ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Solutions to Exercise Session, April 18, 2015

1. **Jacobian matrix.** Calculate the Jacobian matrix of the following maps:

   (a) Let $\mathbf{u} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by

$$\mathbf{u}(x,y) = \begin{pmatrix} -y \\ x \\ x+y \end{pmatrix}$$

$$\begin{aligned} D_{\mathbf{u}}(x,y) &= \begin{pmatrix} D_1u_1(x,y) & D_2u_1(x,y) \\ D_1u_2(x,y) & D_2u_2(x,y) \\ D_1u_3(x,y) & D_2u_3(x,y) \end{pmatrix} \\ &= \begin{pmatrix} D_xu_1(x,y) & D_yu_1(x,y) \\ D_xu_2(x,y) & D_yu_2(x,y) \\ D_xu_3(x,y) & D_yu_3(x,y) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

   (b) Let $\mathbf{v} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $\mathbf{w} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by

$$\mathbf{v}(x,y) = \begin{pmatrix} -y \\ x \\ xy \end{pmatrix}$$

$$\mathbf{w}(x,y,z) = \begin{pmatrix} x^2+y^2-2z \\ x^2+y^2+2z \end{pmatrix}$$

   Calculate the Jacobian matrix of $\mathbf{w} \circ \mathbf{v}$ by calculating first this composition and then by the rule of composition.

$$J_{\mathbf{v}}(x,y) = \begin{pmatrix} D_1v_1(x,y) & D_2v_1(x,y) \\ D_1v_2(x,y) & D_2v_2(x,y) \\ D_1v_3(x,y) & D_2v_3(x,y) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ y & x \end{pmatrix}$$

   and

$$J_{\mathbf{w}}(x,y,z) = \begin{pmatrix} D_1w_1(x,y,z) & D_2w_1(x,y,z) & D_3w_1(x,y,z) \\ D_1w_2(x,y,z) & D_2w_2(x,y,z) & D_3w_2(x,y,z) \end{pmatrix} = \begin{pmatrix} 2x & 2y & -2 \\ 2x & 2y & 2 \end{pmatrix}$$

   So

$$\begin{aligned} J_{\mathbf{w}\circ\mathbf{v}}(x,y) &= \begin{pmatrix} 2v_1(x,y) & 2v_2(x,y) & -2 \\ 2v_1(x,y) & 2v_2(x,y) & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ y & x \end{pmatrix} \\ &= \begin{pmatrix} -2y & 2x & -2 \\ -2y & 2x & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ y & x \end{pmatrix} \\ &= \begin{pmatrix} 2x-2y & 2y-2x \\ 2x+2y & 2x+2y \end{pmatrix} \end{aligned}$$

By calculating $\mathbf{w} \circ \mathbf{v}$, we can easily check this result:

$$(\mathbf{w} \circ \mathbf{v})(x, y) = \begin{pmatrix} x^2 + y^2 - 2xy \\ x^2 + y^2 + 2xy \end{pmatrix}$$

(c) Let $\mathbf{v} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $\mathbf{w} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$\mathbf{v}(x, y, z) = \begin{pmatrix} e^{y+2z} \\ x^2 + yz \end{pmatrix}$$

$$\mathbf{w}(x, y) = \begin{pmatrix} \cos x \\ \sin y \end{pmatrix}$$

Calculate the Jacobian matrix of $\mathbf{w} \circ \mathbf{v}$ by calculating first this composition and then by using the rule of composition.

Then,

$$J_{\mathbf{v}}(x, y, z) = \begin{pmatrix} 0 & e^{y+2z} & 2e^{y+2z} \\ 2x & z & y \end{pmatrix}$$

and

$$J_{\mathbf{w}}(x, y) = \begin{pmatrix} -\sin x & 0 \\ 0 & \cos y \end{pmatrix}$$

