ÉCOLE POLYTECHNIQUE
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Solutions to Exercise Session, April 11, 2016

1. **Implicit functions I.** Show that the equation

$$\ln x + e^{\frac{y}{x}} = 1$$

defined in the neighborhood of the point 1 is an implicit function $y = g(x)$ such that $g(1) = 0$. Give the equation of the tangent to the curve $y = g(x)$ at 1.

Solution. We define the function $f : U \longrightarrow \mathbb{R}$ with $U = \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}$ by

$$f(x, y) = \ln x + e^{\frac{y}{x}} - 1$$

Then, the function f is of class $C^1(U)$ (it is even of class $C^k(U)$ for all $k \geq 1$) and for all $(x, y) \in U$:

$$D_2 f(x, y) = \frac{e^{\frac{y}{x}}}{x}.$$

Moreover, $f(1, 0) = 0$ and $D_2 f(1, 0) = 1 \neq 0$. So, by the theorem of implicit functions, there exists an interval $I =]1 - \epsilon, 1 + \epsilon[$ and a unique function $g : I \longrightarrow \mathbb{R}$ of class $C^1(I)$ such that $g(1) = 0$ and $f(x, g(x)) = 0$. The derivative of g is given by

$$g'(x) = -\frac{D_1 f(x, g(x))}{D_2 f(x, g(x))} = \frac{g(x)}{x} - e^{\frac{-g(x)}{x}}$$

So $g'(1) = -1$. Hence, the equation of the tangent to the curve $y = g(x)$ at $x = 1$ is

$$y = g(1) + g'(1)(x - 1) = 1 - x.$$

2. **Implicit functions II.** Show that the equation

$$\cos(x^2 + y) + \sin(x + y) + e^{x^3 y} = 2$$

defined in the neighborhood of the point 0 is an implicit function $y = g(x)$ such that $g(0) = \pi/2$. Show that the function g has a local maximum at 0.

Solution. We define the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f(x, y) = \cos(x^2 + y) + \sin(x + y) + e^{x^3 y} - 2$$

So, the function f is of class C^k, for all $k \geq 1$, and for all $(x, y) \in \mathbb{R}^2$:

$$D_2 f(x, y) = -\sin(x^2 + y) + \cos(x + y) + x^3 e^{x^3 y}.$$

Moreover, $f(0, \pi/2) = 0$ and $D_2 f(0, \pi/2) = -1 \neq 0$. So, by the theorem of implicit functions, there exists an interval $I =] - \epsilon, \epsilon[$ and a unique function $g : I \longrightarrow \mathbb{R}$ of class $C^1(I)$ such that $g(0) = \pi/2$ and $f(x, g(x)) = 0$. The derivative of g is given by

$$g'(x) = -\frac{D_1 f(x, g(x))}{D_2 f(x, g(x))}$$

and $D_1 f(x, y) = -2x \sin(x^2 + y) + \cos(x + y) + 3x^2 y e^{x^3 y}$. So $g'(0) = 0$. The second derivative in $x = 0$ is

$$g''(0) = -\frac{D_{11} f(0, \pi/2)}{D_2 f(0, \pi/2)} = -3.$$

3. **Implicit functions III.** Show that the equation

$$x^5 + xyz + y^3 + 3xz^4 = 2$$

defined in the neighborhood of the point $(1, -1)$ is an implicit function $z = g(x, y)$ such that $g(1, -1) = 1$. Give the equation of the plane tangent to the surfuce $z = g(x, y)$ in $(1, -1)$.

