Analysis II 15 May 2017 Integrals. ①

Exercise 1.

(a) $\iint_D |f(x,y)|\, dx\,dy$

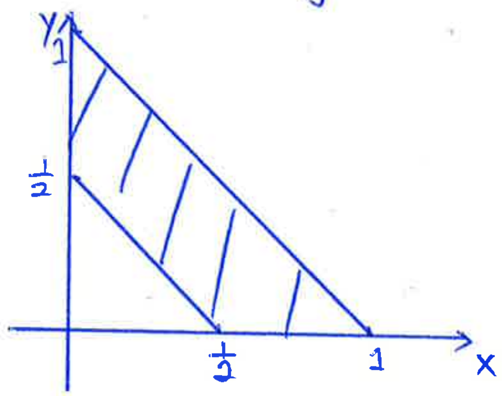
$= \iint_D |e^{\frac{x}{x+y}}| \cdot |e^{\frac{-y}{x+y}}|\, dx\, dy$

$\leq \iint_D |e^{\frac{x}{x+y}}|\, dx\,dy$

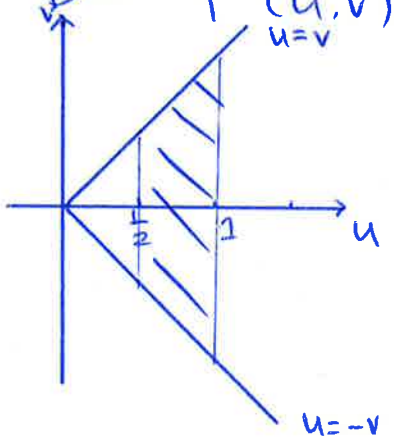
$\leq e^1 \cdot |D| < \infty$, $|D|$ is the area of D

$f(x,y)$ is integrable on D

(b)



(c) $D = \{ (u,v) : \frac{1}{2} < u < 1, \ -u < v < u \}$



$$J_{x,y}(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$|\det J_{x,y}(u,v)| = \frac{1}{2}$

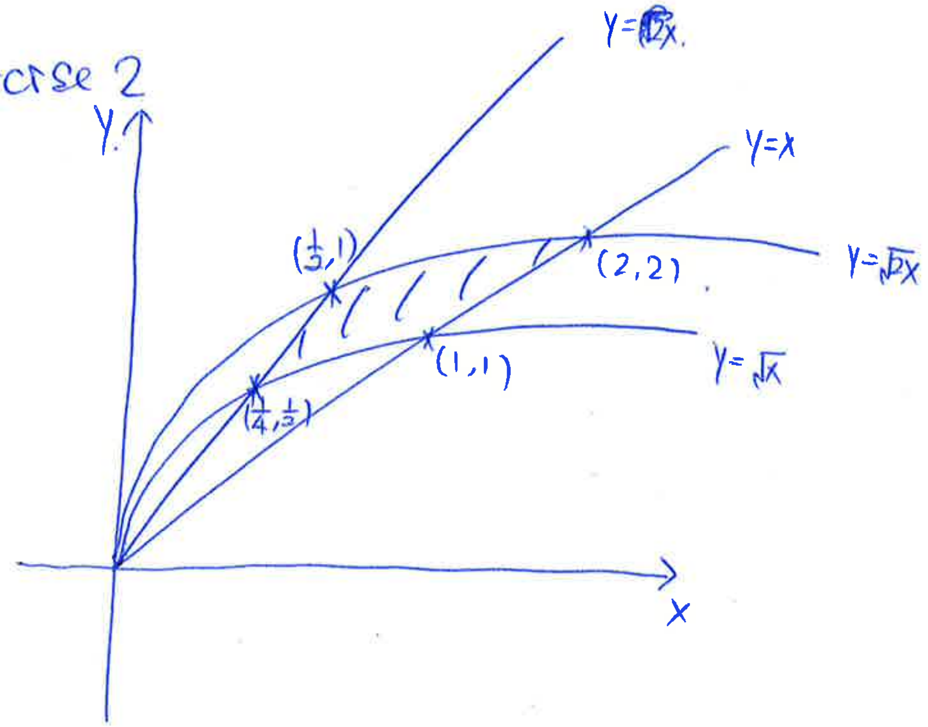
$$\iint_D f(x,y)\,dx\,dy = \frac{1}{2} \int_{\frac{1}{2}}^{1} du \int_{-u}^{u} e^{\frac{v}{u}}\, dv$$

$$= \frac{1}{2}\int_{\frac{1}{2}}^{1} u(e - e^{-1})\,du$$

$$= \frac{3}{16}(e - e^{-1})$$

Exercise 2

(a)



(b) $D = \{(u,v) : \frac{1}{2} < u < 1, \quad 1 < v < 2\}$.

