Multiple choice test 10 Answers

[1.](i)

Let $g(y) = \int\limits_{ln(y)}^{0} xln(e^x - x + x^2)dx \to 0$ when $y \to 1^-$.
$g'(y) = -\frac{1}{y}.ln(y).ln(y - ln(y) + ln(y)^2) \to 0$ when $y \to 1^-$.

$((y-1)^3)' = 3(y-1)^2 \to 0$ when $y \to 1^-$.

$g'(y) \sim -\frac{1}{y}(y-1)ln((y-(y-1)+\frac{3}{2}(y-1)^2 \sim -\frac{3}{2y}(y-1)^3$ when $y \to 1^-$.
Then $\lim_{y\to 1^-} \frac{g(y)}{(y-1)^3} = \lim_{y\to 1^-} \frac{g'(y)}{3(y-1)^2} = \lim_{y\to 1^-} \frac{y-1}{2y} = 0$
Where we used L'Hospital.

[2.](i)TT
$D := \{(x,y) \in \mathbb{R}^2 : 0 < x < \frac{\pi}{2}, 0 < y < x\}$.
$I = \int\limits_0^{\pi}\int\limits_0^{x} xcos(x+y)dydx = \int\limits_0^{\pi}[xsin(x+y)]_0^x dx = \int\limits_0^{\pi} x(sin(2x) - sin(x))dx$
By integrating by parts, we obtain :
$I = [x(cos(x) - \frac{cos(2x}{2}]_0^{\pi} - \int\limits_0^{\pi} cos(x) - \frac{1}{2}cos(2x)dx = -\frac{3}{2}\pi - [sin(x) - \frac{1}{4}sin(2x)]_0^{\pi} = -\frac{3}{2}\pi$
On the other hand, the area of D is equal to $\frac{\pi^2}{2}$. By the mean value theorem, there is (x_0, y_0) such that $-\frac{3}{2}\pi =\mid D \mid .x_0cos(x_0+y_0)$ which means : $\mid D \mid .x_0cos(x_0+y_0) = -\frac{3}{\pi}$

[3.] (ii)
Let $g(y) = \int\limits_1^{e^{y^2}} cos(\sqrt{ln(x)})dx \to 0$ when $y \to 0$
$g'(y) = 2ye^{y^2}cos(y) \to 0$ when $y \to 0$.
$g''(y) = 2e^{y^2}cos(y) + 4y^2e^{y^2}cos(y) - 2ye^{y^2}sin(y) \to 2$ when $y \to 0$.
By L'hospital,
$\lim_{y\to 0} \frac{g(y)}{cos(y)-1} = \lim_{y\to 0} \frac{g'(y)}{-sin(y)} = \lim_{y\to 0} \frac{g''(y)}{-cos(y)} = -2$

[4.] (iii)
$\int\int\limits_D f(x,y)dxdy = \int\limits_0^{\xi} \frac{x}{1+x^2}dx \int\limits_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} \frac{ycos(y^2)}{sin^2(y^2}dy$.

But $\int\limits_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} \frac{ycos(y^2)}{sin^2(y^2}dy = [-\frac{1}{2sin(y^2)}]_{\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{3\pi}}{2}} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0.$

[5.](iv)

$$\int_0^3\int_0^3 \frac{1}{(1+y^2)^2}dydx = \int_0^3[-\frac{1}{1+y^2}]_0^x dx = \int_0^3 -\frac{1}{10}+\frac{1}{1+x^2}dx = -\frac{3}{10}+arctan(3).$$

[6.](iv)

2 conditions for (x,y) to be in D : $\begin{cases} 1 < xy < 2 \\ x^2 < y < 2x^2 \end{cases}$

Which gives us $2x^3 > 1$ and $x^3 > 2 \iff \begin{cases} x > \frac{1}{2^{\frac{1}{3}}} \\ x < 2^{\frac{1}{3}} \end{cases}$ (We notice that $y > 0$, hence $x > 0$).

