Relationship Capital and Financing Decisions*

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Abstract

Lending relationships matter for firm financing. In a model of debt dynamics, we study how lending relationships are formed and how they impact leverage and debt maturity choices. In the model, lending relationships evolve through repeated interactions between firms and debt investors. Stronger lending relationships lead firms to adopt higher leverage ratios, issue longer term debt, and raise funds from non-relationship lenders when relationship quality is sufficiently high. The maturity of debt contracts issued to non-relationship investors is higher than that of relationship investors. Negative shocks to relationship lenders drastically affect the financing choices of firms with intermediate relationship quality.

Keywords: relationship lending, capital structure, debt maturity, default.
JEL Classification: G20, G32, G33.

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Over the past 20 years, outstanding U.S. corporate debt has nearly tripled from $2.5 trillion in 2000 to $7.2 trillion in 2020. Corporate debt is often closely held by banks or large institutional investors. Detragiache, Garella, and Guiso (2000) document for instance that 44.5% of firms have a relationship with only one bank and that the median number of relationships is two. Chen, Schuerhoff, and Seppi (2020) find that on average more than half of a new bond issuance gets placed with issuers' incumbent institutional bondholders. Strong relationships with debt investors improve financing terms and affect financing choices. For example, Bharath, Dahiya, Saunders, and Srinivasan (2011) and Karolyi (2018) show that firms with existing banking relationships are able to obtain larger loans at lower interest rates (see also Engelberg, Gao, and Parsons (2012) or Herpfer (2021)). Zhu (2021) finds that bond mutual funds that hold a firm's existing bonds are five times more likely to provide capital in future bond issues and do so at lower yields.[1]

Even though there is mounting evidence that issuer-investor (or lending) relationships matter for firm financing, most existing capital structure theories implicitly assume that credit supply is perfectly elastic, so that financing decisions depend solely on firm characteristics. That is, although the Modigliani and Miller irrelevance does not hold on the demand side of the market in these models, it is assumed to hold on the supply side. Our objective in this paper is to relax this assumption and examine how issuer-investor relationships are built over time and shape firms' leverage and debt maturity choices.

To do so, we build on the classic models of Fischer, Heinkel, and Zechner (1989) and Leland (1994) and consider a firm with assets that generate a continuous cash flow stream. The firm pays taxes on corporate income and, thus, has incentives to issue debt. Debt financing however increases the likelihood of costly financial distress and is subject to financing frictions, the severity of which depends on the quality of the firm's lending relationships.

[1]Recent evidence from the COVID-19 crisis also highlights the importance of lending relationships in times of market stress. Halling, Yu, and Zechner (2020) and Goel and Serena (2020) show that due to well-established relationships with underwriters, borrowers with more bond issuance experience before the pandemic were able to issue bonds at lower spreads during the crisis. Relationship borrowers also received larger loans and faster approvals (Amiram and Rabetti (2020)).

Notably, a debt investor's willingness to invest in the firm's debt depends on the quality of its relationship with the firm while the cost of issuing debt depends on the (endogenous) composition of the pool of debt investors, in line with the evidence in Yasuda (2005) and Chen, Schuerhoff, and Seppi (2020).[2] Our analysis encompasses both bank loans and bond issues. The "relationship investor" is the firm's bank in case of a loan issue and existing bondholders in case of a bond issue. "Outside investors" are the other banks (respectively bond investors) involved in a syndicated loan (respectively in a bond issue) if the firm intends to issue more debt than the "relationship investor" is willing (or able) to purchase.

In the model, management acts in the best interests of shareholders and maximizes shareholder value by selecting the amount of debt to issue with relationship and non-relationship investors, the maturity of corporate debt, and the firm's default policy. The firm repeatedly interacts with debt investors who differ in their ability to supply credit (or in their appetite for the firm's debt). The quality of the relationship between the firm and the debt investor is unknown ex ante, but the firm learns over time about the investor's type from the history of debt issuance (i.e. from past purchases) and can always opt to start a relationship with a new debt investor. Notably, if the creditor purchases all the firm's new debt, the firm positively updates its beliefs about the relationship investors ability or willingness to supply capital. If the creditor is unable or unwilling to purchase all the firm's debt, the firm negatively updates its beliefs. If the relationship deteriorates sufficiently, it is terminated and the debt investor gets replaced.[3]

How do lending relationships affect debt dynamics and maturity choices? Because stronger lending relationships lead to an increase in the creditor's willingness to invest in the firm's debt (or to lend) and to a decrease in debt issuance costs, they lead to an increase

[2] Graham and Harvey (2001) highlight the central role played by transaction costs and fees in the decision to issue debt. Their survey of corporate CEOs shows that transaction costs and fees come just after interest tax savings—and much before bankruptcy costs or personal taxes—as a determinant of capital structure.

[3] In practice, debt investors differ in their ability to purchase the new issuance due to their asset-holding capacity or investment preferences. That is, the quality of the lending relationship may be unknown ex ante because the investor's availability to supply credit is related to its capital in- and outflows and to the match between its risk strategy and the risk of the firm's debt at the time of issuance.

in the value of refinancing options and in firm value, thereby lowering default risk for any given debt level. The decrease in issuance costs and in default risk leads to an increase in optimal leverage and pushes the firm to issue more debt as lending relationships improve (i.e. leverage increases as credit supply uncertainty decreases, in line with Massa, Yasuda, and Zhang (2013)). A striking result of the model is that stronger lending relationships allow the firm not only to issue more debt from relationship lenders but also to raise additional debt from outside investors. In effect, by decreasing default risk and expected default costs, stronger lending relationships make it optimal for the firm to also raise debt with outside investors at a higher issuance cost. As a result, we find that optimal leverage increases from 21% to 31% as the relationship between the firm and its creditors improves.

In the model, the firm chooses not only the size of debt issues but also their maturity. When the quality of the lending relationship is low, the relationship investor is willing (or able) to purchase lower amounts of debt. This leads the firm to issue shorter maturity debt, allowing it to refinance and adjust leverage sooner, possibly at better terms. As the relationship quality improves, the relationship investor purchases larger amounts of debt, default risk decreases (for any given amount of debt), and the firm issues longer maturity debt. This decrease in the cost of debt makes debt issuance to outside investors more attractive. When the relationship becomes sufficiently strong, the firm issues debt to both relationship and non-relationship investors and optimal debt maturity jumps upwards reflecting the higher costs of issuing debt with non-relationship investors. As the relationship keeps on improving, the firm issues a larger portion of its debt with the relationship investor leading to a decrease in issuance costs and to a shortening of debt maturity. The model therefore predicts that the maturity of debt contracts issued to non-relationship investors is higher than that of relationship investors, in line with the evidence in Bharath et al. (2011). It also predicts that average debt maturity decreases with the share of the debt held by relationship investors. In our base case environment, optimal debt maturity varies between 4 years and 8 years depending on the quality of the lending relationship.

Our analysis also illustrates how the value of lending relationships varies with firm characteristics. We find that stronger relationships are more valuable for firms with lower cash flow volatility, lower default costs, and for firms facing a higher corporate tax rate. The reason is that these firms have higher target leverage ratios and therefore request more debt financing from investors. Since stronger lending relationships lower the cost of debt, they add more value to these firms.

We also find that the wedge between the costs of debt issuance with external and internal investors is an important driver of leverage, debt composition, and debt maturity choice. This wedge can be related for instance to the cost of attracting loan participants and structuring and originating a syndicated loan (Berg, Saunders, and Steffen (2016)) or to the severity of search frictions in the bond market (Chen, Schuerhoff, and Seppi (2020)). When the wedge is large, debt issuance with non-relationship investors is relatively more costly, and firms issue debt mainly to the relationship investor, maintain a lower leverage ratio, and issue shorter term debt. In this case, firms increase both optimal leverage and debt maturity as the lending relationship improves. When the wedge is small, firms issue debt to both the relationship debt investor and external investors. As the relationship quality improves, the share of debt financing coming from the relationship investor increases. As a result, average costs of issuance decrease, the firms increases its leverage ratio and issues shorter term debt. Our analysis also demonstrates that the benefits of having better relationships with debt investors decrease as debt issuance with outside investors becomes less costly, in line with the empirical evidence in Karolyi (2018) and the recent evidence from the COVID19 crisis (see, e.g., Halling, Yu, and Zechner (2020) or Amiram and Rabetti (2020)).

In an important extension of our baseline model, we study how idiosyncratic shocks to the relationship lender—such as the lending cut by Commerzbank due to losses on its international trading book during the financial crisis—affect financing decisions. We show that the effects of such shocks depend on the quality of the lending relationship. Specifically, firms with intermediate relationship quality are most affected as they optimally choose to maintain the lending relationship, leading to a significant drop in leverage and to a sharp shortening of debt maturity. Firms with weak lending relationships terminate their current relationship and borrow from a new relationship lender. Firms with strong lending relationships are almost unaffected. In a second extension, we study the impact of costly relationship formation and rent extraction by the relationship investor on firms' financing decisions. We find that costly relationship formation has almost no impact on capital structure choices due to the low frequency of creditor turnover. We additionally find that rent extraction lengthens debt maturity choice, as shareholders capture a smaller part of the releveraging surplus at maturity, but leaves leverage largely unaffected.

Our paper contributes to several strands of the literature. First, we contribute to the literature on firms' dynamic capital structure choice; see, e.g., Fischer et al. (1989), Leland (1998), Hackbarth, Miao, and Morellec (2006), Strebulaev (2007), Morellec, Nikolov, and Schurhoff (2012), Dangl and Zechner (2021), or DeMarzo and He (2021). In that literature, our paper is most closely related to Hugonnier, Malamud, and Morellec (2015), which introduces capital supply uncertainty in Leland (1998)'s dynamic capital structure framework.[4] In this model, firms search for debt investors when seeking to raise new debt and can only be matched once with a given debt investor. Our paper instead allows firms to build relationships with debt investors and shows how lending relationships impact capital structure decisions and debt maturity choices. We find that optimal leverage is increasing while credit spreads are decreasing in the quality of the relationship between the firm and debt investors, in line with the empirical results in Karolyi (2018) and Zhu (2021).

Second, we advance the literature on dynamic debt maturity choice; see e.g. He and Xiong (2012), Cheng and Milbradt (2012), He and Milbradt (2016), Huang, Oehmke, and Zhong (2019), Geelen (2020), or Chen, Xu, and Yang (2021) for recent contributions. Our work builds on the dynamic capital structure model with endogenous maturity choice developed by Geelen (2016) and Chen et al. (2021). In these models, debt maturity is lumpy, in that all outstanding debt matures simultaneously, and the firm can freely readjust its

[4]Our paper also relates to the recent literature that examines the effects of market liquidity on the pricing of risky bonds (see e.g. Ericsson and Renault (2006), He and Xiong (2012), or He and Milbradt (2014)). These models generally focus on the analysis of secondary market frictions on default risk and the pricing of risky bonds, given some exogenous financing and restructuring strategies.

capital structure at maturity. That is, firms are not required to roll over the matured debt immediately. Instead, they can optimally readjust their debt level and maturity based on the prevailing condition of the firm and the quality of its relationship with debt investors. We advance this literature by allowing firms to build relationships with debt investors, which impacts optimal leverage, the debt maturity choice, and default risk. We show that firms initially issue short maturity debt and increase debt maturity as the quality of the lending relationship improves. We also show that the maturity of debt contracts issued to non-relationship investors is higher than that of relationship investors and that optimal debt maturity decreases with the share of the debt held by relationship investors.

Third, we add to the literature on relationship lending (see, e.g., Diamond (1991), Petersen and Rajan (1994), or Boot and Thakor (1994)) by showing how firms' debt maturity choice impacts the relationships building process and how these relationships in turn affect the joint choice of leverage and debt maturity as well as the decision to default. As in classic studies, lending relationships decrease the cost of borrowing in our model, leading to relationship stickiness. In early papers this stickiness arises from hard-to-verify private information that is acquired by the relationship bank and that is unobservable to other lenders. While early empirical studies provide evidence in support of this mechanism (see, e.g., Slovin, Sushka, and Polonchek (1993)), recent evidence based on U.S. syndicated loans suggests that this information gap is small and thus that this friction is unlikely to explain relationship stickiness in this market (see e.g. Darmouni (2020)).

Lastly, there exists a large empirical literature documenting the role of debt investors in shaping many aspects of firm financing (see e.g. Lemmon and Roberts (2010), Faulkender and Petersen (2006), Leary (2009) for early contributions). In this literature, several recent studies show that firms with stronger relationships with debt investors benefit from better financing terms (see e.g. Bharath et al. (2011), Engelberg et al. (2012), Karolyi (2018), or Herpfer (2021)). Most of the early literature on supply-side frictions in corporate debt markets focuses on bank loans. Recent studies find that capital supply conditions and relationships are also important in primary bond markets. Zhu (2021) for instance finds that when a bond mutual fund experiences capital inflows, firms in the portfolio of that fund are more likely to issue bonds and at a lower spread, compared to other firms. Chen et al. (2020) show that existing institutional bondholders are more likely to purchase new bond issues of holding firms, thus decreasing financing frictions in the primary bond market. Our model captures some key features of primary debt markets and demonstrates how credit supply conditions and lending relationships shape leverage and debt maturity choices.

Section I presents the model. Section II analyzes the model implications for optimal leverage, debt maturity choice, and default risk. Section III extends the model to allow for idiosyncratic shocks to a relationship investor's capital supply. Section IV examines the impact of rent extraction by the relationship investor on firms' financing decisions. Section V concludes. Technical developments are gathered in the Appendix.

I Model

A Assumptions

Throughout the paper, agents are risk-neutral and discount cash flows at the constant rate $r > 0$. Time is continuous and uncertainty is modeled by a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where the filtration $\{\mathcal{F}_t : t \geq 0\}$ satisfies the usual conditions.

We consider a firm with assets in place that generate a cash flow X_t at time $t \geq 0$ as long as the firm is in operation. This operating cash flow is independent of financing choices and governed by the process:

$$dX_t = \mu X_t dt + \sigma X_t dB_t,$$

where $\mu < r$ and $\sigma > 0$ are constant parameters and $(B_t)_{t\geq 0}$ is a Brownian motion.