So

$$\begin{aligned} J_{\mathbf{w}\circ\mathbf{v}}(x, y, z) &= \begin{pmatrix} -\sin\ v_1(x, y) & 0 \\ 0 & \cos\ v_2(x, y) \end{pmatrix} \cdot \begin{pmatrix} 0 & e^{y+2z} & 2e^{y+2z} \\ 2x & z & y \end{pmatrix} \\ &= \begin{pmatrix} -\sin(e^{y+2z}) & 0 \\ 0 & \cos(x^2 + yz) \end{pmatrix} \cdot \begin{pmatrix} 0 & e^{y+2z} & 2e^{y+2z} \\ 2x & z & y \end{pmatrix} \\ &= \begin{pmatrix} 0 & -e^{y+2z} \sin(e^{y+2z}) & -2e^{y+2z} \sin(e^{y+2z}) \\ 2x \cos(x^2 + yz) & z \cos(x^2 + yz) & y \cos(x^2 + yz) \end{pmatrix} \end{aligned}$$

By calculating $\mathbf{w} \circ \mathbf{v}$ we easily check the result:

$$(\mathbf{w} \circ \mathbf{v})(x, y) = \begin{pmatrix} \cos(v_1(x, y, z)) \\ \sin(v_2(x, y, z)) \end{pmatrix} = \begin{pmatrix} \cos(e^{y+2z}) \\ \sin(x^2 + yz) \end{pmatrix}.$$

2. **Jacobian matrix.** The Jacobian matrix of $\mathbf{v}$ defined by

$$\mathbf{v}(r, \theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$$

is

$$J_{\mathbf{v}}(r, \theta, \phi) = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

So

$$\begin{aligned} \det J_{\mathbf{v}}(r, \theta, \phi) &= r\cos\theta\cos\phi\, r\sin\theta\cos\phi\cos\theta \\ &\quad + r\sin\theta\sin\phi\sin\theta\sin\phi\, r\sin\theta \\ &\quad + r\sin\theta\sin\phi\, r\cos\theta\sin\phi\cos\theta \\ &\quad + \sin\theta\cos\phi\, r\sin\theta\cos\phi\, r\sin\theta \\ &= r^2\sin\theta(\cos^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta \\ &\quad + \cos^2\phi\sin^2\theta + \cos^2\phi\sin^2\theta) \\ &= r^2\sin\theta \end{aligned}$$

3.

$$
\begin{aligned}
J_{\mathbf{v}}(x,y) &= \begin{pmatrix} D_x v_1(x,y) & D_y v_1(x,y) \\ D_x v_2(x,y) & D_y v_2(x,y) \\ D_x v_3(x,y) & D_y v_3(x,y) \end{pmatrix} \\
&= \frac{1}{(1+x^2+y^2)^2} \begin{pmatrix} 2(1-x^2+y^2) & -4xy \\ -4xy & 2(1+x^2-y^2) \\ -4x & -4y \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
J_{\mathbf{w}}(x,y,z) &= \begin{pmatrix} D_x w_1(x,y,z) & D_y w_1(x,y,z) & D_z w_1(x,y,z) \\ D_x w_2(x,y,z) & D_y w_2(x,y,z) & D_z w_2(x,y,z) \end{pmatrix} \\
&= \frac{1}{(1+z)^2} \begin{pmatrix} 1+z & 0 & -x \\ 0 & 1+z & -y \end{pmatrix}
\end{aligned}
$$

We note that $f(x,y) = \langle \mathbf{v}(x,y), \mathbf{v}(x,y) \rangle = 1$ and hence

$$J_f(x,y) = (0,0).$$

By the composition rule

$$
\begin{aligned}
J_{\mathbf{w}\circ\mathbf{v}}(x,y) &= \frac{1}{(1+v_3(x,y))^2} \begin{pmatrix} 1+v_3(x,y) & 0 & -v_1(x,y) \\ 0 & 1+v_3(x,y) & -v_2(x,y) \end{pmatrix} \\
&\quad \cdot \frac{1}{(1+x^2+y^2)^2} \begin{pmatrix} 2(1-x^2+y^2) & -4xy \\ -4xy & 2(1+x^2-y^2) \\ -4x & -4y \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

