ÉCOLE POLYTECHNIQUE
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Solution to Exercise Session, April 25, 2016

1. **Divergence, Curl and Laplacian**

   (a) Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $\mathbf{f}(x, y, z) = (y + x^2, z, x^2)$. Compute $\nabla \cdot \mathbf{f}$, $\nabla(\nabla \cdot \mathbf{f})$ and $\nabla \times f$.

   (b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be $f(x, y, z) = x^3 + y^2 + z$. Find $\Delta f + \nabla \cdot (\nabla \times (\nabla f))$.

   Solution.

   (a) For $\mathbf{f} = (f_1, f_2, f_3)$:

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\nabla \times \mathbf{f} = \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix}$$

   So

$$\nabla$$

f = 2x + 0 + 0 = 2x

$$\nabla(\nabla \cdot \mathbf{f}) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\nabla \times \mathbf{f} = \begin{pmatrix} -1 \\ -2x \\ -1 \end{pmatrix}$$

   (b)

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 6x + 2$$

$$\nabla \cdot (\nabla \times (\nabla f)) = 0$$

   So

$$\Delta f + \nabla \cdot (\nabla \times (\nabla f)) = 6x + 2$$

2. Let $\mathbf{f}(x, y, z) = (3xyz^2, 2xy^3, -x^2yz)$ and $\phi(x, y, z) = 3x^2 - yz$. Find $\nabla \cdot \mathbf{f}$, $\nabla \times \mathbf{f}$, $\mathbf{f} \cdot \nabla\phi$, $\nabla \cdot (\nabla\phi)$ and $\nabla \cdot (\phi\mathbf{f})$ at point $(1, -1, 1)$.

   Solution. For $\nabla \cdot \mathbf{f}$ we have

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 3yz^2 + 6xy^2 - x^2y = 4$$

   For $\nabla \times \mathbf{f}$ we have

$$\nabla \times \mathbf{f} = \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix} = \begin{pmatrix} -x^2z \\ 8xyz \\ 2y^3 - 3x^2 \end{pmatrix} = \begin{pmatrix} -1 \\ -8 \\ -5 \end{pmatrix}$$

For $\mathbf{f} \cdot \nabla\phi$ we have

$$\nabla\phi = \begin{pmatrix} 6x \\ -z \\ -y \end{pmatrix}$$

so

$$\mathbf{f} \cdot \nabla\phi = 18x^2yz^2 - 2xy^3z + x^2y^2z = -15$$

For $\nabla \cdot (\nabla\phi)$ we have

$$\nabla \cdot (\nabla\phi) = \Delta\phi = 6$$

And for $\nabla \cdot (\phi\mathbf{f})$ we have

$$\phi\mathbf{f} = \begin{pmatrix} 9x^3yz^2 - 3xy^2z^3 \\ 6x^3y^3 - 2xy^4z \\ -3x^4yz + x^2y^2z^2 \end{pmatrix}$$

so

$$\nabla \cdot (\phi\mathbf{f}) = 27x^2yz^2 - 3y^2z^3 + 18x^3y^2 - 8xy^3z - 3x^3y + 2x^2y^2z = 1$$

3. Transport equation (aka. convection-diffusion equation) is used in physics and engineering to describe phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion (spreading) and convection (movement). The general transport equation is

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\vec{v}) = \nabla \cdot (D\nabla c) + S$$

where $\vec{v} = (v_1, v_2, v_3)$ and c, D, S, v_1, v_2 and v_3 are all real functions of t, x, y and z.

   (a) Write the equation without using the gradient operator ∇ and the divergence operator $(\nabla\cdot)$.

   (b) Verify that $c(x, y, z) = x^2 + y^2 + z$ satisfy the steady state equation, i.e. when all derivatives with respect to time t is zero, when $D = 5$, $v = (1, 2, 2)$ and $S = 2x+4y-18$.

Solution.