Operating cash flows are taxed at the constant rate $\gamma < 1$, providing the firm with an incentive to issue debt. Debt contracts are characterized by a principal ρ and a coupon c

and mature with Poisson intensity $\frac{1}{m}$, all of which are endogenously chosen. We model debt maturity as lumpy, in that all outstanding debt matures simultaneously, as in Geelen (2016) and Chen et al. (2021). This assumption is consistent with the finding in Choi, Hackbarth, and Zechner (2018) that lumpiness in maturity structure is a prevalent feature in the data. Denoting the next time that debt matures by τ_m, we have that (barring default) expected debt maturity is given by $\mathbb{E}(\tau_m) = m$. We allow the firm to re-optimize its capital structure when debt matures.[5] This feature differs from the assumption used, e.g., in Leland and Toft (1996), He and Milbradt (2014), or Della Seta, Morellec, and Zucchi (2020), that firms are committed to roll over any retired debt and continuously issue debt. We also allow the firm to default on its debt, which can occur when the debt matures at time τ_m or before maturity at the endogenous time τ_D. In default, creditors recover a fraction $(1 - \alpha)$ of the unlevered asset value, where $\alpha \in (0, 1)$ is a frictional default cost.

We are interested in building a model in which capital structure and debt maturity depend not only on firm characteristics but also on frictions in primary debt markets and the quality of issuer-debt investor relationship. Indeed, as documented by Petersen and Rajan (1994), Berger and Udell (1995), Bharath et al. (2011), Karolyi (2018), and Zhu (2021), issuer-debt investor relationships are first-order determinants of debt issuance decisions, leverage ratios, and credit spreads. In addition, as discussed for example in Chen, Schuerhoff, and Seppi (2020), the primary market for corporate debt is subject to significant issuance frictions that are reflected in the cost of debt. A typical example is the U.S. corporate bond market, an over-the-counter market which is illiquid and subject to search frictions. Chen, Schuerhoff, and Seppi (2020) find that, as a result of search frictions, on average more than half of a new bond issuance gets placed with issuers' incumbent bondholders and that firms with reduced search frictions face lower costs of issuing new bonds. Relatedly, Zhu (2021) finds that bond

[5] We can also extend our model to allow for debt restructuring as in Fischer et al. (1989) or Hugonnier et al. (2015). As Geelen (2016) shows, having the ability to increase leverage by buying back outstanding debt and issuing new debt lengthens the firm's optimal debt maturity. The reason is that the ability to restructure the firm's debt lowers the value of the option to adjust the firm's capital structure at maturity, which gives firm's an incentive to shorten their debt maturity. To keep our analysis tractable, we focus on the case where the firm cannot restructure its debt.

mutual funds that hold a firm's existing bonds are five times more likely to provide capital in future bond issues and do so at a lower cost.

To capture these features of primary debt markets, we consider that debt issuance at time $t > 0$ works as follows. The firm initially has a relationship with a single debt investor i (or a single pool of investors in the case of bond issues). Throughout the paper, we call this (pool of) investor(s) the relationship investor. When the firm needs to refinance existing debt and potentially change its leverage, it contacts its relationship investor (directly for a loan issue or via its underwriter for a bond issue) for a debt issue of endogenous size $\hat{\rho}$ and maturity $m \in \mathcal{M}$, where $\mathcal{M} \subseteq (\underline{m}, \infty]$ with $\underline{m} > 0$ is the set of available maturities. There is uncertainty regarding the relationship investor's appetite for debt. We denote by $\beta_t \in [0, 1]$ the fraction of the new issue that this investor purchases. In this specification, β_t captures the relationship investor's ability (or willingness) to supply capital, that depends for instance on inflows and outflows to and from the relationship investor and on its investment (risk) strategy. We consider that β_t is drawn from a distribution that depends on the investor's quality $\theta \in \{H, L\}$, where a high quality relationship investor (H) has a higher chance of filling the firm's demand for debt than a low-quality relationship investor (L).

The quality of the relationship between the firm and the relationship investor is unknown ex ante but both the firm and the investor learn over time from the realized capital supply, i.e. from past debt purchases by the relationship investor. In practice, this quality may be unknown to the firm and the relationship investor because the investor's ability to supply credit is related to both the investor's in- and outflows and to the match between its risk strategy and the risk of the debt issued by the firm at time τ_m. We denote by $q_t^i = \mathbb{P}_t(\theta^i = H)$ the probability that the firm assigns to the relationship with (current) investor i being of high quality. This probability captures the strength of the relationship between the issuer and debt investor i and, as we show below, it is governed by an endogenous adapted stochastic process. The firm's prior belief about the quality of the relationship with any new relationship investor is $\tilde{q}$ (i.e. $q_t^j = \tilde{q}$ for all $j > i$). The firm can replace its existing relationship investor with a new one at no frictional cost. Section IV introduces costs of forming relationships.

Debt issuance with the relationship investor i incurs a proportional issuance (underwriting) cost $\psi_R \geq 0$. After the relationship investor has announced the amount of debt it will purchase, the firm decides whether it wants to issue additional debt to outside investors at a proportional issuance cost ψ_O. We assume that the cost of raising debt from non-relationship lenders is higher in that $\psi_O \geq \psi_R$, so that better issuer-investor relationships reduce financing frictions. This assumption is consistent with the finding in Yasuda (2005) that banks charge lower fees to those firms with which they have relationships than to other firms. The additional cost incurred by the firm when raising debt from multiple investors may be due, e.g., to additional search cost incurred by the underwriter for a bond issue or to costs incurred by the lead bank when securing the funds and syndicating the loan (see e.g. Ivashina (2009)). When issuing new debt at time τ_m, the firm issues a single debt contract that is purchased by both the relationship investor and outside investors, so all investors have the same seniority. The firm is always able to sell its full debt issue, but the cost of issuance decreases with the fraction of the issue purchased by the relationship investor.

After the size ρ and the maturity m of the debt issue are chosen and the composition of the pool of investors and the quality of the issuer-investor relationship are determined, the coupon rate is set such that debt is issued at par. Finally, the relationship investor and outside debt investors impose the following restriction on the size of the debt issue, which guarantees that firm value is finite.[6]

Assumption 1. *When the firm issues debt at time τ_m, debt investors require the interest coverage ratio $\frac{X_{\tau_m}}{c}$ to be above some strictly positive constant, which can be arbitrarily small.*

The timeline for debt issuance is therefore as follows:

1. The firm contacts its relationship investor for a debt issue of endogenous size $\hat{\rho}$ and maturity $m \in \mathcal{M}$.

2. The relationship investor signals the fraction $\beta \leq 1$ of the debt issue with maturity m that it is willing to purchase.

[6] This constraint does not bind in equilibrium and plays no role in the analysis.

3. The firm chooses the face value $\rho \in [\beta\hat{\rho}, \hat{\rho}]$ of the debt contract. The proceeds from the debt issue are given by

$$\underbrace{(1-\psi_R)\beta\hat{\rho}}_{\textit{Relationship investor}} + \underbrace{(1-\psi_O)\,(\rho-\beta\hat{\rho})}_{\textit{Outside investors}},$$

and total debt issuance costs amount to $\psi_R\rho + (\psi_O - \psi_R)(\rho - \beta\hat{\rho})$.[7]

4. The coupon rate is set such that debt is issued at par, given the face value and the maturity of the debt contract, the level of cash flows, and the quality of the issuer-debt investor relationship at the time of issuance.

B Learning, Issuer-Investor Relationship, and Firm Financing

Repeated borrowing from the same lender provides information about the quality of the relation between the firm and the lender and therefore feeds back into the pricing of debt and equity and financing decisions. In particular, the amount of debt purchased by the relationship investor at time τ_m provides a signal about the quality θ of the match between this investor and the firm. Therefore, after observing β_t at time t, the firm updates its beliefs about the relationship investor using Bayes' rule:

$$q_t^i = Q(q_{t-}^i, \beta_t) = \frac{q_{t-}^i \mathbb{P}(\beta = \beta_t | \theta = H)}{q_{t-}^i \mathbb{P}(\beta = \beta_t | \theta = H) + (1 - q_{t-}^i)\, \mathbb{P}(\beta = \beta_t | \theta = L)},$$

where $\mathbb{P}(\beta = \beta_t | \theta)$ is the probability that the current relationship investor purchases a fraction β_t of the debt issue conditional of being of type $\theta = L, H$. Since the firm only learns about the quality of the relationship when new debt is issued, the process describing

[7] In practice, bond issues are often oversubscribed and the coupon is set such that the market clears given debt investors demand (Nikolova, Wang, and Wu (2020)). In our model, the effective coupon that the firm pays (that takes into account the issuance costs and clears the market) is

$$\textit{Effective Coupon} = \textit{Coupon} + \frac{\textit{Issuance Costs}}{\textit{Expected Maturity} * (1-\gamma)} = c + \frac{\psi_R\rho + (\psi_O - \psi_R)(\rho - \beta\hat{\rho})}{m(1-\gamma)}.$$

beliefs about the issuer-debt investor relationship is piece-wise constant. In our model, the quality of the issuer-investor relationship is therefore positively related to the duration of the relationship (the proxy used for the quality of the relationship in the empirical studies of Petersen and Rajan (1994) or Berger and Udell (1995)) and the number of loans by the bank to the issuer (the proxy used in the empirical study of Bharath et al. (2011)). Beliefs about the quality of the issuer-investor relationship q_t^i represent a state variable for the firm's problem, i.e. for the choice of leverage and debt maturity and for the decision to default.

The firm replaces relationship investor i with a new relationship investor $i+1$, which is high quality with probability $\tilde{q}$, whenever this increases equity value. In the remainder of the paper, we omit the superscript i and denote by q_t the probability that the relationship between the current relationship investor and the firm at time t is of high quality.[8] Additionally, we denote by τ_q the next time that the firm parts ways with the current debt investor and starts a relationship with a new one.

Figure 1 illustrates the firm's financing choices (top panel) and the dynamics of the issuer-investor relationship (bottom panel) for a given path of operating cash flows. The stopping times $(\tau_m^n)_{n=1}^{+\infty}$ indicate the dates at which debt matures and new debt is issued and therefore the dates when the coupon rate, the face value of debt, debt maturity, and beliefs change. The characteristics of the debt issue (face value, maturity, number of investors) at any time τ_m depend on the realization of the two state variables $(X_{\tau_m^n}, q_{\tau_m^n})_{n=1}^{+\infty}$. Notably, the processes describing the coupon, face value, and maturity of corporate debt are piece-wise constant and only change when outstanding debt matures and new debt is issued. The vertical line indicates a change in relationship investor (from relationship investor i to relationship investor $i+1$) at time τ_q. At this time, the firm learns from its relationship investor that it will be unable to supply the requested amount of debt, which leads to a decline in beliefs about the quality of the relationship and to a replacement of the relationship investor. At time τ_D, the firm defaults on its debt.

[8]While q_t^i is a martingale, q_t is only a submartingale if the firm prefers a high quality relation with its debt investor because the firm replaces its existing relationship investor whenever beliefs fall below $\tilde{q}$.

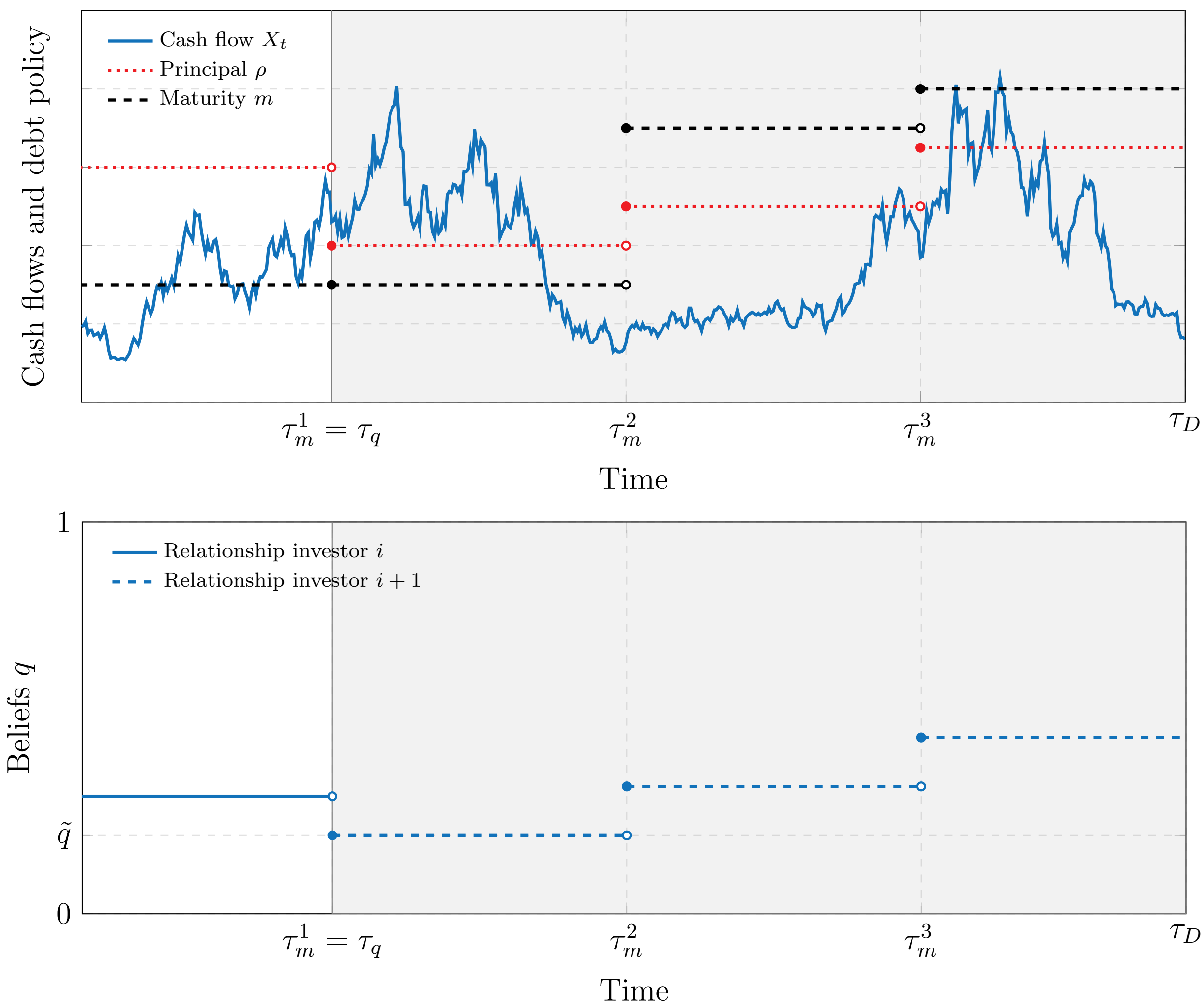


Figure 1: **Capital structure and lending relationship dynamics**. Jumps in endogenous quantities (principal, coupon, maturity, beliefs) occur on maturity dates $(\tau_m^n)_{n=1}^{+\infty}$. The vertical line indicates a change in relationship investor. Default occurs at time τ_D.

