Analysis II
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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Exercise Session, April 25, 2016

1. **Divergence, Curl and Laplacian**

   (a) Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $\mathbf{f}(x, y, z) = (y + x^2, z, x^2)$. Compute $\nabla \cdot \mathbf{f}$, $\nabla(\nabla \cdot \mathbf{f})$ and $\nabla \times f$.

   (b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be $f(x, y, z) = x^3 + y^2 + z$. Find $\Delta f + \nabla \cdot (\nabla \times (\nabla f))$.

2. Let $\mathbf{f}(x, y, z) = (3xyz^2, 2xy^3, -x^2yz)$ and $\phi(x, y, z) = 3x^2 - yz$. Find $\nabla \cdot \mathbf{f}$, $\nabla \times \mathbf{f}$, $\mathbf{f} \cdot \nabla\phi$, $\nabla \cdot (\nabla\phi)$ and $\nabla \cdot (\phi\mathbf{f})$ at point $(1, -1, 1)$.

3. Transport equation (aka. convection-diffusion equation) is used in physics and engineering to describe phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion (spreading) and convection (movement). The general transport equation is

$$\frac{\partial c}{\partial t} + \nabla \cdot (c\vec{v}) = \nabla \cdot (D\nabla c) + S$$

   where $\vec{v} = (v_1, v_2, v_3)$ and c, D, S, v_1, v_2 and v_3 are all real functions of t, x, y and z.

   (a) Write the equation without using the gradient operator ∇ and the divergence operator $(\nabla\cdot)$.

   (b) Verify that $c(x, y, z) = x^2 + y^2 + z$ satisfy the steady state equation, i.e. when all derivatives with respect to time t is zero, when $D = 5$, $v = (1, 2, 2)$ and $S = 2x+4y-18$.

4. Verify that

$$\nabla \times \nabla \times \mathbf{f} = \nabla(\nabla \cdot \mathbf{f}) - \Delta\mathbf{f}$$

   where $\mathbf{f}(x, y, z) = (f_1, f_2, f_3)$ and $\Delta\mathbf{f} = (\Delta f_1, \Delta f_2, \Delta f_3)$.

5. Study the change of variable given by

$$x = \sin s \cosh t, \quad y = \cos s \sinh t.$$

   Give the Jacobian matrix, denoted $J_{\mathbf{v}}$, and calculate $J_{\mathbf{v}}^T J_{\mathbf{v}}$. Let $f(x, y) = f(\sin s \cosh t, \cos s \sinh t)$ a function of class C^2. Calculate

$$\frac{\partial^2 f(x, y)}{\partial s^2} + \frac{\partial^2 f(x, y)}{\partial t^2}.$$

   Use this result to give the Laplacian of a function $f(x, y)$ of class C^2 in terms of coordinates (s, t).

6. **Change of coordinates between spherical and Cartesian coordinates in** $\mathbb{R}^3$. On $U := \{(r, \theta, \phi) : (r > 0, 0 < \theta < \pi, 0 < \phi < 2\pi\}$ we consider the map

$$x = v_1(r, \theta, \phi) = r \sin\theta \cos\phi$$
$$y = v_2(r, \theta, \phi) = r \sin\theta \sin\phi$$
$$z = v_3(r, \theta, \phi) = r \cos\theta$$

Show that $\mathbf{v}$ is locally invertible. Then show that for $(x, y, z) \in W := \{(x, y, z) : x > 0, y > 0, z > 0\}$ the reciprocal map $\mathbf{w} = \mathbf{v}^{-1}$ is given by

$$\begin{aligned} r &= w_1(x, y, z) = \sqrt{x^2 + y^2 + z^2} \\ \theta &= w_2(x, y, z) = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \phi &= w_3(x, y, z) = \arcsin \frac{y}{\sqrt{x^2 + y^2}} \end{aligned}$$

Calculate the Jacobian matrix and the Jacobian determinant $\mathbf{w}$. Give the set $\mathbf{w}(W)$.

7. **Change of coordinates between spherical and Cartesian coordinates in $\mathbb{R}^3$.** On $U := \{(r, \theta, \phi) : (r > 0, 0 < \theta < \pi, 0 < \phi < 2\pi\}$ we consider the map

$$\begin{aligned} x &= v_1(r, \theta, \phi) = r \sin\theta \cos\phi \\ y &= v_2(r, \theta, \phi) = r \sin\theta \sin\phi \\ z &= v_3(r, \theta, \phi) = r \cos\theta \end{aligned}$$

Let $g(r, \theta, \phi)$ be a function of class $C^2(U)$. Using the previous exercise ,calculate

$$||\nabla_{x,y,z}\, g(r, \theta, \phi)||_2^2.$$

Show that

$$\begin{aligned} &\Delta_{x,y,z}\, g(r, \theta, \phi) \\ &= \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right)\right] g(r, \theta, \phi). \end{aligned}$$

8. For each of the following, compute J_f, J_g and $J_{f \circ g}$.

   (a)
$$f(x, y) = \begin{pmatrix} \sin x \\ x - y \\ xy \end{pmatrix}, \quad g(x, y) = \begin{pmatrix} x + y \\ xy \end{pmatrix}$$

   (b)
$$f(x, y) = \begin{pmatrix} x + y \\ x^3 + 2xy \end{pmatrix}, \quad g(x) = \begin{pmatrix} x \\ x^2 \end{pmatrix}$$

9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$f(x, y) = \begin{pmatrix} \frac{x^2+y^2}{2} \\ \frac{x^2-y^2}{2} \end{pmatrix}, \quad g(x, y) = \begin{pmatrix} \sqrt{x + y} \\ \sqrt{x - y} \end{pmatrix}$$

Compute J_f, J_g and $J_{f \circ g}$. Is g the inverse function of f?