The map $\mathbf{v}$ is a map from the plane to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ since $\langle \mathbf{v}(x,y), \mathbf{v}(x,y) \rangle = 1$. More precisely, the image of $\mathbf{v}$ is the whole sphere except the south pole $(0,0,-1)$. The map $\mathbf{w}$ restricted to $\mathbb{S}^2 \setminus \{(0,0,-1)\}$ gives the reciprocal map of $\mathbf{v}$. The map $\mathbf{w} : \mathbb{S}^2 \setminus \{(0,0,-1)\} \longrightarrow \mathbb{R}^2$ is called the stereographic projection.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x,y) = \frac{x^2 y \sin(\sqrt{x^2+y^2})}{(x^2+y^2)^{3/2}}, \quad (x,y) \neq (0,0)$$

Then

(a) $\lim_{(x,y)\to(0,0)} f(x,y) = 1$

(b) $\lim_{(x,y)\to(0,0)} f(x,y) = y$

(c) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

(d) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Solution. (d) is correct.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y \sin(\sqrt{x^2+y^2})}{(x^2+y^2)^{3/2}} = \lim_{r\to 0} \frac{r^3 \cos^2\theta \sin\theta \sin r}{r^3} = \lim_{r\to 0} \cos^2\theta \sin\theta \sin r = 0$$

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x,y) = \begin{cases} \dfrac{y^2}{\sqrt{y^4+x^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then

(a) $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = 1$

(b) $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = 0$

(c) $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y)$ does not exist

(d) $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = +\infty$

Solution. (c) is correct.

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = \lim_{(x,y)\to(0,0)} \frac{-y^2 x}{(y^4+x^2)^{3/2}} = \lim_{r\to 0} \frac{-r^3\sin^2\theta\cos\theta}{(r^4\sin^4\theta + r^2\sin^2\theta)^{3/2}} = \lim_{r\to 0} \frac{-\sin^2\theta\cos\theta}{(r^2\sin^4\theta+\cos^2\theta)^{3/2}}$$

The right hand side depend on θ.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x,y) = x^3 - 2xy + y^2$. Then the point $p = (2/3, 2/3)$

(a) is a local maximum of f

(b) is not a stationary point of f

(c) is a saddle point of f

(d) is a local minimum of f

Solution. (d) is correct. We have

$$\nabla f = \begin{pmatrix} 3x^2 - 2y \\ -2x + 2y \end{pmatrix}, \quad \nabla f(2/3, 2/3) = 0$$

$$\text{Hess}_f(x,y) = \begin{pmatrix} 6x & -2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$$

We have that $\det(\text{Hess}_f(2/3,2/3)) > 0$ and $f_{xx} > 0$ so $(2/3, 2/3)$ is a local minimum.

7. Let $f \in C^2(\mathbb{R}^2)$ and $\mathbf{p} \in \mathbb{R}^2$. If $\text{Hess}_f(\mathbf{p}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ then

(a) $\mathbf{p}$ is necessarily a local maximum

(b) $\mathbf{p}$ is necessarily a local minimum

(c) $\mathbf{p}$ is necessarily a saddle point

(d) None of above

Solution. (d) is correct. Take $f(x,y) = x^3$ at point $(1/3, 0)$. This function has no maximum or minimum.

8. Let the function $f : \mathbb{R}^3 \to \mathbb{R}$ be $f(x,y,z) = 2x^2y^3z^4 + 2x^3y^2 - 3y^2z - 1$ and consider $p = (1,1,1)$. Since $f(p) = 0$ and $\partial f/\partial x(p) \neq 0$, the equation $f(x,y,z) = 0$ defines in the neighbourhood of $(y,z) = (1,1)$ a function $x = g(y,z)$ which satisfies $g(1,1) = 1$ and $f(g(y,z), y, z) = 0$ as well as:

(a) $\frac{\partial g}{\partial z}(1,1) = -\frac{4}{5}$

(b) $\frac{\partial g}{\partial z}(1,1) = -\frac{1}{2}$

(c) $\frac{\partial g}{\partial z}(1,1) = -2$

(d) $\frac{\partial g}{\partial z}(1,1) = \frac{1}{2}$

Solution. (b) is correct. $\frac{\partial g}{\partial z}(1,1) = -F_z(1,1,1)/F_x(1,1,1)$ where $F(x,y,z) = 2x^2y^3z^4 + 2x^3y^2 - 3y^2z - 1$