(a)

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\vec{v}) = \nabla \cdot (D\nabla c) + S \Longrightarrow$$

$$\frac{\partial c}{\partial t} + \nabla \cdot (cv_1, cv_2, cv_3) = \nabla \cdot (D\frac{\partial c}{\partial x}, D\frac{\partial c}{\partial y}, D\frac{\partial c}{\partial z}) + S \Longrightarrow$$

$$\frac{\partial c}{\partial t} + \frac{\partial(cv_1)}{\partial x} + \frac{\partial(cv_2)}{\partial y} + \frac{\partial(cv_3)}{\partial z} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial c}{\partial z}\right) + S$$

(b) We set $\partial c/\partial t = 0$ in part (a) to get

$$\frac{\partial(cv_1)}{\partial x} + \frac{\partial(cv_2)}{\partial y} + \frac{\partial(cv_3)}{\partial z} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial c}{\partial z}\right) + S \Longrightarrow$$

$$\frac{\partial c}{\partial x} + 2\frac{\partial c}{\partial y} + 2\frac{\partial c}{\partial z} = 5\frac{\partial^2 c}{\partial x^2} + 5\frac{\partial^2 c}{\partial y^2} + 5\frac{\partial^2 c}{\partial z^2} + 2x + 4y - 18 \Longrightarrow$$

$$2x + 4y + 2 = 5 \cdot 2 + 5 \cdot 2 + 0 + 2x + 4y - 18$$

4. Verify that

$$\nabla \times \nabla \times \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \Delta \mathbf{f}$$

where $\mathbf{f}(x,y,z) = (f_1, f_2, f_3)$ and $\Delta \mathbf{f} = (\Delta f_1, \Delta f_2, \Delta f_3)$.

Solution.

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla \times \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 f_2}{\partial x \partial y} - \frac{\partial^2 f_1}{\partial y^2} - \left(\frac{\partial^2 f_1}{\partial z^2} - \frac{\partial^2 f_3}{\partial x \partial z} \right) \\ -\left(\frac{\partial^2 f_2}{\partial x^2} - \frac{\partial^2 f_1}{\partial x \partial y} \right) + \frac{\partial^2 f_3}{\partial z \partial y} - \frac{\partial^2 f_2}{\partial z^2} \\ \frac{\partial^2 f_1}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial x^2} - \left(\frac{\partial^2 f_3}{\partial^2 y} - \frac{\partial^2 f_2}{\partial x \partial y} \right) \end{pmatrix}$$

Now we add and subtract the vector

$$\begin{pmatrix} \frac{\partial^2 f_1}{\partial x^2} \\ \frac{\partial^2 f_2}{\partial y^2} \\ \frac{\partial^2 f_3}{\partial z^2} \end{pmatrix}$$

to get

$$\nabla \times (\nabla \times \mathbf{f}) = \begin{pmatrix} \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial^2 f_1}{\partial x \partial z} \\ \frac{\partial^2 f_2}{\partial x \partial y} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial y \partial z} \\ \frac{\partial^2 f_3}{\partial x \partial z} + \frac{\partial^2 f_3}{\partial y \partial z} + \frac{\partial^2 f_3}{\partial z^2} \end{pmatrix} - \begin{pmatrix} \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} \\ \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial z^2} \\ \frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2} \end{pmatrix}$$

$$= \nabla(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}) - \Delta \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \Delta \mathbf{f}$$

5. Study the change of variable given by

$$x = \sin s \cosh t, \quad y = \cos s \sinh t.$$

Give the Jacobian matrix, denoted $J_{\mathbf{v}}$, and calculate $J_{\mathbf{v}}^T J_{\mathbf{v}}$. Let $f(x,y) = f(\sin s \cosh t, \cos s \sinh t)$ a function of class C^2. Calculate

$$\frac{\partial^2 f(x,y)}{\partial s^2} + \frac{\partial^2 f(x,y)}{\partial t^2}.$$

Use this result to give the Laplacian of a function $f(x,y)$ of class C^2 in terms of coordinates (s,t).