C Optimal Financing and Default Policies

To determine the effects of issuer-debt investor relationships on leverage and debt maturity choices, we need to determine the prices of corporate debt and equity. To aid in the understanding of the pricing formulas, Figure 2 shows the cash flows shareholders and creditors (relationship and outside debt investors) receive at different points in time. The middle row in the blue boxes indicates the cash flow to shareholders while the bottom row indicates the cash flow to creditors. The gray area describes the decisions made by shareholders at maturity/issuance and their effects on cash flows. On the maturity date τ_m, shareholders decide whether to default on maturing debt or not. If there is no default, maturing debt is repaid and new debt with face value ρ' and maturity m' is issued.

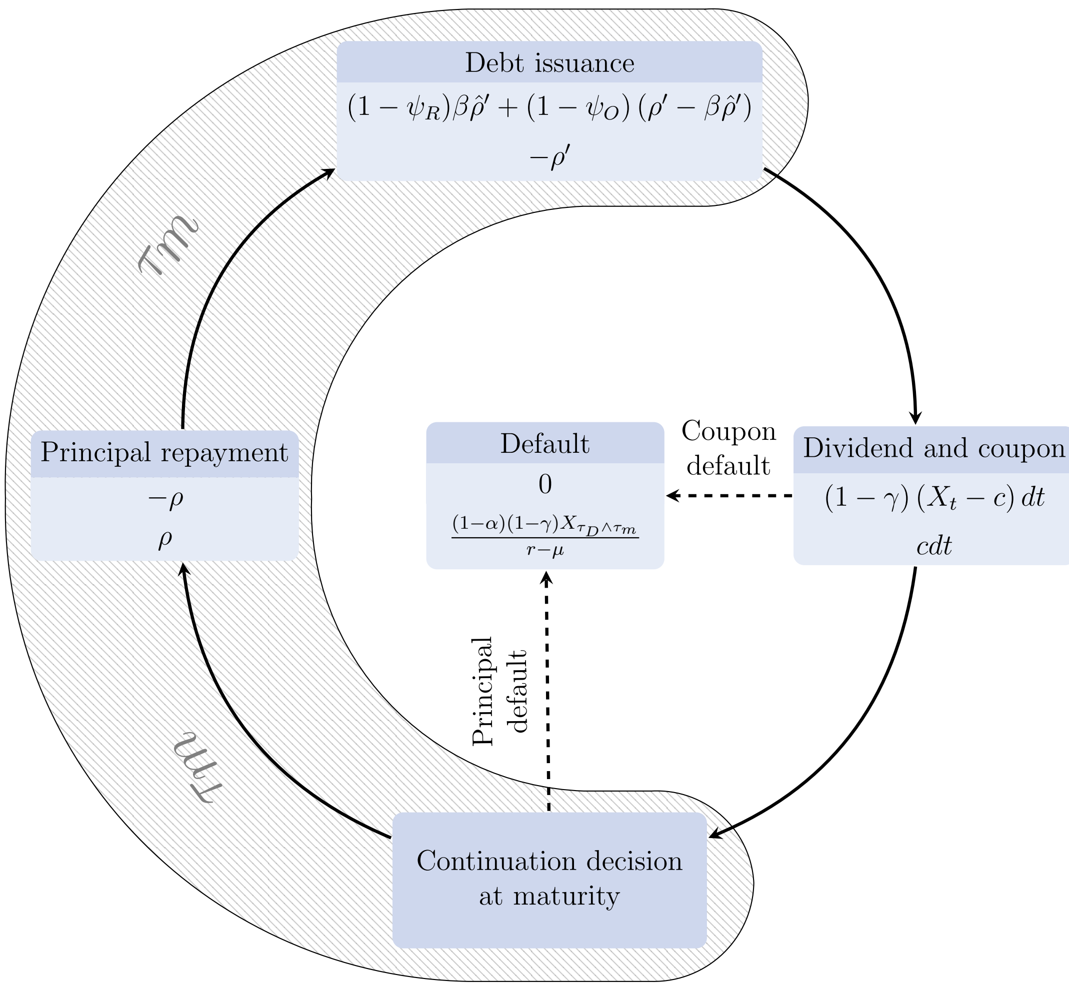


Figure 2: **Cash flows to and from shareholders and creditors (relationship + outside investors).** The middle row in the blue boxes indicates the cash flow to shareholders while the bottom row indicates the cash flow to creditors. The gray area describes the decisions made by shareholders at maturity/issuance and their effects on cash flows. In this figure, $\hat{\rho}'$ and ρ' respectively indicate the quantity of debt requested and issued at time τ_m.

Debt value is given by the present value of the cash flows that creditors expect to receive and depends on the firm's current cash flow x, debt coupon c, the belief about the quality of the relationship investor q, debt principal ρ, and debt maturity m. Specifically, the value of outstanding debt is given by

$$
\begin{aligned}
& D(x, c, q, \rho, m) \\
& = \mathbb{E}_x \left[\int_0^{\tau_m \wedge \tau_D} e^{-rt} c dt + \mathbb{I}_{\{\tau_D \leq \tau_m\}} e^{-r\tau_D} \frac{(1-\alpha)(1-\gamma) X_{\tau_D}}{r-\mu} \right] \\
& + \mathbb{E}_x \left[\mathbb{I}_{\{\tau_m < \tau_D\}} e^{-r\tau_m} \left(\mathbb{I}_{\{F(X_{\tau_m}, q) \geq \rho\}} \rho + \mathbb{I}_{\{F(X_{\tau_m}, q) < \rho\}} \frac{(1-\alpha)(1-\gamma) X_{\tau_m}}{r-\mu} \right) \right]
\end{aligned}
\tag{1}
$$

where $F(x, q)$ is the continuation value of shareholders defined in equation (3) below, $\mathbb{I}_{\{x \leq y\}}$ is the indicator function of the event $x \leq y$, and $\tau_m \wedge \tau_D \equiv \inf \{\tau_m, \tau_D\}$ is the first time that debt matures or the firm defaults. As shown by this equation, creditors receive coupon payments until the debt matures or the firm defaults. The firm can default either if cash flows deteriorate sufficiently before maturity (in which case $\tau_D \leq \tau_m$) or when the debt matures if the continuation value of shareholders is less than the debt principal (i.e. if $F(x, q) < \rho$). If the firm defaults before debt maturity, creditors recover a fraction $(1 - \alpha)$ of the unlevered asset value. If debt matures and the firm does not default, creditors receive the principal ρ. Otherwise, they get the recovery value. As we show below, beliefs feed back not only in financing decisions (i.e. on the choice of (ρ, c, m)) but also in the decision to default for given (ρ, c, m) by affecting the continuation value of equity.

Shareholders' levered equity value, denoted by $E(x, c, q, \rho, m)$, is in turn given by:

$$
E(x, c, q, \rho, m) = \sup_{\tau_D} \mathbb{E}_x \left[\int_0^{\tau_m \wedge \tau_D} e^{-rt} (1-\gamma) (X_t - c) \, dt + \mathbb{I}_{\{\tau_m < \tau_D\}} e^{-r\tau_m} \left(F(X_{\tau_m}, q) - \rho \right)^+ \right]
\tag{2}
$$

where $x^+ = \max\{0, x\}$. As shown by this equation, shareholders receive the firm's cash flow minus coupon payments net of taxes until either the debt matures at τ_m or the firm defaults at τ_D. If the firm defaults before maturity (i.e. $\tau_D \le \tau_m$), absolute priority is enforced and shareholders receive zero. When the debt matures, the firm decides whether or not to repay the principal ρ. If the principal is repaid then shareholders get the continuation value defined in equation (3). Otherwise, they get zero.

Lastly, the unlevered equity value (or firm value at issuance) is given by

$$F(x,q) = \tag{3}$$
$$\sup_{(\hat{\rho},m)\in\mathbb{R}_+\times\mathcal{M}} \mathbb{E}_q\left[\sup_{(\rho,q')\in[\beta\hat{\rho},\hat{\rho}]\times\{Q(q,\beta),\bar{q}\}}\{E(x,c,q',\rho,m)+(1-\psi_R)\beta\hat{\rho}+(1-\psi_O)\left(\rho-\beta\hat{\rho}\right)\}\right]$$

such that

$$c = \left\{c'\middle|D(x,c',q',\rho,m)=\rho\right\}. \tag{4}$$

Equation (3) shows that shareholders first decide on the amount of debt to request from their relationship investor $\hat{\rho}$ and on the maturity m of this debt. The inner maximization operator shows that shareholders decide on how much debt to issue $\rho \in [\beta\hat{\rho},\hat{\rho}]$ after observing the relationship investor's supply β_t. In addition, after observing β_t, shareholders (or the firm's underwriter in case of a bond issue) update their beliefs about the quality of the relationship with the debt investor and can decide to replace the current relationship investor and start a relationship with a new debt investor. Finally, equation (4) indicates that the coupon is set such that debt is issued at par.

Given the functional forms of issuance costs, default costs, and taxes, shareholder's optimization problem is homogeneous of degree one in x. Notably, we can establish the following result (see the Appendix for a proof):

Proposition 1 (Firm value)**.** *Firm value exists, is finite, and satisfies $F(x,q) = xf(q)$.*

This homogeneity property implies that shareholders are better off having a high quality investor when $f(q)$ is increasing in q. As a result, as soon as beliefs drop below $\tilde{q}$ (the prior about the quality of a new relationship), the firm replaces its existing relationship investor and starts a new relationship. The homogeneity of the firm value function in x then works through into the levered equity and debt values, which can be written as

$$E(x, c, q, \rho, m) = xe\left(\frac{c}{x}, q, \frac{\rho}{c}, m\right), \text{ and } D(x, c, q, \rho, m) = xd\left(\frac{c}{x}, q, \frac{\rho}{c}, m\right),$$

where the functions e and d are defined in the Appendix (see Lemma 1).

Proposition 1 implies that in our model, all claims to cash flows scale with the level of cash flows as in Leland (1998), Strebulaev (2007), or Morellec et al. (2012). Using this scaling property, it can be shown that shareholders' optimal default strategy is given by:

Proposition 2 (Optimal default)**.** *The optimal strategy for shareholders is to default:*

1. ***Coupon default**: Before maturity if the ratio of the coupon payment to the firm cash flow* $z = \frac{c}{x}$ *rises above an endogenous threshold* $z_D(q, \frac{\rho}{c}, m)$*, that is determined by the equity value's smooth pasting condition.*

2. ***Principal default**: On the maturity date of the debt contract if the debt principal* ρ *exceeds the continuation value of equity* $F(x, q)$*.*

Proposition 2 shows that there are two types of default in our model: 1) The firm can default when its current cash flow drops sufficiently and shareholders are unwilling to cover additional losses, which we call a *coupon default*, and 2) At maturity the principal needs to be repaid and shareholders are unwilling to do so, which we call a *principal default*.

II Model Analysis

A Parameters

This section examines how supply side frictions and the quality of lending relationships impact leverage and debt maturity choices, creditor turnover, and default risk. To do so, we first need to select values for the model's parameters. We calibrate the model to reflect a typical U.S. public firm. Parameter values are reported in Table 1.

Parameter	**Symbol**	**Value**
Interest rate	r	4.2%
Tax rate	γ	15%
Default costs	α	45%
Issuance costs (relationship investor)	ψ_R	0.6%
Issuance costs (outside investors)	ψ_O	2.7%
Cash flow drift	μ	3%
Cash flow volatility	σ	25%
Maturity set	$\mathcal{M}$	$[1, \infty]$
Relationship investor debt appetite	β	$\{0.4, 1\}$
Probabilities high-quality relationship investor	$\mathbb{P}(\beta = 1 \mid \theta = H)$	0.9
Probabilities low-quality relationship investor	$\mathbb{P}(\beta = 1 \mid \theta = L)$	0.2
Prior quality new relationship investor	$\tilde{q}$	0.1

Table 1: **Baseline parameters.**

The risk-free rate is set equal to $r = 4.2\%$ as in Morellec et al. (2012). The tax benefits of debt are set equal to $\tau = 15\%$. As in Graham (1999), this estimate reflects the adjustment of the marginal corporate tax rate for the personal tax disadvantage of holding debt relative to equity. Glover (2016) finds that firms lose around 45% of their value in default. Therefore, we set α to 45%. In our base case parametrization, we set the costs of debt issuance with the relationship investor ψ_R and outside investors ψ_O to 0.6% and 2.7%, respectively. This produces an average cost of debt issuance between 0.6% and 1% of the issue size under the optimal financing policy, which is in the range reported by Altınkılıç and Hansen (2000). The homogeneity property of the model implies that we can set the initial level of the cash flow to $X_0 = 1$ without loss of generality. We additionally set the growth rate and the volatility of the cash flow process to $\mu = 3\%$ and $\sigma = 25\%$.

We set the distribution for β such that the relationship investor acquires either all the firm's debt or only 40% of it. A high-quality (respectively low-quality) relationship investor has a 90% (20%) probability of purchasing the entire debt issue. A new relationship investor has a $\tilde{q} = 0.1$ chance of being high-quality. Given these parameters, if the firm deals with a high-quality investor and issues debt, on average, every 5 years, then the investor does not provide the full amount requested once every 50 years. A large part of our analysis is dedicated to studying the effects of varying these parameters on outcome variables.

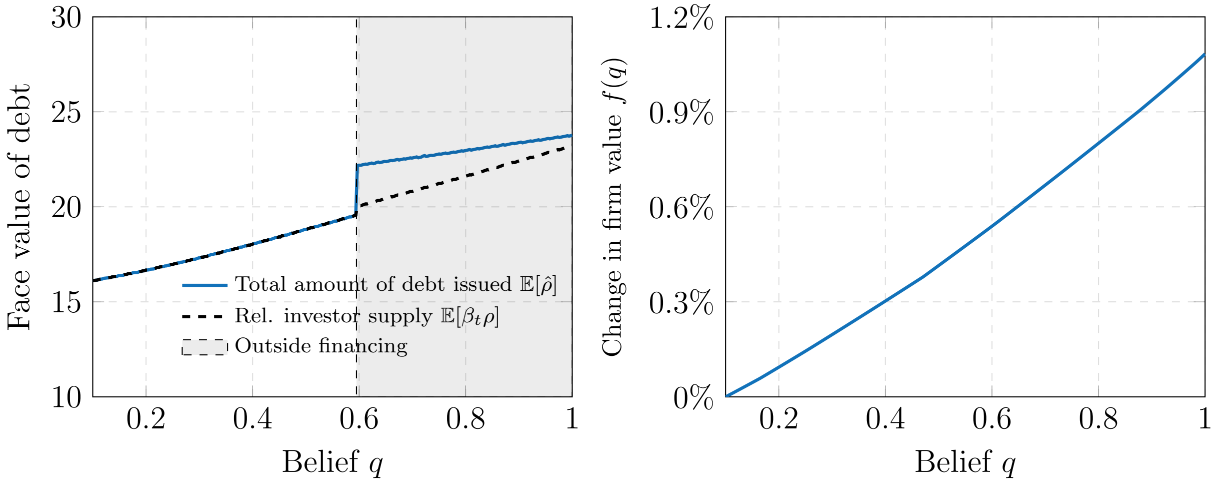
With these baseline parameters, the model predicts an optimal debt maturity between 3.64 and 8.37 years and leverage ratios (at issuance) between 21% and 31% depending on the quality of issuer-investor relationship, in line with empirical estimates. For instance, Choi et al. (2018) show that the average of firms' debt maturities in the Compustat database is 5.15 years. Colla, Ippolito, and Li (2020) report that mean market leverage ratios for firms covered by Compustat from 2002 to 2018 are between 18.1% and 31.6%.

