ÉCOLE POLYTECHNIQUE
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Solutions to Exercise Session, March 21, 2016

1. **Tangent hyperplane.** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function given by

$$f(x, y) = x^2 + y \sin x + y^2 \cos^2 x.$$

   (a) Show that f is partially differentiable and give the gradient of f.

   (b) Give the equation of the tangent plan at the point $(x, y) = (0, 1)$.

Solution. The function is partially differentiable since polynomials and trigonometric functions are differentiable.

$$\nabla f(x, y) = \begin{pmatrix} 2x + y \cos x - 2y^2 \sin x \cos x \\ \sin x + 2y \cos^2 x \end{pmatrix}$$

Note that $f(0, 1) = 1$ and

$$\nabla f(0, 1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The equation of the tangent plane is given by

$$z = 1 + x + 2(y - 1) = -1 + x + 2y.$$

2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function of class C^2. Let $\mathbf{x} \in \mathbb{R}^n$ and $r = ||\mathbf{x}||_2$.

   (a) Show that for all $\mathbf{x} \neq \mathbf{0}$ we get

$$\Delta f(r) = f''(r) + \frac{n-1}{r} f'(r)$$

Solution. By the composition rule,

$$D_k f(r) = f'(r) d_k r = f'(r) \frac{x_k}{r}$$

$$\begin{aligned} D_{kk} f(r) &= D_k \left(f'(r) \frac{x_k}{r} \right) \\ &= f''(r) \frac{x_k}{r} \frac{x_k}{r} + f'(r) \frac{1}{r} - f'(r) \frac{x_k^2}{r^3} \\ &= f''(r) \frac{x_k^2}{r^2} + f'(r) \frac{1}{r} - f'(r) \frac{x_k^2}{r^3} \end{aligned}$$

So

$$\Delta f(r) = \sum_{k=1}^{n} D_{kk} f(r) = f''(r) + \frac{n-1}{r} f'(r)$$

as $r^2 = \sum_{k=1}^n x_k^2$.

   (b) Let $f'(0) = 0$. Give

$$\lim_{r \to 0} \Delta f(r).$$

Solution.

$$\begin{aligned}\lim_{r\to 0}\Delta f(r) &= \lim_{r\to 0} f''(r) + \frac{n-1}{r} f'(r)\\ &= f''(0) + (n-1)\lim_{r\to 0}\frac{f'(r)-f'(0)}{r}\\ &= (n-1)f''(0) + f''(0)\\ &= nf''(0)\end{aligned}$$

(c) Let $f : \mathbb{R}^3 \setminus \{\mathbf{0}\} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x,y,z) = \frac{\sin(\sqrt{x^2+y^2+z^2})}{\sqrt{x^2+y^2+z^2}}.$$

Calculate $\Delta f(x,y,z)$.

Solution. $\Delta f(x,y,z) = -f(x,y,z)$. Either we calculate the partial derivatives D_{xx}, D_{yy} and D_{zz} or we use the fact that f is a function with spherical symmetry:

$$f(x,y,z) = g(r) = \frac{\sin r}{r}$$

So

$$\Delta f(x,y,z) = g''(r) + \frac{2}{r} g'(r)$$

The calculation is easier if we note that

$$g''(r) + \frac{2}{r} g'(r) = r^{-2}(r^2 g'(r))' = r^{-2}\frac{d}{dr}\left(r^2 \frac{d\,g(r)}{dr}\right)$$

since

$$r^2 g'(r) = r\,\cos r - \sin r$$

and so

$$(r^2 g'(r))' = (r\,\cos r - \sin r)' = -r\,\sin r.$$

i.e.

$$g''(r) + \frac{2}{r} g'(r) = -\frac{\sin r}{r} = -g(r).$$

3. Give the Hessian matrix and the Laplacian of

$$f(x,y) = (x-y)\cos(x+y).$$

Solution.