Solution. The Jacobian matrix is given by

$$J_{\mathbf{v}}(s,t) = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \cos s \cosh t & \sin s \sinh t \\ -\sin s \sinh t & \cos s \cosh t \end{pmatrix}$$

Hence, $\det J_{\mathbf{v}}(s,t) = \cos^2 s \cosh^2 t + \sin^2 s \sinh^2 t > 0$ for all $(s,t) \neq ((2k+1)\pi/2, 0)$, $k \in \mathbb{Z}$. Then, $J_{\mathbf{v}}(s,t)$ is locally invertible. We get

$$J_{\mathbf{v}}(s,t) J_{\mathbf{v}}(s,t)^T = \begin{pmatrix} (\frac{\partial x}{\partial s})^2 + (\frac{\partial x}{\partial t})^2 & \frac{\partial x}{\partial s}\frac{\partial y}{\partial s} + \frac{\partial x}{\partial t}\frac{\partial y}{\partial t} \\ \frac{\partial x}{\partial s}\frac{\partial y}{\partial s} + \frac{\partial x}{\partial t}\frac{\partial y}{\partial t} & (\frac{\partial y}{\partial s})^2 + (\frac{\partial y}{\partial t})^2 \end{pmatrix}$$
$$= \begin{pmatrix} \det J_{\mathbf{v}}(s,t) & 0 \\ 0 & \det J_{\mathbf{v}}(s,t) \end{pmatrix}$$

Note that this metric tensor is a multiply of the identity matrix. For $f(x,y) = f(\sin s \cosh t, \cos s \sinh t)$ a function of class C^2 we find by the rule of composition (see lecture notes)

$$\frac{\partial f(x,y)}{\partial s} = \frac{\partial x}{\partial s}\frac{\partial f(x,y)}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial f(x,y)}{\partial y}$$

and

$$\frac{\partial^2 f(x,y)}{\partial s^2} = \frac{\partial^2 x}{\partial s^2}\frac{\partial f(x,y)}{\partial x} + \left(\frac{\partial x}{\partial s}\right)^2 \frac{\partial^2 f(x,y)}{\partial x^2} + 2\frac{\partial x}{\partial s}\frac{\partial y}{\partial s}\frac{\partial^2 f(x,y)}{\partial x \partial y} + \frac{\partial^2 y}{\partial s^2}\frac{\partial f(x,y)}{\partial y} + \left(\frac{\partial y}{\partial s}\right)^2 \frac{\partial^2 f(x,y)}{\partial y^2}$$

and some corresponding expressions for the derivatives for the variable t. Using the relations

$$\frac{\partial^2 x}{\partial s^2} = -x, \ \frac{\partial^2 x}{\partial t^2} = x$$

and

$$\frac{\partial^2 y}{\partial s^2} = -y, \ \frac{\partial^2 y}{\partial t^2} = y$$

we get

$$\frac{\partial^2 f(x,y)}{\partial s^2} + \frac{\partial^2 f(x,y)}{\partial t^2} = \det J_{\mathbf{v}}(s,t)\left(\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\right).$$

6. **Change of coordinates between spherical and Cartesian coordinates in $\mathbb{R}^3$.** On $U := \{(r,\theta,\phi) : (r > 0, 0 < \theta < \pi, 0 < \phi < 2\pi\}$ we consider the map

$$\begin{aligned} x &= v_1(r,\theta,\phi) = r\sin\theta\cos\phi \\ y &= v_2(r,\theta,\phi) = r\sin\theta\sin\phi \\ z &= v_3(r,\theta,\phi) = r\cos\theta \end{aligned}$$

Show that $\mathbf{v}$ is locally invertible. Then show that for $(x,y,z) \in W := \{(x,y,z) : x > 0, y > 0, z > 0\}$ the reciprocal map $\mathbf{w} = \mathbf{v}^{-1}$ is given by

$$\begin{aligned} r &= w_1(x,y,z) = \sqrt{x^2+y^2+z^2} \\ \theta &= w_2(x,y,z) = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}} \\ \phi &= w_3(x,y,z) = \arcsin\frac{y}{\sqrt{x^2+y^2}} \end{aligned}$$

Calculate the Jacobian matrix and the Jacobian determinant $\mathbf{w}$. Give the set $\mathbf{w}(W)$.

