Analysis II
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Spring Semester 2015–2016

ÉCOLE POLYTECHNIQUE
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Exercise Session, April 4, 2016

1. Consider the surface Γ of the equation $z = f(x, y) = xy$.
   (a) At which point(s) of Γ the tangent plane is parallel to the plane
   $$z = p(x, y) = -x/6 + y + 5/3$$
   Give the equation of the tangent plane.
   (b) Determine the equation of tangent plane(s) of Γ that pass through the points $Q = (4, 2, 8)$ and $R = (6, 0, 2)$.
   (*Hint: we search for point(s)* (x_0, y_0, z_0) on Γ with tangent plane passing through Q and R)

2. A company manufactures right circular cylindrical storage tanks that are 25 meters high with a radius of 5 meters. How sensitive are the tank's volumes to small variations in height and radius. (*Hint: Linearize the equation for the volume of a cylinder at point* $(h = 25, r = 5)$ *using Taylor expansion of first order).*

3. Consider the function
   $$f(x, y) = \begin{cases} \dfrac{3xy^2 - y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$
   (a) Study the continuity of the function on $\mathbb{R}^2$.
   (b) Calculate the partial derivatives: Can we say that the function is differentiable on $\mathbb{R}^2 \backslash \{(0, 0)\}$?
   (c) Compute the directional derivative of f at points $(x_0, y_0) = (0, 0)$ and $(x_0, y_0) = (1, 1)$ along vector $v = (v_1, v_2)$, what can you say about differentiability of f at these points.
   (d) calculate the limit
   $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - f(x_0, y_0) - \nabla f(x_0, y_0) \cdot (x - x0, y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}$$
   for $(x_0, y_0) = (0, 0)$ and $(x_0, y_0) = (1, 1)$). Why are the values different?

4. Let $f(x, y) = e^x \log(1 - y) + \sin(2xy)$ and $x_0 = (0, 0)$
   (a) Calculate the Taylor expansion of order 2 ($t_2(x, y)$) of f.
   (b) Consider $g(x) = f(x, 3x)$ and calculate its Taylor expansion of order 2 at $x = 0$ and compare the result with $t_2(x, 3x)$.

5. Study the nature of the stationary points of the functions
   (a) $f(x, y) = x^3 + 6xy^2 - 12x^2 - 18y^2 + 21x$
   (b) $f(x, y) = y^2 + y\cos(x) - \sin(x) - 2$

6. A T-shirt shop carries two competing shirts, one with Batman vs. Superman theme and the other with Captain America: Civil War theme. The owner can obtain both at a cost of 2 CHF per shirt and estimates that if Batman vs. Superman shirts are sold for x CHF apiece and Civil War for y CHF apiece, consumers will buy $40 - 50x + 40y$ of the first shirt and $20 + 60x - 70y$ of the second shirt each day.

   (a) Express as a function of x and y the revenue of selling Batman vs. Superman shirts, the revenue of selling Civil War shirts, the costs for shirts and the overall profit.

   (b) Find the critical point of the profit function.

   (c) How should the owner price the shirts in order to generate the largest possible profit?

   (d) Calculate he Hessian matrix for this problem and its determinant. Is the solution in (b) indeed an absolute maximum?

7. Consider the function

$$f(x, y) = x^2 + y^\alpha, \quad \alpha \geq 0$$

   (a) Determine the stationary points of the system.

   (b) Study the nature of the stationary points for $\alpha = 2$, $\alpha = 3$ and $\alpha = 4$.