Analysis II
Prof. Jan Hesthaven
Spring Semester 2014–2015

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Exercise Session, April 18, 2015

1. **Jacobian matrix.** Find the Jacobian matrix of the following maps:

   (a) Let $\mathbf{u} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined as:

   $$\mathbf{u}(x, y) = \begin{pmatrix} -y \\ x \\ x + y \end{pmatrix}$$

   (b) Let $\mathbf{v} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $\mathbf{w} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined as:

   $$\mathbf{v}(x, y) = \begin{pmatrix} -y \\ x \\ xy \end{pmatrix}$$

   $$\mathbf{w}(x, y, z) = \begin{pmatrix} x^2 + y^2 - 2z \\ x^2 + y^2 + 2z \end{pmatrix}$$

   Find the Jacobian matrix of $\mathbf{w} \circ \mathbf{v}$ by *(i)* computing the composition and then its Jacobian matrix; *(ii)* using the chain rule.

   (c) Let $\mathbf{v} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $\mathbf{w} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by

   $$\mathbf{v}(x, y, z) = \begin{pmatrix} e^{y+2z} \\ x^2 + yz \end{pmatrix}$$

   $$\mathbf{w}(x, y) = \begin{pmatrix} \cos x \\ \sin y \end{pmatrix}$$

   Find the Jacobian matrix of $\mathbf{w} \circ \mathbf{v}$ by *(i)* computing the composition and then its Jacobian matrix; *(ii)* using the chain rule.

2. **Jacobian matrix.** Let $\mathbf{v} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined as

   $$\mathbf{v}(r, \theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta).$$

   Find the Jacobian matrix $J_{\mathbf{v}}$ and the Jacobian, i.e., the determinant $\det J_{\mathbf{v}}$.

3. Find the Jacobian matrices of the following maps:

   $$\mathbf{v} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad \mathbf{v}(x, y) = \begin{pmatrix} \frac{2x}{1+x^2+y^2} \\ \frac{2y}{1+x^2+y^2} \\ \frac{1-x^2-y^2}{1+x^2+y^2}, \end{pmatrix}$$

   $$\mathbf{w} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \mathbf{w}(x, y, z) = \begin{pmatrix} \frac{x}{1+z} \\ \frac{y}{1+z}, \end{pmatrix}$$

   $$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad f(x, y) = \langle \mathbf{v}(x, y), \mathbf{v}(x, y) \rangle$$

   and

   $$\mathbf{w} \circ \mathbf{v} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

   Give an interpretation of this result. (Hint: interpret $\mathbf{w}$ as a bijection from $\mathbb{S}^2 \setminus \{0, 0, -1\}$ onto $\mathbb{R}^2$. Then, since the Jacobian matrix of the composition is the identity, the relation between $\mathbf{w}$ and $\mathbf{v}$ is obvious.)

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x, y) = \frac{x^2 y \sin(\sqrt{x^2 + y^2})}{(x^2 + y^2)^{3/2}}, \quad (x, y) \neq (0, 0)$$

Then

(a) $\lim_{(x,y)\to(0,0)} f(x, y) = 1$

(b) $\lim_{(x,y)\to(0,0)} f(x, y) = y$

(c) $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist

(d) $\lim_{(x,y)\to(0,0)} f(x, y) = 0$

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be

$$f(x, y) = \begin{cases} \dfrac{y^2}{\sqrt{y^4 + x^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then

(a) $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x, y) = 1$

(b) $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x, y) = 0$

(c) $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x, y)$ does not exist

(d) $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x, y) = +\infty$

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x, y) = x^3 - 2xy + y^2$. Then the point $p = (2/3, 2/3)$

(a) is a local maximum of f

(b) is not a stationary point of f

(c) is a saddle point of f

(d) is a local minimum of f

7. Let $f \in C^2(\mathbb{R}^2)$ and $\mathbf{p} \in \mathbb{R}^2$. If $\text{Hess}_f(\mathbf{p}) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ then

(a) $\mathbf{p}$ is necessarily a local maximum

(b) $\mathbf{p}$ is necessarily a local minimum

(c) $\mathbf{p}$ is necessarily a saddle point

(d) None of above

8. Let the function $f : \mathbb{R}^3 \to \mathbb{R}$ be $f(x, y, z) = 2x^2y^3z^4 + 2x^3y^2 - 3y^2z - 1$ and consider $p = (1, 1, 1)$. Since $f(p) = 0$ and $\partial f/\partial x(p) \neq 0$, the equation $f(x, y, z) = 0$ defines in the neighbourhood of $(y, z) = (1, 1)$ a function $x = g(y, z)$ which satisfies $g(1, 1) = 1$ and $f(g(y, z), y, z) = 0$ as well as:

(a) $\frac{\partial g}{\partial z}(1, 1) = -\frac{4}{5}$

(b) $\frac{\partial g}{\partial z}(1, 1) = -\frac{1}{2}$

(c) $\frac{\partial g}{\partial z}(1, 1) = -2$

(d) $\frac{\partial g}{\partial z}(1, 1) = \frac{1}{2}$

9. Let $D = \{(x, y) \in \mathbb{R}^2 : x > 1 \text{ and } y > -1\}$ and let the function $f : D \to \mathbb{R}$ be $f(x, y) = \ln(x^2 + y)$. Then a vector v in the perpendicular direction to the level curve of f passing through point $(2, 0)$ is

(a) $v = (-1/4, -1)^T$
(b) $v = (-4, 1)^T$
(c) $v = (4, 1)^T$
(d) $v = (1, -4)^T$

10. State if the following statements are true or false.

(a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(0,0) = 0$. If for all $m \in \mathbb{R}$ we have $\lim_{x\to 0} f(x, mx) = 0$, then f is continuous at $(0,0)$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $f \in C^2(\mathbb{R}^2)$, then for all points $p \in \mathbb{R}^2$ we have

$$\frac{\partial^2 f}{\partial x \partial y}(p) = \frac{\partial^2 f}{\partial y \partial x}(p)$$

(c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \in C^2(\mathbb{R}^2)$ and let a point $p \in \mathbb{R}^2$. If p is a stationary point of f and if determinant of the Hessian matrix $H_f(p)$ is strictly positive, then f admits a minimum at p.

(d) Let $f : \mathbb{R}^2 \to \mathbb{R}$. If $f \in C^2(\mathbb{R}^2)$, then

$$\frac{\partial f}{\partial x}(x, y) = \lim_{(h,k)\to(0,0)} \frac{f(x+h, y+k) - f(x, y)}{\sqrt{h^2 + k^2}}$$

(e) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and $p \in \mathbb{R}^2$. Then f is differentiable at p if and only if $\partial f/\partial x$ and $\partial f/\partial y$ exist at p.

(f) if $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. If f is differentiable at all points of $\mathbb{R}^2$, then f is of class $C^1(\mathbb{R}^2)$

(g) Let $f : \mathbb{R}^3 \to \mathbb{R}$, be a function that is differentiable at a point $p \in \mathbb{R}^3$. Then the vector

$$v = (-\frac{\partial f}{\partial x}(p), -\frac{\partial f}{\partial y}(p), -\frac{\partial f}{\partial z}(p), 1)$$

is perpendicular to the tangent hyperplane to the graph of f at the point $(p, f(p))$.