Analysis II
Prof. Jan Hesthaven
Spring Semester 2015–2016

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Exercise Session, April 11, 2016

1. **Implicit functions I.** Show that the equation

$$\ln x + e^{\frac{y}{x}} = 1$$

defined in the neighborhood of the point 1 is an implicit function $y = g(x)$ such that $g(1) = 0$. Give the equation of the tangent to the curve $y = g(x)$ at 1.

2. **Implicit functions II.** Show that the equation

$$\cos(x^2 + y) + \sin(x + y) + e^{x^3 y} = 2$$

defined in the neighborhood of the point 0 is an implicit function $y = g(x)$ such that $g(0) = \pi/2$. Show that the function g has a local maximum at 0.

3. **Implicit functions III.** Show that the equation

$$x^5 + xyz + y^3 + 3xz^4 = 2$$

defined in the neighborhood of the point $(1, -1)$ is an implicit function $z = g(x, y)$ such that $g(1, -1) = 1$. Give the equation of the plane tangent to the surfuce $z = g(x, y)$ in $(1, -1)$.

4. **Quadratic form.** Let $A \in M_{n,n}(\mathbb{R})$ be a positive-definite, symmetric matrix. Let $\mathbf{v} \in \mathbb{R}^n$. Show that the function $f(\mathbf{x})$ defined by

$$f(\mathbf{x}) = \frac{1}{2} \langle A\mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{v}, \mathbf{x} \rangle$$

has a unique stationary point at $\mathbf{a} = A^{-1}\mathbf{v}$. Then show that $f(\mathbf{x}) - f(\mathbf{a}) > 0$ for all $\mathbf{x} \neq \mathbf{a}$.

5. Study the nature of the stationary points of the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by

$$f(x, y) = (1 - x^2) \sin y.$$

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = (x - y)^3 + 4x^2 - 3x + 3y.$$

   (a) Give the stationary points of f and study their nature. Calculate f at these points.

   (b) Let T be the domain given by:

$$T = \{(x, y) \in \mathbb{R} : y \geq 0, y \leq x \leq 4 - y\}.$$

   Give the minimum and the maximum of f on T. In particular,

   i. Show that T is bounded.
   ii. Show that $\partial T \subset T$ and conclude that T is closed.
   iii. Show that T is a triangle and give its summits.
   iv. Explain why f has its maximum and minimum on T.
   v. Give f on the boundary of T, i.e. $f|_{\partial T}$ and then study $f|_{\partial T}$.

vi. Give the minimum and the maximum of f on T.

7. Calculate the extrema of the function

$$f(x, y) = x^4 + y^4$$

under the constrain $g(x, y) = xy - 1 = 0$.

(a) Find the extrema directly (by replacing the constrain g in f).

(b) Find the extrema using Lagrange multiplier.

8. Compute the extrema of the function $f(x, y) = x^2 + y^2$ under the constraint $g(x, y) = (x - 1)^2 + (y - 1)^2 - 4$.

9. The atmospheric pressure in a region of space near the origin is given by the formula $P = 30 + (x + 1)(y + 2)e^z$. Approximately where is the point closest to the origin at which the pressure is 31.1. (*Hint: linearize the equation around the origin. Then find the point closest to the origin that satisfy the linearized equation.*)