Now, if x is fixed in $[\frac{1}{2^{\frac{1}{3}}}, 2^{\frac{1}{3}}]$. We need $\begin{cases} xy \in]1,2[\\ y \in]x^2, 2x^2[\end{cases}$.

Hence $y \in]max(\frac{1}{x}, x^2), min(\frac{2}{x}, 2x^2)[\neq \emptyset$. Now, we may notice that :

$$max(\frac{1}{x}, x^2) = \begin{cases} \frac{1}{x} \text{ if } x \in]\frac{1}{2^{\frac{1}{3}}}, 1[\\ x^2 \text{ if } x \in]1, 2^{\frac{1}{3}}[\end{cases} \quad \text{and } min(\frac{2}{x}, 2x^2) = \begin{cases} 2x^2 \text{ if } x \in]\frac{1}{2^{\frac{1}{3}}}, 1[\\ \frac{2}{x} \text{ if } x \in]1, 2^{\frac{1}{3}}[\end{cases}$$

Hence : $\int\int_D x^3+y^3dydx = \int_{\frac{1}{2^{\frac{1}{3}}}}^{1}\int_{\frac{1}{x}}^{2x^2} x^3+y^3dydx + \int_1^{2^{\frac{1}{3}}}\int_{x^2}^{\frac{2}{x}} x^3+y^3dydx =$

$$= \int_{\frac{1}{2^{\frac{1}{3}}}}^{1} [x^3y+\frac{1}{4}y^4]_{\frac{1}{x}}^{2x^2}dx + \int_1^{2^{\frac{1}{3}}}[x^3y+\frac{1}{4}y^4]_{x^2}^{\frac{2}{x}}dx = \int_{\frac{1}{2^{\frac{1}{3}}}}^{1} 2x^5+4x^8-x^2-\frac{1}{4x^4}dx + \int_1^{2^{\frac{1}{3}}} 2x^2+\frac{4}{x^4}-x^5-\frac{1}{4}x^8 =$$

$$= [\frac{1}{3}x^6+\frac{4}{9}x^9-\frac{1}{3}x^3+\frac{1}{12x^3}]_{\frac{1}{2^{\frac{1}{3}}}}^{1} + [\frac{2}{3}x^3-\frac{4}{3x^3}-\frac{1}{6}x^6-\frac{1}{36}x^9]_1^{2^{\frac{1}{3}}} = \frac{37}{36}$$

[7](i)

$$| D | = \int\int_D f(x,y)dydx = \int_0^1\int_0^2 4+x^2-y^2dydx = \int_0^1[4y+x^2y-\frac{1}{3}y^3]_0^2dx = \int_0^1 8+2x^2-\frac{8}{3}dx =$$

$$= [\frac{16}{3}x+\frac{2}{3}x^3]_0^1 = \frac{18}{3} = 6$$

[8.](ii)

$$\int_0^{\pi}\int_0^{cos(\theta)} 4e^{sin(\theta)}drd\theta = \int_0^{\pi}[4re^{sin(\theta)}]_0^{cos(\theta)}d\theta = \int_0^{\pi} 4cos(\theta)e^{sin(\theta)}d\theta = [4e^{sin(\theta)}]_0^{\pi} = 4-4 = 0$$

[9.](ii)

$$\int_0^1 \int_0^{\sqrt{x}} \frac{y}{x^2+1} dydx = \int_0^1 [\frac{y^2}{2(1+x^2}]_0^{\sqrt{x}} dx = \int_0^1 \frac{x}{2(1+x^2} dx = [\frac{1}{4} ln(1+x^2)]_0^1 = \frac{ln(2)}{4}.$$

[10.] (ii)

The desired quantity is : $\int \int_D f(x,y)dydx = \int_0^2 \int_0^{\sqrt{4-x^2}} 4xydydx = \int_0^2 [2xy^2]_0^{\sqrt{4-x^2}} dx =$

$= \int_0^2 2x(4-x^2)dx = [4x^2 - \frac{1}{x}x^4]_0^2 = 16 - 8 = 8$