B Lending Relationships and Financing Decisions

I Lending relationships, debt supply, and firm value

We start by examining the effects of lending relationships on the size of debt issues. To do so, we plot in Figure 3a the amount of debt purchased by relationship and non-relationship investors as a function of the relationship quality q. The figure shows that, as expected, the relationship investor's willingness to purchase the firm's debt increases with q. Higher availability of debt financing at a lower cost leads a firm with better lending relationships to issue more debt, in line with empirical findings. For instance, Bharath et al. (2011) find that relationship borrowers receive larger loans. Figure 3a therefore shows that debt issuance is driven not only by a firm's demand for debt but also by credit supply, in line with the evidence in Lemmon and Roberts (2010), Leary (2009), and Zhu (2021).

A striking result in Figure 3a is that stronger lending relationships allow the firm not only to issue more debt from relationship lenders but also to raise additional debt from outside investors, due to the associated decrease in the cost of debt. In our base case environment, this occurs whenever $q \geq 0.6$. In the context of our model, debt issuance with both the



(a) Face value of debt

(b) Change in firm value

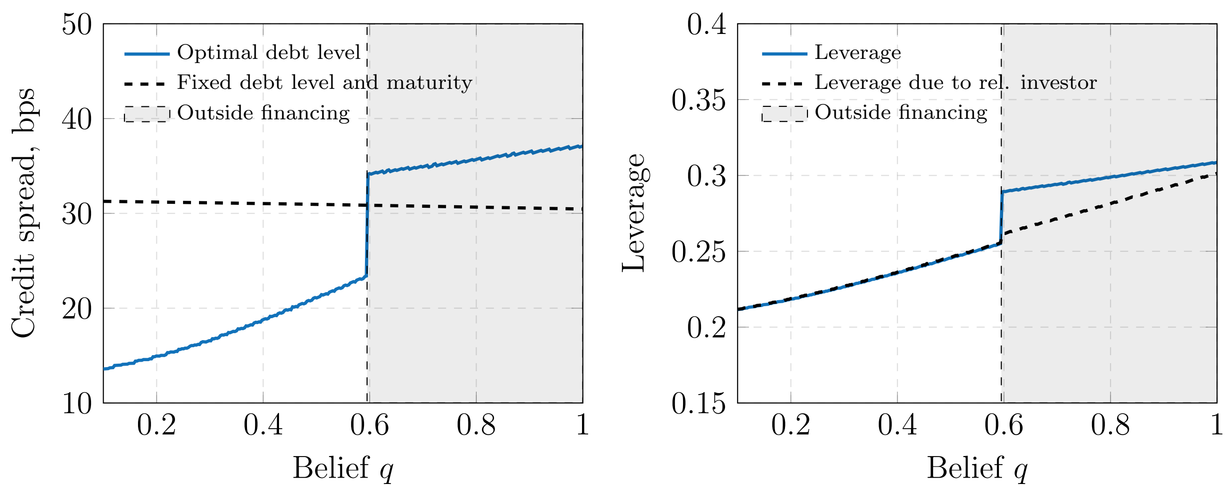
Figure 3: **Lending relationship, debt supply, and firm value.** The grey area indicates the region in which the firm also raises debt from outside investors. Parameters are as in Table 1.

relationship and outside investors can be interpreted as the issuance of a syndicated loan. Among the firms that issue debt to outside investors, the fraction of the debt issue acquired by the relationship investor is lower for firms with weaker relationship quality. Therefore, the model predicts that stronger lending relationships lead to a higher likelihood of issuing syndicated loans. It also predicts that, conditional on issuing a syndicated loan, the loan structure becomes more concentrated as the quality of the lending relationship improves.

Stronger lending relationships allow the firm to borrow more at better terms. As a result, they are associated with a higher firm value, as illustrated by Figure 3b. In our base case environment, a firm with a high-quality relationship investor ($q = 1$) has a value that exceeds by 1.08% the value of a firm that issues debt to a new relationship investor ($q = 0.1$). This suggests that lending relationships contribute significantly to the net benefits of leverage estimated between 3.5% and 5.5% of firm value by Korteweg (2010) and Van Binsbergen, Graham, and Yang (2010). The reason is that the better the quality of the lending relationship is, the higher is the likelihood that the relationship investor provides the entire amount of debt requested. As a result, firms with a strong lending relationship are more likely to issue debt at a lower cost. In addition, firms with better lending relationships may also borrow more from outside investors (as shown by Figure 3a) and may benefit from a reduced credit spread (as we show later in the section), which translates into a higher firm value.

II Leverage and credit spreads

An additional prediction of the model is that repeated interactions with the same investor lead to a reduction in the cost of borrowing (black dashed line in Figure 4a), in line with the evidence in Karolyi (2018) and Bharath et al. (2011). The intuition for this result is that better relationship quality raises the likelihood of obtaining funds from the relationship investor. For a fixed amount of debt, this decreases refinancing risk and the average cost of debt issuance and makes the option to refinance at maturity more valuable. This effect is reflected in the higher firm value and lower probability of default at maturity. As a result, firms with better lending relationships are able to issue debt at lower spreads (Figure 4a).



(a) Credit spreads

(b) Leverage

Figure 4: **Lending relationships, leverage, and credit spreads.** The grey area indicates the region in which the firm also raises debt from outside investors. Parameters are as in the Table 1.

As shown in Figure 3a, firms with stronger lending relationships are able to borrow more from both the relationship and non-relationship investors. This reduces financing risk and allows the firm to sustain a higher leverage ratio (Figure 4b). With our baseline parameters, the leverage ratio increases from 21% to 31% as the lending relationship improves. Figure 4a shows that the increase in leverage (combined with the longer debt maturity associated with better lending relationships) outweighs the increase in firm value so that the credit spread at issuance is increasing in the quality of the relationship. This result highlights the need to control for leverage when determining the effects of lending relationships on financing costs.

III Debt maturity

In the model, the firm chooses not only how much debt to issue but also the maturity of this debt. By issuing shorter maturity debt, the firm can change its capital structure more frequently by repaying existing debt and optimally adjust its leverage ratio. Shorter debt maturity also allows the firm to adjust the terms of its debt contracts faster if relationships improve. On the other hand, shorter debt maturities also imply that the firm incurs debt issuance costs more frequently. Optimal debt maturity balances these different effects.

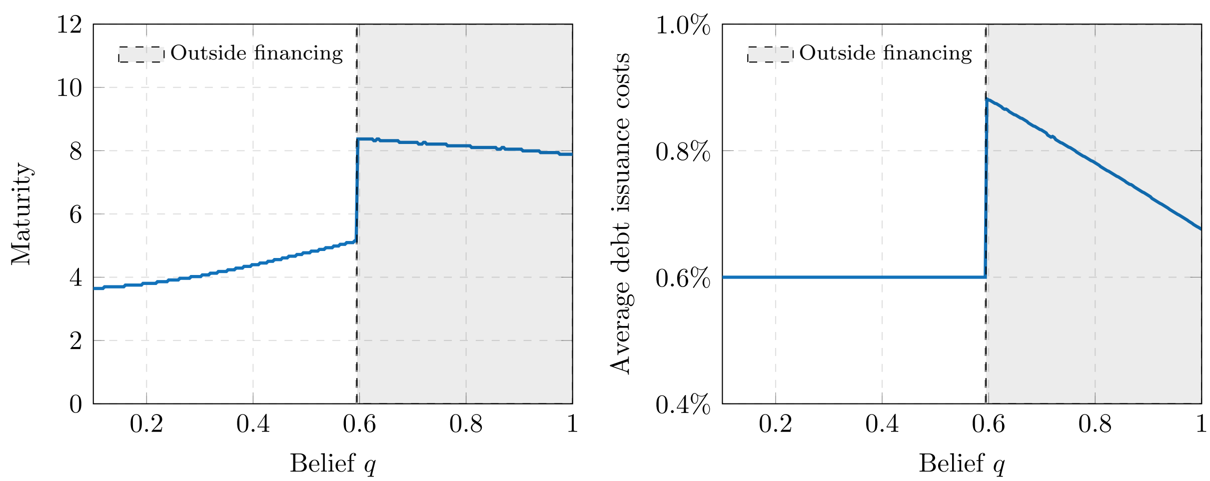


Figure 5: **Impact of the firm-investor relationship on debt maturity and average costs of debt issuance.** The grey area indicates the region in which the firm also raises debt from outside investors. Parameters are as in the Table 1.

Figure 5 illustrates the effects of lending relationships on optimal debt maturity. We note several results. First, at the beginning of the relationship (when q is low), the firm issues shorter maturity debt. Indeed, when the quality of the relationship is low, the relationship investor is able (or willing) to provide a lower amount of credit to the firm. As a result, the firm abstains from issuing longer maturity debt. Debt contracts with shorter maturity allow the firm to refinance debt at better terms sooner if the relationship improves or to terminate the relationship sooner if it deteriorates. Once the relationship quality improves, more debt financing is available from the relationship investor, which reduces default risk (for a given amount of debt) and leads the firm to issue longer maturity debt.

Second, firms that issue shorter maturity debt incur debt issuance costs more frequently. Since the firm only receives tax benefits over the interest payments but pays issuance costs over all the debt's cash flows, debt issuance costs are relatively larger for shorter maturity debt. This effect makes debt issuance with outside investors relatively less attractive for the firms with weak lending relationships that issue shorter maturity debt. As a result, these firms abstain from raising funding from outside investors (see Figure 4b).

As the relationship quality improves, default risk and the cost of debt decrease and the firm issues longer maturity debt. When the cost of debt decreases sufficiently, the marginal cost of issuing debt to outside investors falls below the marginal benefit of issuing additional debt and it becomes optimal for the firm to issue debt with outside investors. At that point, the average issuance cost jumps and, as a result, so does the optimal maturity as illustrated by Figure 5. As the relationship quality keeps on improving, the firm issues a larger portion of its debt with the relationship investor leading to a decrease in issuance costs (right panel of Figure 5) and to a shortening of debt maturity (left panel of Figure 5). The model therefore predicts that the maturity of debt contracts issued to non-relationship investors is higher than that of relationship investors, in line with the evidence in Bharath, Dahiya, Saunders, and Srinivasan (2011). It also predicts that average debt maturity decreases with the share of the debt held by relationship investors. Overall, Figure 4b and Figure 5 highlight the central role played by transaction costs and fees in leverage and maturity choices. This is consistent with the survey evidence in Graham and Harvey (2001) where transaction costs and fees come just after interest tax savings—and much before bankruptcy costs or personal taxes—as a determinant of capital structure choice.

C Comparative Statics

Figure 6 shows optimal maturity, leverage ratio, and the fraction of the debt issued to outside investors as functions of the relationship quality for different values of cash flow volatility σ, default costs α, and the tax rate γ.

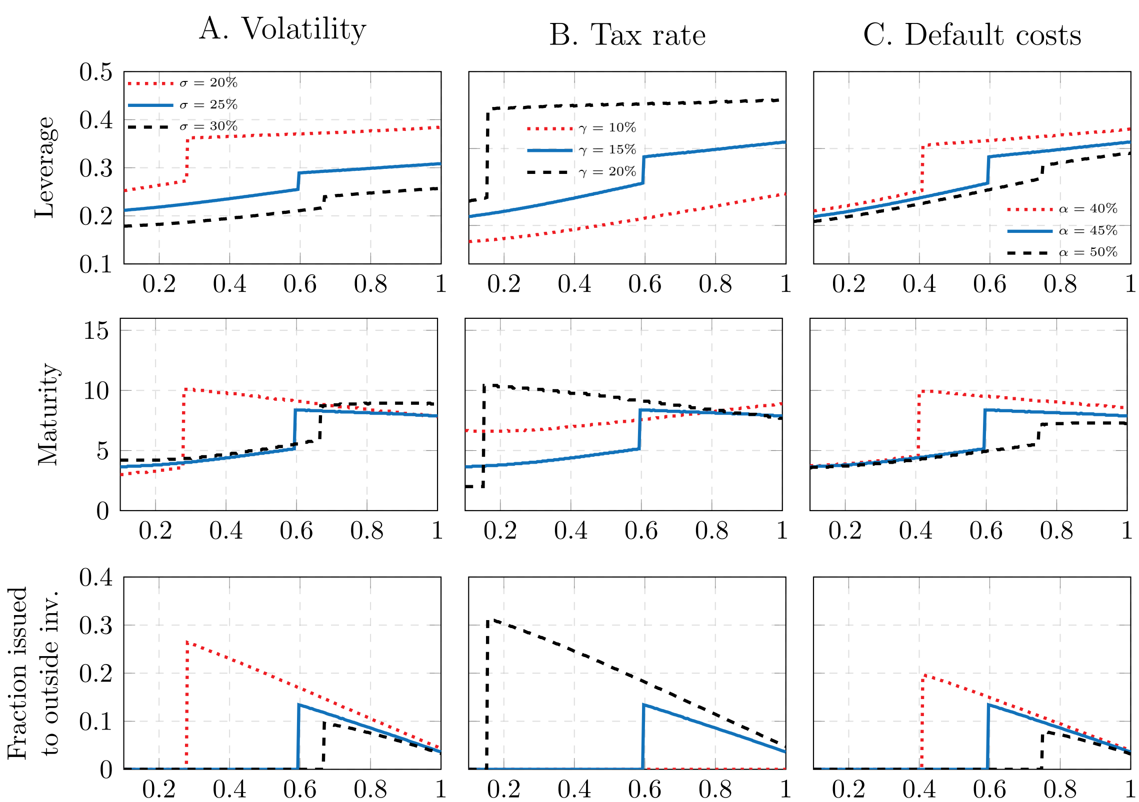


Figure 6: **The effects of lending relationships on financing decisions for varying levels of the cash flow volatility σ, default costs α and tax rate γ.** The base case (Table 1) is depicted by the blue solid line.

Optimal leverage is determined by the trade-off between the costs and benefits of debt.