Solution. We define the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ by

$$f(x, y, z) = x^5 + xyz + y^3 + 3xz^4 - 2$$

Then, the function f is of class C^k, for all $k \geq 1$, and for all $(x, y, z) \in \mathbb{R}^3$:

$$D_3 f(x, y, z) = xy + 12xz^3.$$

Moreover, $f(1, -1, 1) = 0$ and $D_3 f(1, -1, 1) = 11 \neq 0$. Then, by the theorem of implicit functions, there exists a neighborhood $B_\epsilon(1, -1) \subset \mathbb{R}^2$ and a unique function $g : B_\epsilon(1, -1) \longrightarrow \mathbb{R}$ of class $C^k(B_\epsilon(1, -1))$ such that $g(1, -1) = 1$ and $f(x, y, g(x, y)) = 0$. The equation of the tangent plane to the surface $z = g(x, y)$ in $(1, -1)$ is given by

$$0 = \langle \nabla f(1, -1, 1), \begin{pmatrix} x - 1 \\ y + 1 \\ z - 1 \end{pmatrix} \rangle = 0$$

i.e. using

$$\nabla f(x, y, z) = \begin{pmatrix} 5x^4 + yz + 3z^4 \\ xz + 3y^2 \\ xy + 12xz^3 \end{pmatrix}$$

we find

$$7x + 4y + 11z = 14.$$

4. **Quadratic form.** Let $A \in M_{n,n}(\mathbb{R})$ be a positive-definite, symmetric matrix. Let $\mathbf{v} \in \mathbb{R}^n$. Show that the function $f(\mathbf{x})$ defined by

$$f(\mathbf{x}) = \frac{1}{2} \langle A\mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{v}, \mathbf{x} \rangle$$

has a unique stationary point at $\mathbf{a} = A^{-1}\mathbf{v}$. Then show that $f(\mathbf{x}) - f(\mathbf{a}) > 0$ for all $\mathbf{x} \neq \mathbf{a}$.

Solution. Note first that A positive-definite implies that A is invertible. The stationary points of f are given by the solutions of the equation $\nabla f(\mathbf{x}) = \mathbf{0}$ so $A\mathbf{x} - \mathbf{v} = \mathbf{0}$. When A is invertible, this equation has as unique solution, the vector $\mathbf{a} = A^{-1}\mathbf{v}$. It is a strict local minimum as $\text{Hess}\,(f)(\mathbf{a}) = A > 0$. Moreover,

$$f(\mathbf{a}) = \frac{1}{2} \langle AA^{-1}\mathbf{v}, A^{-1}\mathbf{v} \rangle - \langle \mathbf{v}, A^{-1}\mathbf{v} \rangle = -\frac{1}{2} \langle \mathbf{v}, A^{-1}\mathbf{v} \rangle$$

and (note that A^{-1} is also symmetric) hence, writing

$$\langle \mathbf{v}, \mathbf{x} \rangle = \frac{1}{2} \langle A\mathbf{x}, A^{-1}\mathbf{v} \rangle + \frac{1}{2} \langle AA^{-1}\mathbf{v}, \mathbf{x} \rangle$$

we find

$$f(\mathbf{x}) - f(\mathbf{a}) = \frac{1}{2} \langle A(\mathbf{x} - A^{-1}\mathbf{v}), (\mathbf{x} - A^{-1}\mathbf{v}) \rangle > 0$$

for all $\mathbf{x} \neq A^{-1}\mathbf{v}$.

5. Study the nature of the stationary points of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x, y) = (1 - x^2) \sin y.$$

Solution. The stationary points are given by the solutions of

$$\nabla f(x,y) = \mathbf{0} \quad \text{i.e.} \quad \begin{array}{r} -2x\sin y = 0 \\ (1-x^2)\cos y = 0 \end{array}.$$

The function f has four families of stationary points:

$$P_k = (0, \frac{\pi}{2} + 2k\pi),\ Q_k = (0, \frac{3\pi}{2} + 2k\pi),\ S_k = (-1, k\pi),\ T_k = (1, k\pi)$$

for $k \in \mathbb{Z}$.
Proof: If $x = 0$, then $D_x f = 0$, and $D_y f = 0$ if and only if $\cos y = 0$ hence the P_k, Q_k. If $\sin y = 0$, then $y = k\pi$ and $D_y f = 0$ if and only if $x = -1$ or $x = +1$, hence the S_k, T_k.
To study the stationary points, we calculate the Hessian matrix f:

$$(\operatorname{Hess} f)(x,y) = \begin{pmatrix} -2\sin y & -2x\cos y \\ -2x\cos y & -(1-x^2)\sin y \end{pmatrix}$$

We get

$$(\operatorname{Hess} f)(P_k) = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

hence some local maxima,

$$(\operatorname{Hess} f)(Q_k) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

hence some local minima

$$(\operatorname{Hess} f)(S_k) = \begin{pmatrix} 0 & \pm 2 \\ \pm 2 & 0 \end{pmatrix}, (\operatorname{Hess} f)(T_k) = \begin{pmatrix} 0 & \pm 2 \\ \pm 2 & 0 \end{pmatrix}$$

hence some saddle points.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = (x-y)^3 + 4x^2 - 3x + 3y.$$

(a) Give the stationary points of f and study their nature. Calculate f at these points.