$$\nabla f(x,y) = \begin{pmatrix} \cos(x+y) - (x-y)\sin(x+y) \\ -\cos(x+y) - (x-y)\sin(x+y) \end{pmatrix}.$$

$$\text{Hess}\,(f)(x,y) = \begin{pmatrix} -2\sin(x+y) - (x-y)\cos(x+y) & -(x-y)\cos(x+y) \\ -(x-y)\cos(x+y) & 2\sin(x+y) - (x-y)\cos(x+y) \end{pmatrix}.$$

$$\Delta f(x,y) = -2f(x,y).$$

4. **Partial derivatives.**

(a) Let $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$, $\mathbf{a},\mathbf{b} \neq \mathbf{0}$, $g(\mathbf{x}) = \langle \mathbf{x},\mathbf{a}\rangle$, $h(\mathbf{x}) = \langle \mathbf{x},\mathbf{b}\rangle$. Show that the function

$$f(\mathbf{x}) = g(\mathbf{x})h(\mathbf{x})$$

is of class C^2. Give its Hessian matrix and its Laplacian. Give the symmetric matrix $A \in M_{2,2}(\mathbb{R})$ such that $f(\mathbf{x}) = \frac{1}{2}\langle \mathbf{x}, A\mathbf{x}\rangle$.

Solution. By exercise 3 in the March 14 sheet:

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = a_k\langle \mathbf{b}, \mathbf{x}\rangle + b_k\langle \mathbf{a}, \mathbf{x}\rangle$$

hence

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_j x_k} = a_k b_j + a_j b_k.$$

The Laplacian is given by the trace of this matrix, so

$$\triangle f(\mathbf{x}) = \sum_{j=1}^{n} a_j b_j + a_j b_j = 2\langle \mathbf{a}, \mathbf{b}\rangle.$$

The matrix A that gives the quadratic form is given by $A = \text{Hess}\,(f)$.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a function of class C^2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(\mathbf{x}) = \sum_{k=1}^{n} g(x_k)$, where x_k denotes the kth component of the vector $\mathbf{x}$, $x_k = \langle \mathbf{e}_k, \mathbf{x}\rangle$. Give the Hessian matrix and the Laplacian of f.

Solution. By exercise 3 in the March 9 sheet:

$$\frac{\partial f(\mathbf{x})}{\partial x_j} = g'(x_j)$$

so

$$\frac{\partial^2 f(\mathbf{x})}{\partial x_j x_k} = \begin{cases} 0 & \text{if } j \neq k, \\ g''(x_j) & \text{if } j = k. \end{cases}$$

Hence, the Hessian matrix is a diagonal matrix:

$$\text{Hess}\,(f)(\mathbf{x}) = \sum_{k=1}^{n} g''(x_k) E_{kk}$$

and $\triangle f(\mathbf{x}) = \sum_{k=1}^{n} g''(x_k)$.

5. Let $U \subset \mathbb{R}^n$ be an open set. Let $f, g : U \longrightarrow \mathbb{R}$ be two functions of class $C^2(U)$. Check that

$$\Delta(fg)(\mathbf{x}) = f(\mathbf{x})\Delta g(\mathbf{x}) + 2\langle \nabla f(\mathbf{x}), \nabla g(\mathbf{x})\rangle + g(\mathbf{x})\Delta f(\mathbf{x})$$

for all $\mathbf{x} \in U$. Using this identity and exercise 2 (c), then calculate the Laplacian of

$$h(x, y, z) = g(x, y) f(x, y, z) = \frac{xy}{x^2 + y^2} \frac{\sin(\sqrt{x^2 + y^2 + z^2})}{\sqrt{x^2 + y^2 + z^2}}.$$

on $U = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0\}$.