Lower volatility decreases the probability of default and, thus, expected bankruptcy costs. In addition, firms with lower cash flow volatility also benefit from reduced uncertainty regarding tax benefits. This encourages these firms to issue more debt. Panel A of Figure 6 also shows that lowering volatility decreases the threshold for the quality q of the lending relationship above which the firm raises debt with outside investors. That is, lower volatility of cash flows σ implies a lower cost of debt, thus making borrowing from outside investors more attractive. In contrast, when volatility is high ($\sigma = 30\%$), only firms with strong lending relationships issue debt with outside investors. Lastly, for firms that do not issue debt to outside investors (low q), higher cash flow volatility slightly increases debt maturity. This happens because higher volatility increases the probability of default at maturity, making it optimal to postpone the repayment of the principal.

Because firms balance the costs and benefits of issuing debt, Panel B of Figure 6 indicates that the threshold for the relationship quality q above which firms raise debt with outside investors decreases with the tax rate. With the base case issuance costs and a tax rate of 10%, firms abstain from issuing debt with outside investors at all levels of relationship quality. As γ increases to 15%, tax benefits become larger relative to the cost of debt. As a result, outside debt issuance becomes attractive for firms with high relationship quality that pay lower average debt issuance costs. A further increase in tax benefit of debt ($\gamma = 20\%$) makes outside debt issuance attractive for firms with even weaker lending relationships.

The effects of bankruptcy costs on financing choices follow the same logic. Firms with higher default costs optimally choose lower target leverage ratios at all levels of relationship quality. For firms with relatively weaker lending relationships (for q between 0.40 and 0.75), the increase in default costs from 40% to 50% makes debt issuance with outside investors overly expensive relative to the tax benefits of debt. As a result, these firms decide to abstain from issuing debt with outside investors.

Figure 6 and Table 2 demonstrate that the benefits of stronger lending relationships are larger for firms with less volatile cash flows, facing a higher tax rate, and subject to

Cash Flow Volatility		Default costs		Tax Rate	
σ	$\frac{f(1)-f(\tilde{q})}{f(\tilde{q})}$	α	$\frac{f(1)-f(\tilde{q})}{f(\tilde{q})}$	γ	$\frac{f(1)-f(\tilde{q})}{f(\tilde{q})}$
20%	1.60%	40%	1.16%	10%	0.56%
25%	1.09%	45%	1.09%	15%	1.09%
30%	0.77%	50%	1.02%	20%	1.61%

Table 2: **Percentage change in firm value due to lending relationships.** Other parameters are as in Table 1.

lower default costs. Firms with these characteristics have higher target leverage ratios and, therefore, need to raise more debt from investors. As a result, stronger lending relationships are more valuable for these firms. The first and last of these predictions are opposite to those coming out of a mechanism based on informational asymmetries.

D Relationship Versus Outside Investors

The wedge between the costs of issuing debt with the relationship investor versus outside investors reflects the severity of frictions in the primary debt markets. This wedge can arise because underwriters need to search for new investors within a limited time frame when placing a new bond issue (Chen et al. (2020)). In the case of syndicated loans, it can arise because of the upfront fees necessary to compensate lead arrangers for attracting loan participants and structuring and originating the syndicated loan. Berg et al. (2016) show that these fees increase, for instance, with the volatility of borrowers' profits.

Figure 7 shows optimal maturity and leverage choices for different costs of debt issuance with outside investors. When issuing debt with outside investors is relatively more expensive (first column of Figure 7), firms optimally choose to issue debt only with the relationship investor or to raise a small fraction of the debt issue in the outside market. Because of this, the average costs of debt issuance are similar for firms with different relationship qualities. As a result, the optimal maturity choice is driven mainly by the availability of debt financing from the relationship investor, and, thus, monotonically increases as quality improves.

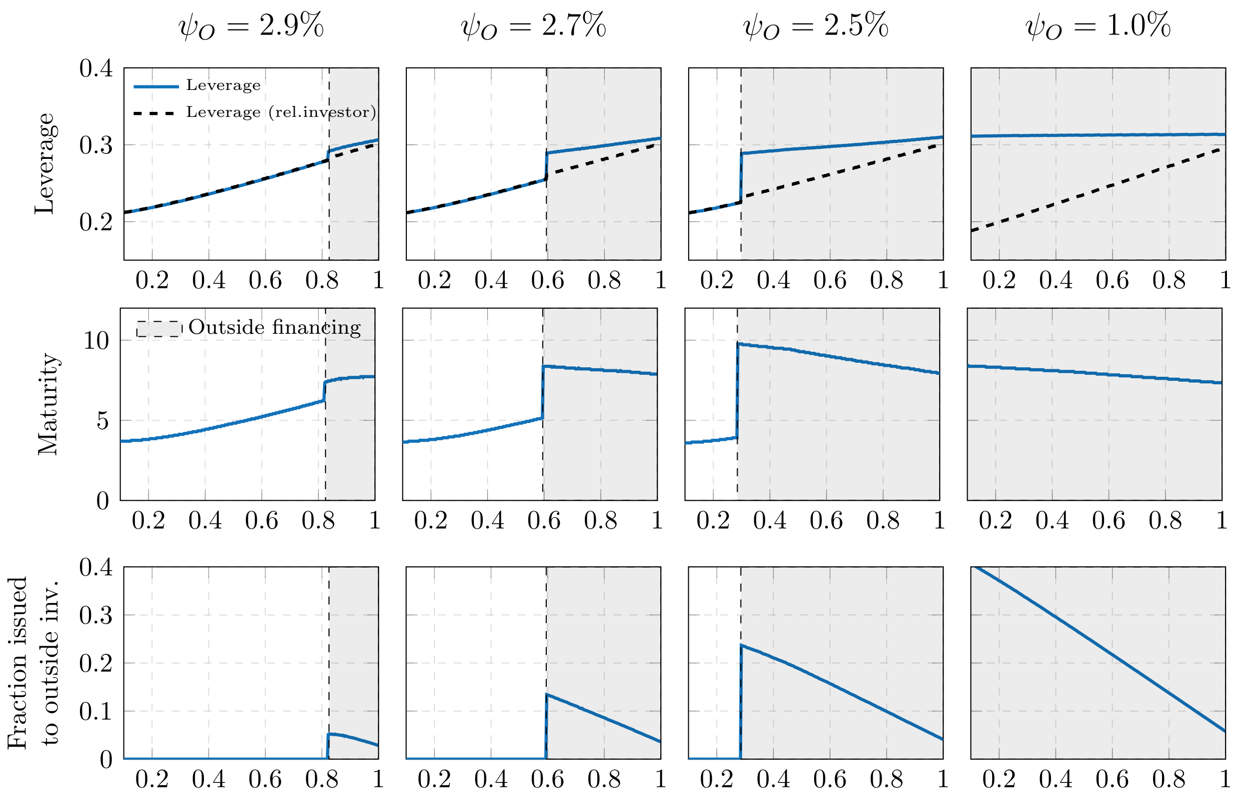


Figure 7: **The impact of the firm-investor relationship for various costs of debt issuance with outside investors.** The grey area indicates the region in which the firm also raises debt from outside investors. Parameters are as in the Table 1.

As the cost of issuing debt with outside investors decreases, firms naturally choose to issue more debt with non-relationship investors. Among the firms that decide to raise debt with both types of investors, firms with lower relationship quality pay higher average costs of debt issuance. As relationship quality improves, the average costs of debt issuance declines and firms issue debt with shorter maturity. As the cost of issuing debt with outside investors decreases further, all firms issue debt in the outside market (the last column of Figure 7). As a result, all firms have approximately the same target leverage ratios at refinancing points.

Figure 7 also shows that the benefits of having better relationships with debt investors decrease as the cost of issuing debt with outside investors decreases. This is in agreement with the empirical evidence in Karolyi (2018) and recent evidence from the COVID-19 crisis

(Halling et al. (2020) or Amiram and Rabetti (2020)) that relationships with debt investors are particularly important in times of economic downturns, i.e. when credit supply is weaker.

III Shocks to the Relationship Investor

A number of empirical studies have shown that relationship investors face shocks that may affect their ability to supply credit and therefore the financing choices of the firms they finance. For instance, Huber (2018) shows that Commerzbank—a major German bank—suffered significant losses on its international trading book during the financial crisis, resulting in a reduction in the bank debt of companies that had a relationship with it before the crisis.[9] Karceski, Ongena, and Smith (2005) and Di Patti and Gobbi (2007) show that bank consolidation—another form of exogenous shock to the lending relationship—negatively impacts firms with which the (target) bank has a relationship.

This section studies the effects of idiosyncratic shocks to the relationship investor's ability to supply credit on firm financing policies. To do so, we assume that the availability of credit from the relationship investor depends on the relationship investor's state s which can be either good, G, or bad, B. We assume that $\beta_t \in \{\beta_B, 1\}$ in state B and $\beta_t \in \{\beta_G, 1\}$ in state G, with $\beta_B < \beta_G$ so that the ability of the relationship investor to purchase decreases in state B. We also assume that $\mathbb{P}_B(\beta_B|\theta) = \mathbb{P}_G(\beta_G|\theta)$. As a result, the expected fraction of a debt issue that is purchased by the relationship investor is lower when she is in state B at all levels of relationship quality. Furthermore, a transition to the bad state increases uncertainty regarding the relationship investor credit supply as it increases the standard deviation of β_t. As in the baseline model, the firm can always start a relationship with a new debt investor, who is in a good state and has a prior quality $\tilde{q}$. The relationship investor transits from the good to the bad state with intensity κ_G and from the bad to the good state with intensity κ_B. If $\kappa_B > 0$ then the shock is transitory. If $\kappa_B = 0$ then the shock is permanent. The

[9] Similarly, Fernando, May, and Megginson (2012) show that firms that had stronger security underwriting relationships with Lehman Brothers before the financial crisis were affected more severely by its collapse.

state s is observable to all agents.

This setup implies that financing policy is a function of the firm's current cash flow x, the quality of the relationship investor q, and, additionally, the relationship investor's state s. The model remains homogeneous of degree one in x so that firm value can be written as $F(x, q, s) = xf(q, s)$. Similarly, the debt and equity values scale linearly in x and additionally depend on the relationship investor's state s; see Lemma 3 in the Appendix.

Because shocks to the relationship investor affect the cost of debt, the firm may decide to request debt from a new relationship investor at the time of refinancing if the current relationship investor is in state B. The firm will do so if the value of starting a new relationship exceeds the value from staying with the current debt investor, i.e. if

$$f(\tilde{q}, G) > f(q, B).$$

Since $f(q, B)$ is monotonically increasing in q, there exists a replacement threshold q_R such that for $q < q_R$ the firm replaces its relationship investor if in state B at the time of refinancing. Using the scaling property of the model, it can be further shown that the optimal strategy for shareholders is to default (i) before maturity if the ratio of the coupon payment to the firm cash flow $z = \frac{c}{x}$ rises above an endogenous threshold $z_D(q, \frac{\rho}{c}, m, s)$; (ii) or on the maturity date of the debt contract if the debt principal ρ exceeds the continuation value of equity $F(x, q, s)$; see Proposition 4 in the Appendix.

Figure 8 plots optimal leverage, debt maturity, average issuance costs, and the fraction of debt issued to outside investors at issuance in the good and bad relationship investor states. The relationship investor credit supply is set to $\beta_G = 0.6$ in the good state and to $\beta_B = 0.2$ in the bad state. The transition intensities are set to $\kappa_B = 0.2$ and $\kappa_G = 0.1$. Other parameters are as in Table 1. The figure shows that when the relationship investor's ability to purchase debt decreases (i.e. when moving to state B), firms with good enough lending relationships ($q > q_R$) do not switch to a new lender. When the relationship quality is below q_R, as shown by the blue area in Figure 8, the negative effects from a lending cut

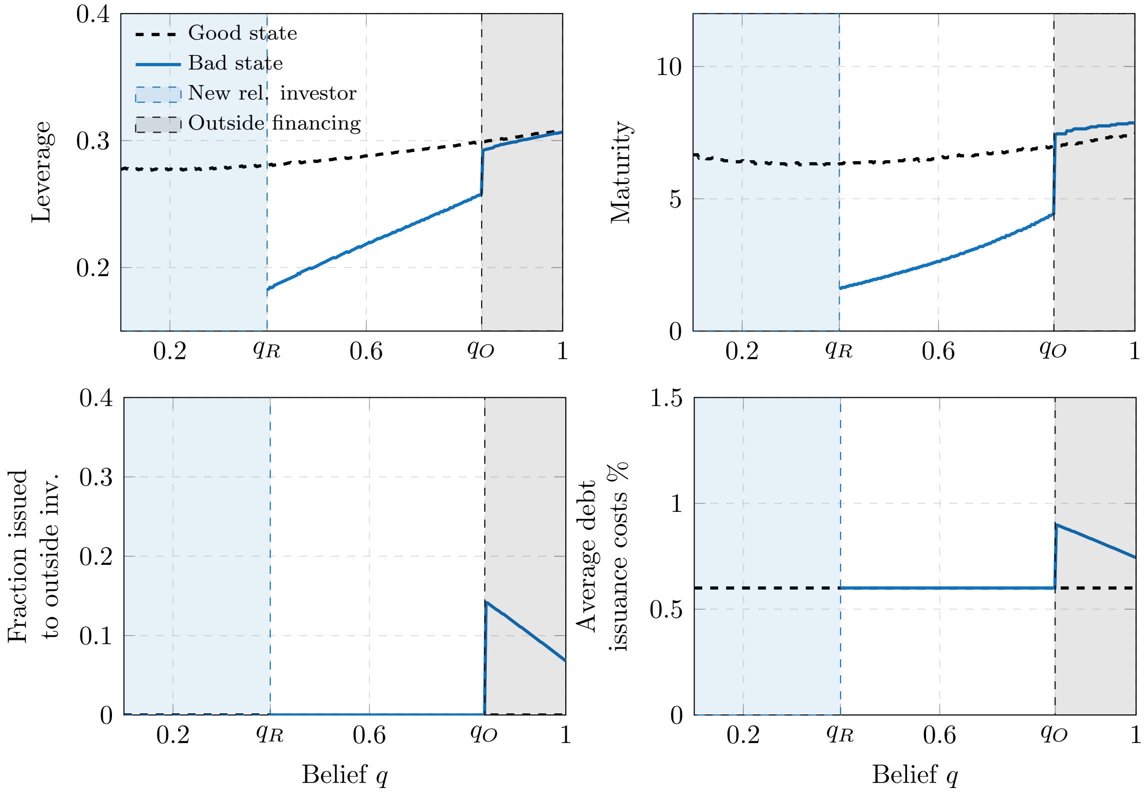


Figure 8: **Effects of an idiosyncratic shocks to the relationship investor.** The blue area indicates the region in which the firm is better off starting a new lending relationship. The grey area indicates the region in which the firm raises debt from outside investors. Debt purchasing capacity is set to $\beta_G = 0.6$ and $\beta_B = 0.2$ in the good and bad states. Transition intensities are set to $\kappa_G = 0.1$ and $\kappa_B = 0.2$. Other parameters are as in Table 1.

on firm value outweigh those from borrowing from a new relationship investor. As a result, firms terminate their current lending relationship and switch to a new lender.