Solution. The Jacobian matrix of $\mathbf{v}$ defined by

$$\mathbf{v}(r,\theta,\phi) = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$$

is

$$J_{\mathbf{v}}(r,\theta,\phi) = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

So

$$\begin{aligned}\det J_{\mathbf{v}}(r,\theta,\phi) &= r\cos\theta\cos\phi\, r\sin\theta\cos\phi\cos\theta\\ &\quad + r\sin\theta\sin\phi\sin\theta\sin\phi\, r\sin\theta\\ &\quad + r\sin\theta\sin\phi\, r\cos\theta\sin\phi\cos\theta\\ &\quad + \sin\theta\cos\phi\, r\sin\theta\cos\phi\, r\sin\theta\\ &= r^2\sin\theta(\cos^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta\\ &\quad + \cos^2\phi\sin^2\theta + \cos^2\phi\sin^2\theta)\\ &= r^2\sin\theta\end{aligned}$$

Then $\mathbf{v}$ is locally invertible. The calculation of $\mathbf{w}$ is trivial. To calculate the Jacobian matrix of $\mathbf{w}$ set $s = \sqrt{x^2+y^2}$. Using $r = \sqrt{x^2+y^2+z^2}$ we have

$$J_{\mathbf{w}}(x,y,z) = \begin{pmatrix} x/r & y/r & z/r \\ zx/r^2s & zy/r^2s & -s/r^2 \\ -y/s^2 & x/s^2 & 0 \end{pmatrix}$$

and $\det J_{\mathbf{w}}(x,y,z) = 1/rs$. We can get the result for the Jacobian determinant from the Jacobian of $\mathbf{v}$:

$$\det J_{\mathbf{w}}(x,y,z) = \frac{1}{\det J_{\mathbf{v}}(r,\theta,\phi)} = \frac{1}{r^2\sin\theta} = \frac{1}{rs}.$$

$$\mathbf{w}(W) = \{(r,\theta,\phi) : r>0, 0<\theta<\pi/2, 0<\phi<\pi/2\}.$$

7. **Change of coordinates between spherical and Cartesian coordinates in $\mathbb{R}^3$.** On $U := \{(r,\theta,\phi) : (r>0, 0<\theta<\pi, 0<\phi<2\pi\}$ we consider the map

$$\begin{aligned} x &= v_1(r,\theta,\phi) = r\sin\theta\cos\phi\\ y &= v_2(r,\theta,\phi) = r\sin\theta\sin\phi\\ z &= v_3(r,\theta,\phi) = r\cos\theta\end{aligned}$$

Let $g(r,\theta,\phi)$ be a function of class $C^2(U)$. Using the previous exercise ,calculate

$$||\nabla_{x,y,z}\, g(r,\theta,\phi)||_2^2.$$

Show that

$$\begin{aligned}&\Delta_{x,y,z}\, g(r,\theta,\phi)\\ &= \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right] g(r,\theta,\phi).\end{aligned}$$

Solution. Note that

$$(J_{\mathbf{w}}(x,y,z))J_{\mathbf{w}}(x,y,z)^T = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1/r^2 & 0\\ 0 & 0 & 1/s^2\end{pmatrix}$$

and $s^2 = r^2\sin^2\theta$. We also know that $\nabla_{x,y,z}\, g(r,\theta,\phi) = J_{\mathbf{w}}^T\nabla_{r,\theta,\phi} g(r,\theta,\phi)$. Hence,

$$\begin{aligned}||\nabla_{x,y,z}\, g(r,\theta,\phi)||_2^2 &= \nabla_{x,y,z}\, g(r,\theta,\phi)^T\nabla_{x,y,z}\, g(r,\theta,\phi) = \nabla g^T\cdot J_{\mathbf{w}}\cdot J_{\mathbf{w}}^T\cdot g\\ &= \big(D_r\, g(r,\theta,\phi)\big)^2 + \frac{1}{r^2}\big(D_\theta\, g(r,\theta,\phi)\big)^2 + \frac{1}{r^2\sin^2\theta}\big(D_\phi\, g(r,\theta,\phi)\big)^2.\end{aligned}$$

