)(\rho-\beta\hat{\rho})\}\right]$$

such that

$$z = \left\{z' \middle| d(z',\rho,m|f(q')) = \rho\right\}$$

where e and d follow from equation (3) and equation (4) and where q' is the relationship quality assuming the firm issued debt another time with the relationship investor. As the relationship quality after issuance q' affects e and d only through the firm value at (the next debt) issuance $f(q')$, we can drop q' from e and d and rewrite these functions conditional on $f(q')$. The mapping $\mathcal{I}(q,f)$ calculates the firm value at issuance assuming that when the debt matures, the firm value at issuance is given by $f(q)$.

We numerically find the fixed point of the mapping $\mathcal{I}(q,f)$ in two steps. First, for a range of values of $\tilde{f}$ we calculate

$$g(\beta,\hat{\rho},m,\tilde{f}) = \sup_{\rho\in[\beta\hat{\rho},\hat{\rho}]}\left\{e(z,\rho,m|f(q')=\tilde{f}) + (1-\psi_R)\beta\hat{\rho} + (1-\psi_O)(\rho-\beta\hat{\rho})\right\}$$

such that

$$z = \left\{z' \middle| d(z',\rho,m|f(q')=\tilde{f}) = \rho\right\}.$$

The function $g(\beta,\hat{\rho},m,\tilde{f})$ is calculated in the following way. Given $\tilde{f}$, z, ρ, m, and the default threshold z_D, we calculate the equity and debt value in closed-form using the Feynman-Kac formula and the appropriate boundary conditions (Harrison, 2013). The optimal default threshold z_D is found using the smooth pasting condition (Proposition 2) and z is found using the condition that debt is issued at par. Finally, ρ can be found using a grid search.

Algorithm 1: Calculate $g(\beta, \hat{\rho}, m, \tilde{f})$

```
// Initialize
```
set grid $\mathcal{P} = \left(\rho_1, ..., \rho_{n_\rho}\right)$

set grid $\mathcal{M} = (m_1, ..., m_{n_m})$

```
// Only calculate firm value at issuance if the principal is in the
   feasible set
```
$\mathcal{P} = \mathcal{P} \bigcap [\beta\hat{\rho}, \hat{\rho}]$

// Loop over debt principal ρ

for $\rho \in \mathcal{P}$ **do**

> // Loop over maturity m
>
> **for** $m \in \mathcal{M}$ **do**
>
> > ```
> > // Solve default threshold
> > ```
> > Find z_D that solves $e_z(z_D, \rho, m|\tilde{f}, z_D) = 0$
> >
> > ```
> > // Find the coupon such that debt is issued at par
> > ```
> > Find $\tilde{z}$ that solves $d(\tilde{z}, \rho, m|\tilde{f}, z_D) = \rho$
> >
> > ```
> > // Save firm value function after debt issuance
> > ```
> > $V(\rho) = e(\tilde{z}, \rho, m|\tilde{f}, z_D) + (1 - \psi_R)\beta\hat{\rho} + (1 - \psi_O)(\rho - \beta\hat{\rho})$
>
> **end**

end

```
// Find and return optimal firm value at issuance
```
$g(\beta, \hat{\rho}, m, \tilde{f}) = \sup_{\rho \in \mathcal{P}} V(\rho)$

return $g(\beta, \hat{\rho}, m, \tilde{f})$

Second, using the function $g(\beta, \hat{\rho}, m, \tilde{f})$, we find a fixed point of the mapping $\mathcal{I}(q, f)$. Note that $\mathcal{I}(q, f)$ can be written as

$$\mathcal{I}(q, f) = \sup_{(\hat{\rho}, m) \in \mathbb{R}_+ \times \mathcal{M}} \mathbb{E}_q \left[g\left(\beta, \hat{\rho}, m, f(q')\right)\right].$$

Given the function $g(\beta, \hat{\rho}, m, \tilde{f})$, we iteratively calculate $f_{n+1} = \mathcal{I}(q, f_n)$. The proof of Proposition 2 shows that convergence of this algorithm is guaranteed if we start from

$f_0(q) = \frac{(1-\gamma)}{r-\mu}$.[12]

Algorithm 2: Calculate $f(q)$

```
// Initialize
load g(·,·,·,·) and f_0(·)
set grid Q = (q_1, ...., q_{n_q})
set i = 0

// Value function iteration
while error > error_bound do
    i = i + 1

    // Loop over q
    for j ∈ {1, ..., n_q} do
        f_i(q_j) = sup_{(ρ̂,m)∈ℝ_+×M} E_{q_j} [g(β, ρ̂, m, f_{i-1}(q'))]
    end

    // Calculate error
    error = Σ_{q∈Q} (f_i(q) − f_{i-1}(q))^2

    // Interpolate
    Interpolate f_i(q) from f_i(Q)
end

// Return results
return f_i(q)
```

Where: $f_i(q_j) = \sup_{(\hat{\rho},m)\in\mathbb{R}_+\times\mathcal{M}} \mathbb{E}_{q_j}\left[g\left(\beta,\hat{\rho},m,f_{i-1}(q')\right)\right]$, $error = \sum_{q\in\mathcal{Q}}\left(f_i(q)-f_{i-1}(q)\right)^2$, $\mathcal{Q} = (q_1, ..., q_{n_q})$.

[12]The solution to the model with shocks to the relationship investor is calculated in a similar way where we need to solve a system of differential equations to calculate the equity and debt value (Hackbarth et al., 2006; Geelen, 2016; Chen et al., 2021) instead of a differential equation.