Firms whose lending relationship ($q \in [q_R, q_O]$) are of intermediate quality are those that are the most affected by a shock to the relationship investor. These firms are better off maintaining their relationship with their current debt investor. As shown in Figure 8, they abstain from switching to a new relationship investor or issuing debt to outside investors and, as a result, experience a sharp drop in leverage (moving from the dashed black line to the solid blue line). These firms also significantly shorten the maturity of their debt. By doing so, they retain the possibility of refinancing at better terms with their existing relationship investor in case it moves back to a good state.

The figure also shows that when the quality of the lending relationship is sufficiently high ($q > q_O = 0.83$, gray area in Figure 8), the shock to the relationship investor has little impact on debt maturity and leverage choices as the capital supply from their existing relationship investor is relatively unaffected. In response to any shortage in debt financing, firms sell to outside investors the debt that relationship investor did not buy. Given the higher cost of outside financing, these firms increase the maturity of their debt in the bad state.

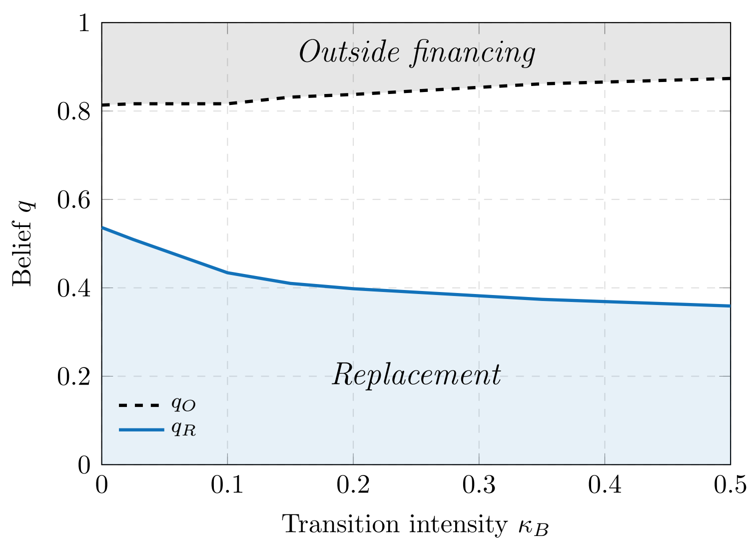


Figure 9: **Transition intensity κ_B, replacement threshold q_R, and outside financing threshold q_O.** For beliefs $q < q_R$, the firm replaces its existing relationship investor in the bad state. For $q > q_O$, the firm issues debt to outside investors if its relationship investor is in the bad state. Debt purchasing capacity is set to $\beta_G = 0.6$ and $\beta_B = 0.2$ in the good and bad states. Transition intensities are set to $\kappa_G = 0.1$ and $\kappa_B = 0.2$. Other parameters are as in Table 1.

Figure 9 shows how varying the expected duration of a negative shock to the relationship lender ($1/\kappa_B$) impacts the decisions to terminate the lending relationship ($q < q_R$) or to raise additional debt from outside investors ($q > q_O$). Decreasing the duration of the negative shock ($1/\kappa_B$) makes the existing relationship investor relatively more valuable to the firm. As a result, the replacement threshold q_R moves down as κ_B increases. Furthermore, when the duration of the shock is shorter, firm issue shorter maturity debt and do not make use of the relatively costlier outside financing. As a result, q_O goes up. In sum, the figure shows that the area in which firms neither change relationship lender nor issue debt with outside investors but instead change drastically their leverage ratio and debt maturity choice grows as the expected duration of the shock decreases. Figure 9 also indicates that relationship stickiness arises even if the shock to the relationship investor is permanent ($\kappa_B = 0$). Overall, these results show that relationship stickiness in debt markets persists even in times when the lender is in distress and the more so when this distress is short-lived.

IV Relationship Investor Rent Extraction

Our analysis so far has assumed that relationship lenders cannot extract rents from borrowing firms and that it is costless for firms to start a relationship with a new debt investor. In practice, relationship lenders may be able to extract rents, in particular from bank-dependent firms that do not have access to public bond markets (see e.g. Rajan (1992), Santos and Winton (2008), Hale and Santos (2009), and Schenone (2010)). Furthermore, it may be costly for firms to form new lending relationships due to the information gathering and processing necessary to assess their creditworthiness.

In this section, we study the impact of rent extraction and of the cost of forming lending relationships on financing choices. To do so, we extend our baseline model by assuming that shareholders and relationship investors bargain over the surplus generated by their relationship. The relationship investor and the firm split the surplus using Nash-bargaining, where $\eta \in [0, 1]$ is the relationship investor's bargaining power. Furthermore, the firm (or the new relationship investor who passes it onto the firm) incurs a cost ϕx when starting a new lending relationship. The surplus $s(q)$ generated by a relationship of quality q is the difference between firm value with the current relationship investor $f(q)$ and firm value with a new relationship investor $f(\tilde{q}) - \phi$: $s(q) = f(q) - (f(\tilde{q}) - \phi)$. Nash-bargaining then implies that shareholders' unlevered equity value is given by

$$f^e(q) = f(q) - \eta s(q) = (1-\eta)f(q) + \eta(f(\tilde{q}) - \phi).$$

Debt and equity values are the same as in equation (1) and (2) with shareholders' unlevered equity value when the debt matures now given by $f^e(q)X_{\tau_m}$. The unlevered firm value $f(q)x$ is defined as before; see equation (3). Finally, the firm starts a new lending relationship only if the benefits exceeds the costs, i.e. if: $f^e(\tilde{q}) - \phi > f^e(q)$. Since $f^e(q)$ is monotonically increasing in q, there exists a threshold $q_R \leq \tilde{q}$ such that for $q < q_R$ the firm terminates the current lending relationship and starts a new one when seeking to issue debt. In our baseline model, we have that $q_R = \tilde{q}$ since $\eta = \phi = 0$.

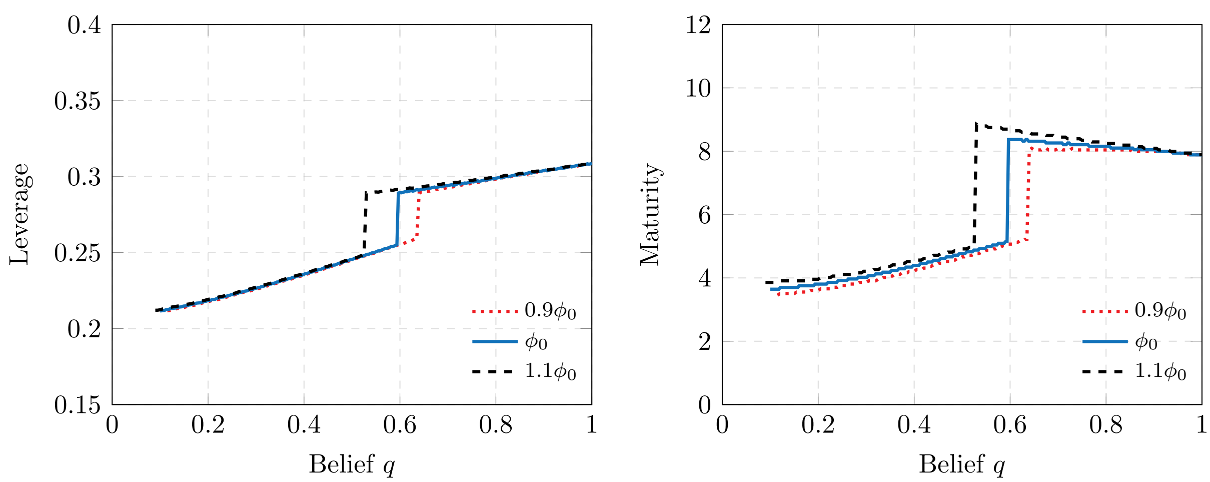


Figure 10: **The impact of changes in the replacement cost ϕ on leverage and debt maturity.** The base case ($\phi_0 = f(0.5|baseline) - f(0.1|baseline)$) is depicted by the blue solid line. The investor's bargaining power η is set to 0 and $\tilde{q} = 0.5$. Other parameters are as in Table 1.

We start by studying the impact of the cost of forming a new relationship ϕ on firms' leverage and debt maturity choices (Figure 10). We take the parameters from the baseline calibration (Table 1) where $\eta = 0$. To allow for variation in q_R, we set $\tilde{q} = 0.5$. The cost of changing the relationship investor ϕ is then set to $\phi_0 = f(0.5|baseline) - f(0.1|baseline)$. This cost implies that firms terminate lending relationships at $q_R = 0.1$ and that, conditional on q, the firm's financing choices are the same as in the baseline model. When the cost of forming new relationship increases, firms terminate the current lending relationship at a slightly lower level of relationship quality, in that q_R decreases from 0.12 to 0.09 as ϕ changes from $0.9\phi_0$ to $1.1\phi_0$. Figure 10 also reveals that capital structure decisions, conditional on q, are relatively unaffected by changes in ϕ. These effects are even smaller for firms with high relationship quality q. The reason is that firms change lenders relatively infrequently and even less so when the relationship quality is high. These results show that allowing for costs of forming lending relationships has almost no influence on firms' financing choices.

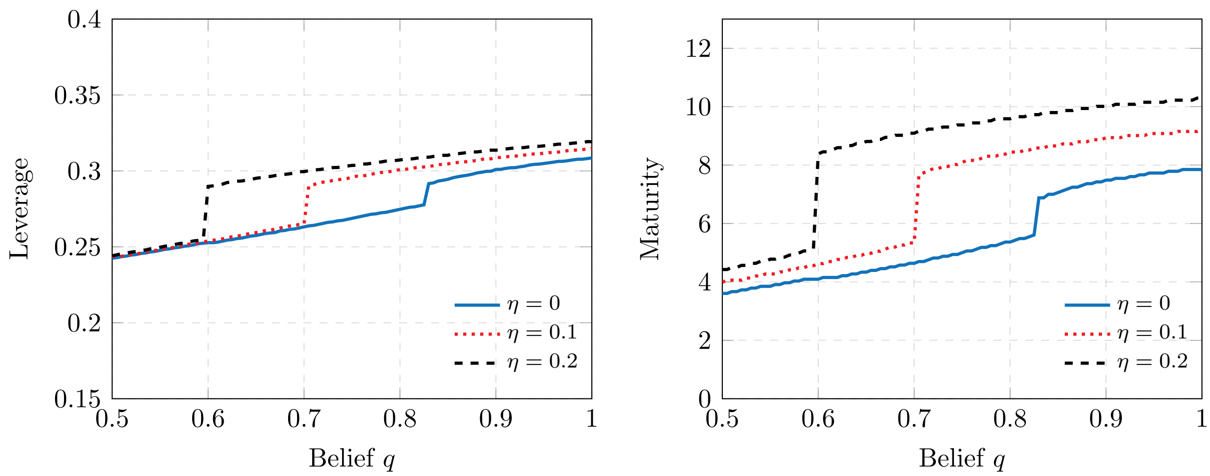


Figure 11: **The impact of changes in the relationship investor's bargaining power η on firm leverage and maturity choices.** We use the same calibration as in Figure 10 with $\phi = 0$.

Next, we study the effects of the relationship investor's bargaining power η on financing decisions. Figure 11 shows that increasing the relationship investor's bargaining power impacts firms' capital structure choices in two ways. First, a higher η means that a smaller fraction of the benefits of the lending relationship $f(q) - f(\tilde{q})$ accrues to shareholders. As a result, the benefits of changing leverage on maturity dates are smaller for shareholders so that they decide to issue longer maturity debt. Second, increasing debt maturity implies that raising financing from outside investors becomes more attractive. As a result, shareholders do so at a lower level of relationship quality. In summary, rent extraction by the relationship investor essentially affects firms' debt maturity choice and debt composition. Leverage remains relatively unaffected.

V Conclusion

In a model of debt dynamics, we study how lending relationships are formed and how they impact leverage and debt maturity choices. In the model, firms build lending relationships through repeated interactions with debt investors. Stronger lending relationships increase firm value by lowering financing costs and by improving access to credit from both relationship and non-relationship investors. Financing risk therefore decreases as lending relationships improve, leading firms with stronger relationships to adopt higher leverage ratios.

Lending relationships are also an important driver of the debt maturity choice. We find that firms with weaker relationship quality issue shorter maturity debt, allowing them to refinance debt at better terms sooner if the relationship improves. Our model makes several predictions about optimal debt maturity that are consistent with the data. For instance, we find that the maturity of debt contracts issued to non-relationship investors is higher than that of debt issued to relationship investors. We also find that average debt maturity decreases with the share of total debt held by relationship investors.

Our analysis also shows that lending relationships are more valuable for firms that have higher target leverage ratios and thus need to raise more debt from investors. In the model, these are the firms with lower cash flow volatility, lower default costs, and higher tax benefits of debt. Finally, our model predicts that idiosyncratic shocks to debt investors that decrease availability of credit supply differently affect firms, depending on the strength of their lending relationships. We find that capital structures of firms with intermediate-quality relationships are affected the most. Overall, our results show that lending relationships have the potential to explain cross-sectional and time-series variation in leverage ratios and debt maturity.

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Appendix

This Appendix includes proofs of the results provided in Section I (the baseline model) and Section III (the two-state model).

A Baseline Model

This section consists of four parts. First, we show that the equity and debt value are homogeneous in X and c (Lemma 1). Second, we establish that the firm value is finite (Lemma 2). Third, we show existence of the firm value (Proposition 1). Fourth, we prove optimality of the default strategy (Proposition 2).

We assume the firm can always issue debt when its outstanding debt matures. In the proves, we establish the results recursively, i.e. the firm can issue debt n more times and we let n go to infinity. Using these recursive arguments simplifies solving the model. This setup implies that if the firm cannot issue debt anymore $n = 0$ then the firm value is given by $f_0(q) = \frac{1-\gamma}{r-\mu}$ and if the firm can issue debt n more times then the firm value is given by $f_n(q)$. The results presented in our paper are those for $f(q) = \lim_{n\to\infty} f_n(q)$. We will abstain from explicitly writing down this recursive argument when it does not lead to confusion.