Solution. By the product rule for all $k = 1, ..., n$.

$$D_{kk}(fg) = fD_{kk} + 2D_k f D_k g + gD_{kk} f.$$

The sum on k gives the wanted identity. By exercise 2 (c), we get

$$\Delta f(x, y, z) = -f(x, y, z)$$

Moreover,

$$\nabla f(x,y,z) = \frac{1}{r}(\frac{\sin r}{r})' \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We calculate

$$\nabla g(x,y) = \begin{pmatrix} \frac{-y(x-y)(x+y)}{(x^2+y^2)^2} \\ \frac{x(x-y)(x+y)}{(x^2+y^2)^2} \end{pmatrix}$$

and

$$\Delta g(x,y) = -\frac{4g(x,y)}{x^2+y^2}.$$

Using $\langle \nabla f(x,y,z), \nabla g(x,y) \rangle = 0$ we get

$$\Delta h(x,y,z) = -h(x,y,z) - \frac{4h(x,y,z)}{x^2+y^2}.$$

6. Find:

$$\lim_{(x,y)\to(0,0)} \frac{3(x^2+y^2)}{\sqrt{x^2+y^2+4}-2}$$

if it exists.

Solution. We have:

$$\lim_{(x,y)\to(0,0)} \frac{3(x^2+y^2)}{\sqrt{x^2+y^2+4}-2} = \lim_{(x,y)\to(0,0)} \frac{3(x^2+y^2)(\sqrt{x^2+y^2+4}+2)}{x^2+y^2} =$$
$$= \lim_{(x,y)\to(0,0)} 3(\sqrt{x^2+y^2+4}+2) = 12.$$

7. Find:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+3yz^2+7xz^2}{x^2+y^2+z^4}$$

if it exists.

Solution. The limit does not exist because, for example, on the curve $y = z = 0$ the function is identically 0, while on the curve $x = y$, $z = 0$ the function is identically $1/2$.

8. Find the Taylor expansion of order two of:

$$f(x,y) = \cos(x-y) + 2\sin(x-y)$$

at $(0,0)$.

(a) $2 + 2x - 2y - 1/2x^2 + xy + 1/2y^2$
(b) $1 + 2x - 2y - 1/2x^2 - xy - 1/2y^2$
(c) $1 + 2x - 2y - 1/2x^2 + xy - 1/2y^2$
(d) $2 - 2x + 2y - 1/2x^2 - xy + 1/2y^2$

Solution. The correct answer is (c). The Taylor expansion of order two of f about $(0,0)$ is given by:

$$f(0,0) + f_x(0,0)x + f_y(0,0)y + 1/2 f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + 1/2 f_{yy}(0,0)y^2.$$

Also:

$$\begin{aligned} f_x &= -\sin(x-y) + 2\cos(x-y) \\ f_y &= \sin(x-y) - 2\cos(x-y) \\ f_{xx} &= -\cos(x-y) - 2\sin(x-y) \\ f_{xy} &= \cos(x-y) + 2\sin(x-y) \\ f_{yy} &= \cos(x-y) - 2\sin(x-y) \end{aligned}$$

so we find that the Taylor expansion is: $1 + 2x - 2y - 1/2x^2 + xy - 1/2y^2$.

9. Find the Taylor expansion of order two of:

$$f(x,y) = e^{2x-y^2}$$

at $(0,0)$.

(a) $1 - y - xy + 2y^2$
(b) $1 + 2x + 2x^2 - y^2$
(c) $1 + x + 2x^2 + y^2$
(d) $1 - x + 2x^2 - y^2$

Solution. The correct answer is (b). The Taylor expansion of order two of f about $(0,0)$ is given by:

$$f(0,0) + f_x(0,0)x + f_y(0,0)y + 1/2 f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + 1/2 f_{yy}(0,0)y^2.$$

Also:

$$\begin{aligned} f_x &= 2e^{2x-y^2} \\ f_y &= -2ye^{2x-y^2} \\ f_{xx} &= 4e^{2x-y^2} \\ f_{xy} &= -4ye^{2x-y^2} \\ f_{yy} &= -2e^{2x-y^2} + 4y^2e^{2x-y^2} \end{aligned}$$

so we find that the Taylor expansion is: $1 + 2x + 2x^2 - y^2$.