(c) $\begin{cases} x = u^2 v \\ y = uv \end{cases}$

$$J_{x,y}(u,v) = \begin{pmatrix} 2uv & u^2 \\ v & u \end{pmatrix}$$

$$|\det J_{x,y}(u,v)| = u^2 v.$$

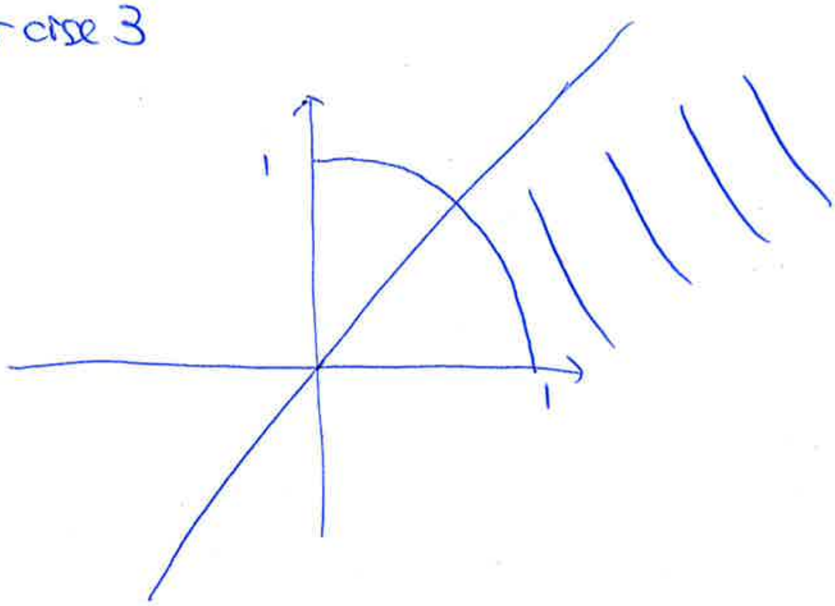
$$\iint_D \frac{y}{x}\,dx\,dy = \int_{\frac{1}{2}}^{1} du \int_1^2 u^{-1} \cdot u^2 v\,dv$$

$$= \int_{\frac{1}{2}}^{1} u\,du \int_1^2 v\,dv$$

$$= \frac{9}{16}$$

Exercise 3

(a)



(b) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$D = \{(r,\theta) : r \geq 1,\ 0 \leq \theta \leq \frac{\pi}{4}\}$

$\iint_D e^{-(x^2+y^2)}\,dx\,dy$

$= \int_1^{\infty} dr \int_0^{\frac{\pi}{4}} e^{-r^2} r\,d\theta$

$= \frac{\pi}{4} \int_1^{\infty} e^{-r^2} r\,dr$

$= \frac{\pi}{8} e^{-1}$

(c) $\iint_D |f(x,y)|\,dx\,dy$

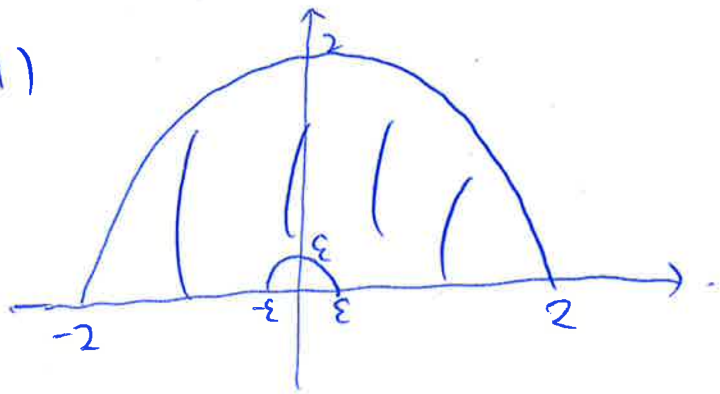
$= \iint_D e^{-(x^2+y^2)}\,dx\,dy$

$= \frac{\pi}{8} e^{-1} < \infty$

$f(x,y)$ is integrable.

Exercise 4.

(a)



(b) $$I_\varepsilon^1 = \int_\varepsilon^2 dr \int_0^\pi \frac{2r^2\cos\theta\sin\theta}{r^4}\, r\, d\theta$$

$$= \int_\varepsilon^2 \frac{2}{r} dr \int_0^\pi \cos\theta\sin\theta\, d\theta.$$

$$= 0$$

$$I_\varepsilon^2 = \int_\varepsilon^2 \frac{2}{r} dr \int_0^\pi |\sin\theta\cos\theta|\, d\theta$$

$$= \int_\varepsilon^2 \frac{2}{r} dr$$

$$= \log 2 - \log \varepsilon.$$

(c) $\lim_{\varepsilon\to 0} I_\varepsilon^1 = 0 \qquad \lim_{\varepsilon\to 0} I_\varepsilon^2 = \infty$

f is not integrable on D.

Exercise 5

(a) $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$

$D = \{(r,\theta,\varphi) : 0 \leq \theta \leq \pi,\ \varphi \in [0,\frac{\pi}{2}] \cup [\frac{3}{2}\pi, 2\pi],\ r \leq \sin\theta\cos\varphi\}$

since $x \geq 0$

(b) $I = \iiint_D \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz$

$= \int_{\varphi\in[0,\frac{\pi}{2}]\cup[\frac{3}{2}\pi,2\pi]} d\varphi \int_0^{\pi} d\theta \int_0^{\sin\theta\cos\varphi} r\cdot r^2 \sin\theta\, dr$

$= \frac{1}{8}\int_0^{2\pi} \cos^4\varphi\, d\varphi \int_0^{\pi} \sin^5\theta\, d\theta$

$\int_0^{\pi} \sin^5\theta\, d\theta$

$= \int_0^{\pi} \sin^4\theta \sin\theta\, d\theta$

$= -\int_0^{\pi} (1-\cos^2\theta)^2\, d\cos\theta$

$= \int_{-1}^{1} (1-x^2)^2\, dx$

$= \frac{16}{15}$

$\int_0^{2\pi} \cos^4\varphi\, d\varphi$

$= \int_0^{2\pi} \cos^3\varphi\, d\sin\varphi$

$= 0 + 3\int_0^{2\pi} \cos^2\varphi \sin^2\varphi\, d\varphi$

$= 3\int_0^{2\pi} \cos^2\varphi(1-\cos^2\varphi)\, d\varphi$

$\Rightarrow \int_0^{2\pi} \cos^4\varphi\, d\varphi$

$= \frac{3}{4}\int_0^{2\pi} \cos^2\varphi\, d\varphi$

$= \frac{3}{4}\pi$

$I = \frac{1}{10}\pi$