Lemma 1. *Assume a Markovian default strategy in $z = c/x$ is used. Then the equity and debt value satisfy*

$$\begin{aligned}
E(x, c, q, \rho, m) =& xe\left(\frac{c}{x}, q, \frac{\rho}{c}, m\right), \\
e(z, q, \rho, m) =& \sup_{\tau_D} \mathbb{E}_z^{\mathbb{Q}}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\left((1-\gamma)(1-Z_t) + \frac{1}{m}\left(f(q) - \rho Z_t\right)^+\right) dt\right], \\
D(x, c, q, \rho, m) =& xd\left(\frac{c}{x}, q, \frac{\rho}{c}, m\right), \\
d(z, q, \rho, m) =& \mathbb{E}_z^{\mathbb{Q}}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\left(Z_t + \frac{1}{m}\left(\mathbb{I}_{\{f(q)\geq\rho Z_t\}}\rho Z_t + \mathbb{I}_{\{f(q)<\rho Z_t\}}(1-\alpha)\frac{(1-\gamma)}{r-\mu}\right)\right) dt\right] \\
&+ \mathbb{E}_z^{\mathbb{Q}}\left[e^{-\left(r-\mu+\frac{1}{m}\right)\tau_D}(1-\alpha)\frac{(1-\gamma)}{r-\mu}\right], \\
F(x, q) =& xf(q),
\end{aligned}$$

where the dynamics of Z_t are given by

$$dZ_t = -\mu Z_t dt - \sigma Z_t dB_t^{\mathbb{Q}}$$

where $B_t^{\mathbb{Q}}$ is a standard Brownian motion under the probability measure $\mathbb{Q}$.

Proof. Observe that the equity value with the debt maturity date integrated out can be written as

$$E(x,c,q,\rho,m) = \sup_{\tau_D} \mathbb{E}_x \left[\int_0^{\tau_D} e^{-\left(r+\frac{1}{m}\right)t} X_t \left((1-\gamma)(1-Z_t) + \frac{1}{m}\left(f(q) - \frac{\rho}{c} Z_t \right)^+ \right) dt \right].$$

Using Girsanov's theorem, we can apply the following change of measure (see Harrison (2013) Theorem 1.17 on page 12)

$$\mathbb{Q}(A) = \mathbb{E}_0 \left[\mathbb{I}_{\{A\}} e^{-\frac{\sigma^2}{2}+\sigma B_t} \right] = \mathbb{E}_0 \left[\mathbb{I}_{\{A\}} e^{-\mu t} \frac{X_t}{X_0} \right] \ \forall A \subseteq \mathcal{F}_t,$$

which yields

$$\begin{aligned} E(x,c,q,\rho,m) &= \sup_{\tau_D} x \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t} \left((1-\gamma)(1-Z_t) + \frac{1}{m}\left(f(q) - \frac{\rho}{c} Z_t \right)^+ \right) dt \middle| Z_0 = \frac{c}{x} \right] \\ &= xe\left(\frac{c}{x}, q, \frac{\rho}{c}, m \right) \end{aligned}$$

where

$$dZ_t = -\mu Z_t dt - \sigma Z_t dB_t + \sigma^2 Z_t dt = -\mu Z_t dt - \sigma Z_t (dB_t - \sigma dt) = -\mu Z_t dt - \sigma Z_t dB_t^{\mathbb{Q}}.$$

The same change of measure can be applied to the debt value. □

Remark 1: for ease of exposition, we will drop $\mathbb{Q}$ from the expectations.

Remark 2: We normalize $c = 1$ in the rest of the proofs with out loss of generality.

Lemma 2. *The firm value, if it exists, is finite.*

Proof. The lower bound for the firm value is the unlevered value of its assets. From above, the firm value is bounded by

$$\begin{aligned} & e\left(z,q,\rho,m\right) + d\left(z,q,\rho,m\right) \\ & \le \mathbb{E}_z \left[\int_0^{\infty} e^{-\left(r-\mu+\frac{1}{m}\right)t} \left((1-\gamma) + \gamma Z_t + \frac{1}{m}\left(f(q) + \frac{(1-\alpha)(1-\gamma)}{r-\mu} \right) \right) dt \right] \\ & \le \frac{(1-\gamma) + \frac{1}{m}\left(\sup_q f(q) + \frac{(1-\alpha)(1-\gamma)}{r-\mu} \right)}{r-\mu+\frac{1}{m}} + \frac{\gamma z}{r+\frac{1}{m}}, \end{aligned} \tag{5}$$

i.e. the firm value is smaller than the present value of all cash flows and tax benefits until maturity plus the payoff at maturity when there is default and when there is no default.

Given Assumption 1 and the fact that $\inf\{\mathcal{M}\} \geq \underline{m} > 0$, we have that[10]

$$\sup_q f(q) \leq \sup_{m\in\mathcal{M}} \frac{(1-\gamma) + \frac{1}{m}\left(\sup_q f(q) + \frac{(1-\alpha)(1-\gamma)}{r-\mu}\right)}{r-\mu+\frac{1}{m}} + \frac{\gamma\bar{z}}{r+\frac{1}{m}},$$

$$\sup_q f(q) \leq \sup_{m\in\mathcal{M}} \frac{(1-\gamma) + \frac{1}{m}\frac{(1-\alpha)(1-\gamma)}{r-\mu} + \frac{\left(r-\mu+\frac{1}{m}\right)\gamma\bar{z}}{r+\frac{1}{m}}}{r-\mu} < \infty.$$

□

Proof of Proposition 1. Let $f_0(q) = \frac{(1-\gamma)}{r-\mu}$ be the unlevered firm value assuming it can no longer issue debt. Furthermore, let $f_n(q)$ be the unlevered firm value assuming the firm can issue debt n times. Given that we know $f_0(q)$, we can construct any $f_n(q)$ recursively since the face value and maturity of the debt requested, the prices, and default decisions are made sequentially. Furthermore, by construction

$$f_{n+1}(q) \geq f_n(q).$$

Since we know from Lemma 2 that the firm value is finite, the monotone convergence theorem then tells us that $f(q) = \lim_{n\to\infty} f_n(q)$ exists. □

The final step is deriving the optimal default strategy.

Proof of Proposition 2. Optimality of the default strategy at maturity follows from the fact that it's a static choice.

Given this default decision at maturity, the equity value can be written as

$$e(z,q,\rho,m|z_D) = \mathbb{E}_z\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\left((1-\gamma)\,(1-Z_t) + \frac{1}{m}(f(q)-\rho Z_t)^+\right)dt\right]$$

where $\tau_D = \inf\{t>0|Z_t > z_D\}$. The goal is to show that the equity value where z_D satisfies the smooth pasting condition exists and that this equity value solves the Hamilton-Jaccobi-Bellman equation for our optimal stopping problem.

We need to show that a solution z_D to the following equation exists

$$e_z(z_D,q,\rho,m|z_D) = 0.$$

[10]In the equation below we implicitly assume that $\sup_q f(q)$ is finite but this argument works because we establish our results recursively (i.e. assuming the firm can only issue debt n more times) and $f_0(q) = \frac{1-\gamma}{r-\mu} < \infty$. We just want to show that $f_n(q)$ cannot out grow a bound.

Define $\hat{z}$ as the solution to

$$\left((1-\gamma)(1-\hat{z})+\frac{1}{m}(f(q)-\rho\hat{z})^{+}\right)=0$$

For $z<z_D<\hat{z}$, we must have that $e(z,q,\rho,m|z_D)>0$ since the cash flow is always strictly positive. This result directly implies that $e_z(z_D,q,\rho,m|z_D)\leq 0$ for $z_D<\hat{z}$.

Furthermore, as $z_D\to\infty$ we must have that stopping is optimal since stopping is optimal for

$$e(z,q,\rho,m|z_D)\leq\sup_{\tau}\mathbb{E}_z\left[\int_0^{\tau}e^{-\left(r-\mu+\frac{1}{m}\right)t}\left((1-\gamma)(1-Z_t)+\frac{1}{m}f(q)\right)dt\right],$$

which is a standard optimal stopping problem, which has a threshold solution (see Harrison (2013) Chapter 5). Let $\hat{z}_D$ be the threshold solution to this auxiliary problem. Then for $z_D>\hat{z}_D$ we must have that for z close enough to z_D

$$e(z,q,\rho,m|z_D)<0$$

and therefore $e_z(z_D,q,\rho,m|z_D)\geq 0$. Continuity in $e_z(z_D,q,\rho,m|z_D)$ with respect to z_D (see Lemma A.6 in Hugonnier et al. (2015)) then implies that a solution to

$$e_z(z_D,q,\rho,m|z_D)=0$$

exists.

The next step is to show that this equity value satisfies the Hamilton-Jacobi-Bellman equation. For $z>z_D$, we have that $e(z|z_D)=0$. For $z<z_D$ it solves the Feynman-Kac ordinary differential equation. Furthermore, at z_D (approaching it from the right) we have that

$$0=(1-\gamma)(1-z_D)+\frac{1}{m}\left(f(q)-\rho z_D\right)^{+}+\frac{1}{2}\sigma^2 z_D^2 e_{zz}(z_D,q,\rho,m|z_D)$$

Assume $e_{zz}(z_D,q,\rho,m|z_D)<0$, then $(1-\gamma)(1-z_D)+\frac{1}{m}\left(f(q)-\rho z_D\right)^{+}>0$. This would imply that $e(z,q,\rho,m|z_D)>0$ for $z<z_D$ since the cash flow is always positive. This result contradicts the fact that $e_z(z,q,\rho,m|z_D)>0$ in some left neighborhood of z_D (since $e_{zz}(z_D,q,\rho,m|z_D)<0$ and $e_z(z_D,q,\rho,m|z_D)=0$) and $e(z_D,q,\rho,m|z_D)=0$. Therefore, we must have that $e_{zz}(z_D,q,\rho,m|z_D)\geq 0$ and, as a result, $(1-\gamma)(1-z_D)+\frac{1}{m}\left(f(q)-\rho z_D\right)^{+}\leq 0$. Since the cash flow is decreasing in z, this proves that the equity value satisfies the Hamilton-Jacobi-Bellman equation for $z\geq z_D$.

In some left neighborhood of z_D it must be that $e(z|z_D)>0$. Assume $e_{zz}(z_D,q,\rho,m|z_D)=0$. Then $(1-\gamma)(1-z_D)+\frac{1}{m}\left(f(q)-\rho z_D\right)^{+}=0$ and, therefore, the cash flow is positive for any $z < z_D$ and the equity value is always positive. Assume $e_{zz}(z_D, q, \rho, m|z_D) > 0$, then $e_z(z|z_D) < 0$ in some left neighborhood of z_D and, therefore, $e(z, q, \rho, m|z_D) > 0$ in this neighborhood.

For $z \leq z_D$, we only need to show that $e(z|z_D) \geq 0$. Assume this is not the case. Then there exists a local minimum $\tilde{z} \in (0, z_D)$ such that[11]

$$\begin{aligned}
e(\tilde{z}, q, \rho, m|z_D) &< 0, \\
e_z(\tilde{z}, q, \rho, m|z_D) &= 0, \\
e_{zz}(\tilde{z}, q, \rho, m|z_D) &\geq 0, \\
(1-\gamma)(1-\tilde{z}) + \frac{1}{m}\left(f(q) - \rho\tilde{z}\right)^+ &\geq 0.
\end{aligned}$$

where the last inequality follows from the fact that for some $z \in [\tilde{z}, z_D]$ the equity value is positive and thus the cash flow must be positive for some $z \in [\tilde{z}, z_D]$ and, as a consequence, also at $\tilde{z}$. But these inequalities lead to a contradiction

$$\begin{aligned}
0 &> \left(r - \mu + \frac{1}{m}\right) e(\tilde{z}, q, \rho, m|z_D) \\
&= (1-\gamma)(1-\tilde{z}) + \frac{1}{m}\left(f(q) - \rho\tilde{z}\right)^+ - \mu\tilde{z}e_z(\tilde{z}, q, \rho, m|z_D) + \frac{1}{2}\sigma^2\tilde{z}^2 e_{zz}(\tilde{z}, q, \rho, m|z_D) \\
&\geq 0.
\end{aligned}$$

Therefore, the equity value must be non-negative for $z \leq z_D$. This proves that the equity value satisfies the Hamilton-Jacobi-Bellman equation for $z \leq z_D$

Finally, the equity value function is piecewise C^2. Therefore, using Theorem 5.1 of Harrison (2013), we conclude that the optimal default strategy is a threshold default strategy where the threshold follows from the smooth pasting condition.

□

B Two-State Model

This section contains proofs of propositions provided in Section III and is organized as follows. First, we show that the equity and debt value are homogeneous in X and c (Lemma 3). Second, we show existence of the firm value (Proposition 3). Third, we show optimality of the default strategy (Proposition 4).

Lets denote by $S_t \in \{G, B\}$ the state the firm's relationship investor is in at time t. We can establish that

[11] Observe that $\lim_{z\to 0} e(z|z_D) > 0$, which follows from the ordinary differential equation the equity value satisfies.

Lemma 3. *Assume a Markovian default strategy in $z = c/x$ and s is used then the equity and debt value satisfy*

$$\begin{aligned}
E(x, c, q, \rho, m, s) =& xe\left(\frac{c}{x}, q, \frac{\rho}{c}, m, s\right), \\
e(z, q, \rho, m, s) =& \sup_{\tau_D}\left\{\mathbb{E}^{\mathbb{Q}}_{z,s}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}(1-\gamma)(1-Z_t)dt\right]\right. \\
&\left. + \mathbb{E}^{\mathbb{Q}}_{z,s}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\frac{1}{m}\left(\max\{f(\tilde{q}, G), f(q, S_t)\} - \rho Z_t\right)^+ dt\right]\right\},
\end{aligned}$$

$$\begin{aligned}
D(x, c, q, \rho, m, s) =& xd\left(\frac{c}{x}, q, \frac{\rho}{c}, m, s\right), \\
d(z, q, \rho, m, s) =& \mathbb{E}^{\mathbb{Q}}_{z,s}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\left(Z_t + \frac{1}{m}\mathbb{I}_{\{\max\{f(\tilde{q},G), f(q,S_t)\}\geq \rho Z_t\}}\rho Z_t\right)dt\right] \\
&+ \mathbb{E}^{\mathbb{Q}}_{z,s}\left[\int_0^{\tau_D} e^{-\left(r-\mu+\frac{1}{m}\right)t}\frac{1}{m}\mathbb{I}_{\{\max\{f(\tilde{q},G), f(q,S_t)\}< \rho Z_t\}}(1-\alpha)\frac{(1-\gamma)}{r-\mu}dt\right] \\
&+ \mathbb{E}^{\mathbb{Q}}_{z,s}\left[e^{-\left(r-\mu+\frac{1}{m}\right)\tau_D}(1-\alpha)\frac{(1-\gamma)}{r-\mu}\right], \\
F(x, q, s) =& xf(q, s),
\end{aligned}$$

where the dynamics of Z_t are given by

$$dZ_t = -\mu Z_t dt - \sigma Z_t dB_t^{\mathbb{Q}}$$

where $B_t^{\mathbb{Q}}$ is a standard Brownian motion under the probability measure $\mathbb{Q}$.