Also simple calculation shows that

$$\Delta_{x,y,z} r = \nabla_{x,y,z} \cdot (\nabla_{x,y,z} r) = \nabla_{x,y,z} \cdot \begin{pmatrix} x/r \\ y/r \\ z/r \end{pmatrix} = \frac{2}{r},$$

$$\Delta_{x,y,z} \theta = \nabla_{x,y,z} \cdot (\nabla_{x,y,z} \theta) = \nabla_{x,y,z} \cdot \begin{pmatrix} \frac{zx}{r^2 s} \\ \frac{zy}{r^2 s} \\ \frac{-s}{r^2} \end{pmatrix} = \frac{z}{r^2 s} = \frac{\cos\theta}{r^2 \sin\theta}$$

and

$$\Delta_{x,y,z} \phi = \nabla_{x,y,z} \cdot \begin{pmatrix} \frac{-y}{s^2} \\ \frac{x}{s^2} \\ 0 \end{pmatrix} = 0.$$

Now for the Laplacia of g we have

$$\begin{aligned} &\Delta_{x,y,z}\, g(r,\theta,\phi) \\ &= \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right. \\ &\left. + \Delta_{x,y,z} r \frac{\partial}{\partial r} + \Delta_{x,y,z} \theta \frac{\partial}{\partial \theta} + \Delta_{x,y,z} \phi \frac{\partial}{\partial \phi} \right] g(r,\theta,\phi). \end{aligned}$$

8. For each of the following, compute J_f, J_g and $J_{f\circ g}$.

(a)

$$f(x,y) = \begin{pmatrix} \sin x \\ x - y \\ xy \end{pmatrix}, \quad g(x,y) = \begin{pmatrix} x + y \\ xy \end{pmatrix}$$

(b)

$$f(x,y) = \begin{pmatrix} x + y \\ x^3 + 2xy \end{pmatrix}, \quad g(x) = \begin{pmatrix} x \\ x^2 \end{pmatrix}$$

Solution.

(a)

$$J_f = \begin{pmatrix} \cos x & 0 \\ 1 & -1 \\ y & x \end{pmatrix}, \quad J_g = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$$

So

$$J_{f\circ g} = J_f(g(x,y)) \cdot J_g = \begin{pmatrix} \cos(x+y) & 0 \\ 1 & -1 \\ xy & x+y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix} = \begin{pmatrix} \cos(x+y) & \cos(x+y) \\ 1 - y & 1 - x \\ 2xy + y^2 & 2xy + x^2 \end{pmatrix}$$

(b)

$$J_f = \begin{pmatrix} 1 & 1 \\ 3x^2 + 2y & 2x \end{pmatrix}, \quad J_g = \begin{pmatrix} 1 \\ 2x \end{pmatrix}$$

So

$$J_{f\circ g} = J_f(g(x,y)) \cdot J_g = \begin{pmatrix} 1 & 1 \\ 3 + 4x & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2x \end{pmatrix} = \begin{pmatrix} 1 + 2x \\ 3 + 8x \end{pmatrix}$$

9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$f(x, y) = \begin{pmatrix} \frac{x^2+y^2}{2} \\ \frac{x^2-y^2}{2} \end{pmatrix}, \quad g(x, y) = \begin{pmatrix} \sqrt{x+y} \\ \sqrt{x-y} \end{pmatrix}$$

Compute J_f, J_g and $J_{f \circ g}$. Is g the inverse function of f?

Solution. We have

$$J_f = \begin{pmatrix} x & y \\ x & -y \end{pmatrix}$$

$$J_g = \frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{x+y}} & \frac{1}{\sqrt{x+y}} \\ \frac{1}{\sqrt{x-y}} & -\frac{1}{\sqrt{x-y}} \end{pmatrix}$$

$$J_{f \circ g} = J_f(g(x, y)) \cdot J_g = \frac{1}{2} \begin{pmatrix} \sqrt{x+y} & \sqrt{x-y} \\ \sqrt{x+y} & -\sqrt{x-y} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{x+y}} & \frac{1}{\sqrt{x+y}} \\ \frac{1}{\sqrt{x-y}} & -\frac{1}{\sqrt{x-y}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Although $J_{f \circ g} = I$, but g is not the inverse function of f. As the matter of fact f is not ever invertible since $f(x, y) = f(-x, -y)$.