(b) Let T be the domain given by:

$$T = \{(x,y) \in \mathbb{R} : y \geq 0, y \leq x \leq 4 - y\}.$$

Give the minimum and the maximum of f on T. In particular,

i. Show that T is bounded.
ii. Show that $\partial T \subset T$ and conclude that T is closed.
iii. Show that T is a triangle and give its summits.
iv. Explain why f has its maximum and minimum on T.
v. Give f on the boundary of T, i.e. $f|_{\partial T}$ and then study $f|_{\partial T}$.
vi. Give the minimum and the maximum of f on T.

Solution. (a)

$$\nabla f(x,y) = \begin{pmatrix} 3(x-y)^2 + 8x - 3 \\ -3(x-y)^2 + 3 \end{pmatrix}$$

$$\operatorname{Hess} f(x,y) = \begin{pmatrix} 6x - 6y + 8 & -6x + 6y \\ -6x + 6y & 6x - 6y \end{pmatrix}$$

Stationary points: $\nabla f(x,y) = 0 \Leftrightarrow x = 0, (x-y)^2 = 1$

$$P_1 = (0,-1), \quad P_2 = (0,1)$$

$$\text{Hess} f(0,-1) = \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix}, \quad \text{Hess} f(0,1) = \begin{pmatrix} 2 & 6 \\ 6 & -6 \end{pmatrix}$$

P_1: min. loc. as det and trace > 0, $f(0,-1) = -2$
P_2: saddle point as det < 0 , $f(0,1) = 2$

(b)
(i). The definition of T gives us the inequalities $0 \leq y \leq x \leq 4 - y \leq 4$. So

$$T \subset [0,4] \times [0,4]$$

which implies that T is bounded.

(ii). The boundary ∂T is given by the segments

$$S_1 = \{(x,y) \in \mathbb{R} : y = 0, x \in [0,4]\}$$
$$S_2 = \{(x,y) \in \mathbb{R} : y = 4 - x, x \in [2,4]\}$$
$$S_3 = \{(x,y) \in \mathbb{R} : y = x, x \in [0,2]\}$$

which are in T (seen by the signs $\leq$ and not $<$ in the definition of T). So T is closed (see exercise 8 of chapter 1).

(iii). The boundary of T is given by 3 segments which have 3 intersection points (the summits):

$$A = (0,0), B = (4,0), C = (2,2)$$

(iv). f is continuous (polynomial) and T bounded and closed hence we conclude that f has its min and max on T.

(v).

$$f_1(x) := f|_{S_1} = f(x,0) = x^3 + 4x^2 - 3x, \quad x \in [2,4]$$
$$f_2(x) := f|_{S_2} = f(x,4-x) = 8x^3 - 44x^2 + 90x - 52, \quad x \in [2,4]$$
$$f_3(x) := f|_{S_3} = f(x,4-2x) = 4x^2, \quad x \in [0,2]$$

$$f_1'(x) := 3x^2 + 8x - 3, \quad x = 1/3, f_1(0) = 0, f_1(\frac{1}{3}) = -14/27, f_1(4) = 116$$
$$f_2'(x) := 24x^2 - 88x + 90, \quad \text{no stationary points.}, f_2(2) = 16, f_2(4) = 116$$
$$f_3'(x) := 8x, \quad x = 0, f_3(0) = 0, f_3(2) = 16$$

(vi). $\min f|_T = -14/27$, $\max f|_T = 116$

7. Calculate the extrema of the function

$$f(x,y) = x^4 + y^4$$

under the constrain $g(x,y) = xy - 1 = 0$.