Proof. The proof is exactly the same as for the baseline model (see the proof of Lemma 1). □

Proposition 3 (Firm value)**.** *Firm value exists, is finite, and satisfies $F(x, q, s) = xf(q, s)$.*

Proof of Proposition 3. The same arguments as in the proof of Lemma 2 imply that $f(q, s)$ is bounded from above and below. In equation (5) we now take the supremum over both q and s.

Let $f_0(q, s) = \frac{(1-\gamma)}{r-\mu}$ be the unlevered firm value assuming it can no longer issue debt. Furthermore, let $f_n(q, s)$ be the unlevered firm value assuming the firm can issue debt n times and the firm's relationship investor is in state s. Given that we know $f_0(q, s)$, we can construct any $f_n(q, s)$ recursively since the face value and maturity of the debt requested, the prices, and default decisions are made sequentially. Furthermore, by construction

$$f_{n+1}(q,s) \geq f_n(q,s).$$

Since we know that the firm value is finite, the monotone convergence theorem then tells us that $f(q,s) = \lim_{n\to\infty} f_n(q,s)$ exists. □

Proposition 4 (Optimal default). *The optimal strategy for shareholders is to default:*

1. ***Coupon default**: Before maturity if the ratio of the coupon payment to the firm cash flow $z = \frac{c}{x}$ rises above an endogenous threshold $z_D(q, \frac{\rho}{c}, m, s)$, that is determined by the equity value's smooth pasting conditions.*

2. ***Principal default**: On the maturity date of the debt contract if the debt principal ρ exceeds the continuation value of equity $F(x,q,s)$.*

Proof of Proposition 4. Optimality of the default strategy at maturity follows from the fact that it's a static choice.

For the coupon default strategy, we establish optimality recursively. Furthermore, we will integrate out the switching of the relationship investor's state. Fixing $f(q,s)$, ρ, and m and normalizing $c = 1$, we define

$$\begin{aligned}
\tau_D^{si} =& \inf\left\{t > 0 | Z_t \geq z_D^{si}\right\},\\
e_{s0}(z|0) =& 0,\\
e_{si}\left(z|z_D^{si}\right) =& \mathbb{E}_z\left[\int_0^{\tau_D^{si}} e^{-\left(r-\mu+\frac{1}{m}+\kappa_s\right)t}\left((1-\gamma)(1-Z_t) + \frac{1}{m}\left(\max\{f(\tilde{q},G), f(q,s)\} - \rho Z_t\right)^+\right)dt\right]\\
&+\mathbb{E}_z\left[\int_0^{\tau_D^{si}} e^{-\left(r-\mu+\frac{1}{m}+\kappa_s\right)t}\kappa_s e_{s'(i-1)}\left(Z_t\middle|z_D^{s'(i-1)}\right)dt\right]
\end{aligned}$$

where s' is the opposite state of s. Observe that by construction $e_{B0}(z|0) = 0$ and therefore $e'_{B0}(z|0) \leq 0$.

Take an uneven i and assume that $e_{B(i-1)}\left(z\middle|z_D^{B(i-1)}\right) \geq 0$ and $e'_{B(i-1)}\left(z\middle|z_D^{B(i-1)}\right) \leq 0$. We first want to establish that there exists a threshold z_D^{Gi} such that

$$e'_{Gi}\left(z_D^{Gi}\middle|z_D^{Gi}\right) = 0.$$

Define $\hat{z}$ as the solution to

$$\left((1-\gamma)(1-\hat{z})+\frac{1}{m}\left(\max\{f(\tilde{q},G),f(q,s)\}-\rho\hat{z}\right)^{+}\right)=0$$

For $z<z_D<\hat{z}$, we must have that $e_{Gi}(z|z_D)>0$ since the cash flow is always strictly positive. This result directly implies that $e'_{Gi}(z_D|z_D)\leq 0$ for $z_D<\hat{z}$.

Furthermore, as $z_D\to\infty$ we must have that stopping is optimal since stopping is optimal for

$$\begin{aligned} & e_{Gi}(z|z_D) \\ & \leq \sup_{\tau}\Bigg\{\mathbb{E}_z\left[\int_0^{\tau} e^{-\left(r-\mu+\frac{1}{m}+\kappa_G\right)t}\left((1-\gamma)(1-Z_t)+\frac{1}{m}\max\{f(\tilde{q},G),f(q,G)\}\right)dt\right] \\ & \qquad +\mathbb{E}_z\left[\int_0^{\tau} e^{-\left(r-\mu+\frac{1}{m}+\kappa_G\right)t}\kappa_G e_{B(i-1)}\left(0\middle|z_D^{B(i-1)}\right)dt\right]\Bigg\} \end{aligned} \tag{6}$$

which is a standard optimal stopping problem, which has a threshold solution (see Harrison (2013) Chapter 5). Let $\hat{z}_D$ be the threshold solution to this auxiliary problem then for $z_D>\hat{z}_D$ we must have that for z close enough to z_D

$$e_{Gi}(z|z_D)<0$$

and therefore $e'_{Gi}(z_D|z_D)\geq 0$. Continuity in $e'_{Gi}(z_D|z_D)$ with respect to z_D (see Lemma A.6 in Hugonnier et al. (2015)) then implies that a solution to

$$e'_{Gi}\left(z_D^{Gi}\middle|z_D^{Gi}\right)=0$$

exists.

The next step is showing that this equity value satisfies the Hamilton-Jacobi-Bellman equation. For $z>z_D^{Gi}$, we have that $e_{Gi}\left(z\middle|z_D^{Gi}\right)=0$ while for $z<z_D^{Gi}$ it solves the Feynman-Kac ordinary differential equation. Furthermore, at z_D^{Gi} (approaching it from the right) we have that

$$\begin{aligned} 0 = & (1-\gamma)\left(1-z_D^{Gi}\right)+\frac{1}{m}\left(\max\{f(\tilde{q},G),f(q,G)\}-\rho z_D^{Gi}\right)^{+}+\kappa_G e_{B(i-1)}\left(z\middle|z_D^{B(i-1)}\right) \\ & +\frac{1}{2}\sigma^2\left(z_D^{Gi}\right)^2 e''_{Gi}\left(z_D^{Gi}\middle|z_D^{Gi}\right) \end{aligned}$$

Assume $e''_{Gi}\left(z_D^{Gi}|z_D^{Gi}\right)<0$, then $(1-\gamma)\left(1-z_D^{Gi}\right)+\frac{1}{m}\left(\max\{f(\tilde{q},G),f(q,G)\}-\rho z_D^{Gi}\right)^{+}+\kappa_G e_{B(i-1)}\left(z_D^{Gi}\middle|z_D^{B(i-1)}\right)>0$. Since the cash flow is always positive, this would imply that

$e_{Gi}\left(z \middle| z_D^{Gi}\right) > 0$ for $z < z_D^{Gi}$. This result contradicts the fact that $e'_{Gi}\left(z \middle| z_D^{Gi}\right) > 0$ in some left neighborhood of z_D^{Gi} (since $e''_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) < 0$ and $e'_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) = 0$) and $e_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) = 0$. Therefore, we must have that $e''_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) \geq 0$ and as a result $(1-\gamma)\left(1 - z_D^{Gi}\right) + \frac{1}{m}\left(\max\{f(\tilde{q}, G), f(q, G)\} - \rho z_D^{Gi}\right)^+ + \kappa_G e_{B(i-1)}\left(z_D^{Gi} \middle| z_D^{B(i-1)}\right) \leq 0$. Since the cash flow is decreasing in z, this proves that the equity value satisfies the Hamilton-Jacobi-Bellman equation for $z \geq z_D^{Gi}$.

In some left neighborhood of z_D^{Gi} it must be that $e_{Gi}\left(z \middle| z_D^{Gi}\right) > 0$. Assume $e''_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) = 0$ then $(1-\gamma)\left(1 - z_D^{Gi}\right) + \frac{1}{m}\left(\max\{f(\tilde{q}, G), f(q, G)\} - \rho z_D^{Gi}\right)^+ + \kappa_G e_{B(i-1)}\left(z_D^{Gi} \middle| z_D^{B(i-1)}\right) = 0$ and therefore the cash flow is positive for any $z < z_D^{Gi}$ and the equity value is always positive. Assume $e''_{Gi}\left(z_D^{Gi} \middle| z_D^{Gi}\right) > 0$ then $e'_{Gi}\left(z \middle| z_D^{Gi}\right) < 0$ in some left neighborhood of z_D^{Gi} and therefore $e\left(z_D^{Gi} \middle| z_D^{Gi}\right) > 0$ in this same neighborhood.

For $z \leq z_D^{Gi}$, we only need to show that $e_{Gi}\left(z \middle| z_D^{Gi}\right) \geq 0$. We will more generally show that $e'_{Gi}\left(z \middle| z_D^{Gi}\right) \leq 0$. For $z \geq z_D^{Gi}$ this result trivially holds. Assume this is not the case for some $z \in \left[0, z_D^{Gi}\right]$. First, it must be the case that

$$
\begin{aligned}
& e_{Gi}\left(0 \middle| z_D^{Gi}\right) \\
&= \frac{(1-\gamma) + \frac{1}{m}\max\{f(\tilde{q}, G), f(q, G)\} + \kappa_G e_{B(i-1)}\left(0 \middle| z_D^{B(i-1)}\right)}{r - \mu + \frac{1}{m} + \kappa_G} \\
&= \int_0^\infty e^{-\left(r-\mu+\frac{1}{m}+\kappa_G\right)t}\left((1-\gamma) + \frac{1}{m}\max\{f(\tilde{q}, G), f(q, G)\} + \kappa_G e_{B(i-1)}\left(0 \middle| z_D^{B(i-1)}\right)\right) dt \\
&\geq e_{Gi}\left(z \middle| z_D^{Gi}\right).
\end{aligned}
$$

Therefore, there must exist a local minimum $z_1 > 0$ and local maximum $z_2 (> z_1)$ such that

$$
\begin{aligned}
& e_{Gi}\left(z_1 \middle| z_D^{Gi}\right) < e_{Gi}\left(z_2 \middle| z_D^{Gi}\right), \\
& e'_{Gi}\left(z_1 \middle| z_D^{Gi}\right) = e'_{Gi}\left(z_2 \middle| z_D^{Gi}\right) = 0, \\
& e''_{Gi}\left(z_1 \middle| z_D^{Gi}\right) \geq 0 \geq e'_{Gi}\left(z_2 \middle| z_D^{Gi}\right), \\
& CF(z) = (1-\gamma)(1-z) + \frac{1}{m}\left(\max\{f(\tilde{q}, G), f(q, G)\} - \rho z\right)^+ + \kappa_G e_{B(i-1)}\left(z \middle| z_D^{B(i-1)}\right) \\
& CF(z_1) > CF(z_2).
\end{aligned}
$$

But these inequalities lead to a contradiction,

$$
\begin{aligned}
0 &> \left(r-\mu+\frac{1}{m}+\kappa_G\right)\left(e_{Gi}\left(z_1|z_D^{Gi}\right)-e_{Gi}\left(z_2|z_D^{Gi}\right)\right) \\
&= CF(z_1)-CF(z_2)+\frac{1}{2}\sigma^2 z_1^2 e''_{Gi}\left(z_1|z_D^{Gi}\right)-\frac{1}{2}\sigma^2 z_2^2 e''_{Gi}\left(z_2|z_D^{Gi}\right)>0,
\end{aligned}
$$

and as a result $e'_{Gi}\left(z\middle|z_D^{Gi}\right)\le 0$. Therefore, the equity value must be non-negative for $z\le z_D^{Gi}$. This proves that the equity value satisfies the Hamilton-Jacobi-Bellman equation for $z\le z_D^{Gi}$

Finally, the equity value function is piecewise C^2. Using Theorem 5.1 (Harrison (2013)), we conclude that the optimal default strategy is a threshold default strategy where the threshold follows from the smooth pasting condition.

The same arguments as for the case $s=G$ and i allow us to establish the existence of a threshold $z_D^{B(i+1)}$ that solves $e'_{B(i+1)}\left(z_D^{B(i+1)}\middle|z_D^{B(i+1)}\right)=0$ and which is the optimal stopping threshold with $e_{B(i+1)}\left(z\middle|z_D^{B(i+1)}\right)\ge 0$ and $e'_{B(i+1)}\left(z\middle|z_D^{B(i+1)}\right)\le 0$. We can then iteratively obtain the default thresholds for all values i (with $s=G$ when i is uneven and $s=B$ when i is even).

There exists an upper bound on the default threshold (using equation (6) and the fact that $e_{si}\left(0\middle|z_D^{si}\right)\le\sup_{q,s}f(q,s)<\infty$) $z_D^{si}\le\bar{z}_D<\infty$). Furthermore, $e_{si}\left(z\middle|z_D^{si}\right)$ follows from the monotone mapping

$$
\begin{aligned}
&\mathcal{T}_s(e)= \\
&\sup_{\tau_D^s}\left\{\mathbb{E}_z\left[\int_0^{\tau_D^s}e^{-\left(r-\mu+\frac{1}{m}+\kappa_s\right)t}\left((1-\gamma)(1-Z_t)+\frac{1}{m}\left(\max\{f(\tilde{q},G),f(q,s)\}-\rho Z_t\right)^+\right)dt\right]\right. \\
&\left.\qquad\qquad+\mathbb{E}_z\left[\int_0^{\tau_D^s}e^{-\left(r-\mu+\frac{1}{m}+\kappa_s\right)t}\kappa_s e(Z_t)dt\right]\right\}.
\end{aligned}
$$

As a result, $e_{B2}\left(z\middle|z_D^{B2}\right)\ge e_{B0}\left(z\middle|z_D^{B0}\right)$ and therefore $e_{G3}\left(z\middle|z_D^{G3}\right)\ge e_{G1}\left(z\middle|z_D^{G1}\right)$. This implies that the sequence $(z_D^{G(2i-1)},z_D^{B(2i)})$ is increasing in i since $e_{G(2i-1)}\left(z\middle|z_D^{G(2i-1)}\right)$ and $e_{B(2i)}\left(z\middle|z_D^{B(2i)}\right)$ are increasing in i. The monotone convergence theorem then implies that $\lim_{i\to\infty}\left(z_D^{G(2i-1)},z_D^{B(2i)}\right)$ converges. Call this limit (z_D^G,z_D^B). We know that both these default thresholds satisfy the smooth pasting condition (simultaneously) and that both stopping times are optimal (since each stopping time in the sequence is optimal given the stopping time in the other state). □