9. Let $D = \{(x, y) \in \mathbb{R}^2 : x > 1 \text{ and } y > -1\}$ and let the function $f : D \to \mathbb{R}$ be $f(x, y) = \ln(x^2 + y)$. Then a vector v in the perpendicular direction to the level curve of f passing through point $(2, 0)$ is

   (a) $v = (-1/4, -1)^T$
   (b) $v = (-4, 1)^T$
   (c) $v = (4, 1)^T$
   (d) $v = (1, -4)^T$

   Solution. (c) is correct. $\nabla f(2, 0) = (1, 1/4)$ which is perpendicular to the level curve of f at $(2, 0)$.

10. State if the following statements are true or false.

   (a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(0, 0) = 0$. If for all $m \in \mathbb{R}$ we have $\lim_{x\to 0} f(x, mx) = 0$, then f is continuous at $(0, 0)$.
   (b) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $f \in C^2(\mathbb{R}^2)$, then for all points $p \in \mathbb{R}^2$ we have

   $$\frac{\partial^2 f}{\partial x \partial y}(p) = \frac{\partial^2 f}{\partial y \partial x}(p)$$

   (c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \in C^2(\mathbb{R}^2)$ and let a point $p \in \mathbb{R}^2$. If p is a stationary point of f and if determinant of the Hessian matrix $H_f(p)$ is strictly positive, then f admits a minimum at p.
   (d) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $f \in C^2(\mathbb{R}^2)$, then

   $$\frac{\partial f}{\partial x}(x, y) = \lim_{(h,k)\to(0,0)} \frac{f(x + h, y + k) - f(x, y)}{\sqrt{h^2 + k^2}}$$

   (e) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and $p \in \mathbb{R}^2$. Then f is differentiable at p if and only if $\partial f/\partial x$ and $\partial f/\partial y$ exist at p.
   (f) if $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. If f is differentiable at all points of $\mathbb{R}^2$, then f is of class $C^1(\mathbb{R}^2)$
   (g) Let $f : \mathbb{R}^3 \to \mathbb{R}$, be a function that is differentiable at a point $p \in \mathbb{R}^3$. Then the vector

   $$v = (-\frac{\partial f}{\partial x}(p), -\frac{\partial f}{\partial y}(p), -\frac{\partial f}{\partial z}(p), 1)$$

   is perpendicular to the tangent hyperplane to the graph of f at the point $(p, f(p))$.

   Solution.

   (a) False, take $f(x, y) = \frac{x^2 y}{x^3}$. Along the line $y = mx$

   $$\lim x \to 0 \frac{x^2 y}{x^3} = \lim x \to 0 \frac{m x^3}{x^3} = m$$

   But along the curve $y = x^2$

   $$\lim x \to 0 \frac{x^2 y}{x^3} = \lim x \to 0 \frac{x^4}{x^3} = 0$$

   (b) True
   (c) False, depending in the sign of f_{xx} the point can be either a local minimum or a local maximum.

(d) False, definition of partial derivative is

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

(e) False, take as an example $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

This function is differentiable everywhere except for $(0, 0)$. On x axis $(y = 0)$ the function is identical to zeros, and on y axis the function is identical to zero. This implies that $\partial f/\partial x(0, 0) = 0$ and $\partial f/\partial y(0, 0) = 0$. On the other hand f is not continuous at $(0, 0)$:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{r \to 0} \frac{r^2 \cos\theta \sin\theta}{r^2} = \cos\theta \sin\theta$$

which depends on θ. So f is not differentiable at $(0, 0)$.

(f) False, this is false because $C^1(\mathbb{R}^2)$ is the class of functions that are differentiable with continuous partial derivatives.

(g) True, perpendicular vector to the tangent hyperplane to f is given by $n = (\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p), -1)$. We see that $n = -v$ so v must be also perpendicular to the tangent hyper plane.