(a) Find the extrema directly (by replacing the constrain g in f).
(b) Find the extrema using Lagrange multiplier.

Solution.

(a) The constrain gives $y = 1/x$. We replace this in f to get

$$f(x, y(x)) = h(x) = x^4 + x^{-4}$$

We have

$$\frac{dh}{dx} = 4x^3 - 4x^{-5} = 0 \Longrightarrow x = \pm 1$$

If $x = \pm 1$ then $y = \pm 1$ and $f(\pm 1, \pm 1) = 2$. Since

$$\frac{d^2 h}{dx^2}(\pm 1) > 0$$

then both points are minimum points.

(b) We formulate the Lagrange function $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$. $\nabla L = 0$ gives

$$\begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\ g = 0 \end{cases}$$

Which gives

$$\begin{cases} 4x^3 = \lambda y & (i) \\ 4y^3 = \lambda x & (ii) \\ xy - 1 = 0 & (iii) \end{cases}$$

Equation (iii) gives $y = 1/x$. If we replace this into (i) we get $4x^4 = \lambda$. We now replace $y = 1/x$ and $4x^4 = \lambda$ in (ii) to get $4 = 4x^8$ which gives $x = \pm 1$ and $y = \pm 1$ and $\lambda = 4$. This is the same result as part (a).

8. Compute the extrema of the function $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = (x-1)^2 + (y-1)^2 - 4$.

Solution. We formulate the Lagrange function $L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$. $\nabla L = 0$ gives the equations

$$\begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \\ g = 0 \end{cases}$$

which are

$$\begin{cases} 2x = 2\lambda(x-1) \rightarrow x = \lambda/(\lambda - 1) \\ 2y = 2\lambda(y-1) \rightarrow y = \lambda/(\lambda - 1) = x \\ (x-1)^2 + (y-1)^2 - 4 = 0 \end{cases}$$

This system of equations has two solutions $(x_1, y_1) = (1 - \sqrt{2}, 1 - \sqrt{2})$ and $(x_2, y_2) = (1 + \sqrt{2}, 1 + \sqrt{2})$. Also $f(1 - \sqrt{2}, 1 - \sqrt{2}) = 6 - 4\sqrt{2}$ and $f(1 + \sqrt{2}, 1 + \sqrt{2}) = 6 + 4\sqrt{2}$, so f attains its minimum at $(1 - \sqrt{2}, 1 - \sqrt{2})$ and its maximum at $(1 + \sqrt{2}, 1 + \sqrt{2})$.

9. The atmospheric pressure in a region of space near the origin is given by the formula $P = 30 + (x+1)(y+2)e^z$. Approximately where is the point closest to the origin at which the pressure is 31.1. (*Hint: linearize the equation around the origin. Then find the point closest to the origin that satisfy the linearized equation.*)

Solution. We have

$$\nabla P = ((y+2)e^z, (x+1)e^z, (x+1)(y+2)e^z)$$

So the first order Taylor's expansion of P is

$$f(0+\delta x, 0+\delta y, 0+\delta z) = 32 + 2x + y + 2z$$

We want $f(\delta x, \delta y, \delta z) = 31.1$ so the solution belongs to the plane $2x + y + 2z = -0.9$. The closest point to the origin will be the solution of the following minimization

$$\begin{aligned}&\text{min } x^2 + y^2 + z^2\\ &\text{subject to } 2x + y + 2z = -0.9\end{aligned}$$

By introducing the Lagrange multiplier λ we construct the Lagrange function

$$L(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(2x + y + 2z + 0.9)$$

We seek for points that $\nabla L = 0$ so,

$$\nabla L = (2x - 2\lambda, 2y - \lambda, 2z - 2\lambda, 2x + y + 2z + 0.9) = (0, 0, 0, 0)$$

These are 4 equations with 4 unknowns which has the unique solution $(x, y, z, \lambda) = (-.2, -.1, -.2, -.2)$. Note that if we actually compute P at $(-.2, -.1, -.2)$ we get $P(-.2, -.1, -.2) = 31.2445$ which is a good approximation.