Analysis II
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1. **Level curves.** Find the equation of the level curve of the function $f(x, y)$ that passes through the given point.

   (a) $f(x, y) = 16 - x^2 - y^2$, $(2\sqrt{2}, \sqrt{2})$

   Solution. $f(2\sqrt{2}, \sqrt{2}) = 6$, So the level curve have the equation $x^2 + y^2 = 10$.

   (b) $f(x, y) = \sqrt{x^2 - 1}$, $(1, 0)$

   Solution. $f(1, 0) = 0$, so the level curve have the equation $x^2 = 1$ which consist of two lines $x = 1$ and $x = -1$.

   (c) $f(x, y) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}}$, $(0, 1)$

   Solution. We have that

$$\int_x^y \frac{d\theta}{\sqrt{1-\theta^2}} = \text{arc sin}y - \text{arc sin}x$$

where $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. We also have that $f(0, 1) = \pi/2$. In order for $\text{arc sin}y - \text{arc sin}x$ to be equal to $\pi/2$ we must have $0 \leq \text{arc sin}y \leq \pi/2$ and $-\pi/2 \leq \text{arc sin}x \leq 0$ meaning $0 \leq y \leq 1$ and $-1 \leq x \leq 0$. So the equation of the curve is given by

$$\text{arc sin}y - \text{arc sin}x = \frac{\pi}{2} \Longrightarrow y = \sin(\frac{\pi}{2} - \text{arc sin}x) \Longrightarrow y = \sqrt{1 - x^2}, \quad x \leq 0$$

2. **Continuous functions.**

   (a) Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$. Show that the function $f(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle \cdot \langle \mathbf{b}, \mathbf{x} \rangle$ is continuous for all $\mathbf{x} \in \mathbb{R}^n$.

   Solution . It is the product of two continuous functions (linear forms), so we get the continuity of f. Indeed, let $\mathbf{x}, \mathbf{x}_j \in \mathbb{R}^n$ s.t. $\lim_{j\to\infty} \mathbf{x}_j = \mathbf{x}$. Then, by continuity of linear forms

$$\lim_{j\to\infty} \langle \mathbf{a}, \mathbf{x}_j \rangle = \langle \mathbf{a}, \mathbf{x} \rangle, \quad \lim_{j\to\infty} \langle \mathbf{b}, \mathbf{x}_j \rangle = \langle \mathbf{b}, \mathbf{x} \rangle,$$

   hence $\lim_{j\to\infty} f(\mathbf{x}_j) = f(\mathbf{x})$ since it is the product of two convergent numerical sequences.

   (b) For $A \in M_{n,n}(\mathbb{R})$ let $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form given by $b(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, A\mathbf{y} \rangle$. Show that b is continuous for all $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{2n}$.

Solution-1. For all sequences of vectors $\begin{pmatrix} \mathbf{x}_j \\ \mathbf{y}_j \end{pmatrix} \in \mathbb{R}^{2n}$ that converge to $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{2n}$, we have $\lim_{j\to\infty} \mathbf{x}_j = \mathbf{x}$ and $\lim_{j\to\infty} \mathbf{y}_j = \mathbf{y}$ in $\mathbb{R}^n$, ie $\lim_{j\to\infty} ||\mathbf{x}_j - \mathbf{x}||_2 = 0$, $\lim_{j\to\infty} ||\mathbf{y}_j - \mathbf{y}||_2 = 0$ (with the Euclidean norm in $\mathbb{R}^n$). In particular, $||\mathbf{x}_j||_2, ||\mathbf{y}_j||_2$ are bounded. By bilinearity of b:

$$b(\mathbf{x}_j, \mathbf{y}_j) - b(\mathbf{x}, \mathbf{y}) = b(\mathbf{x}_j - \mathbf{x}, \mathbf{y}_j) + b(\mathbf{x}, \mathbf{y}_j - \mathbf{y}).$$

By Cauchy-Schwarz's inequality and the inequality $||A\mathbf{x}||_2 \leq ||A||_2||\mathbf{x}||_2$ (see lecture, ch.1.6.1, p.14) we get:

$$|b(\mathbf{x}_j, \mathbf{y}_j) - b(\mathbf{x}, \mathbf{y})| \leq ||A||_2||\mathbf{y}_j||_2||\mathbf{x}_j - \mathbf{x}||_2 + ||A||_2||\mathbf{x}||_2||\mathbf{y}_j - \mathbf{y}||_2 \to 0$$

hence the result.

Solution-2. The game $\epsilon - \delta$. We have to show that for all $\epsilon < 0$ there exists $\delta < 0$ such that for all vectors $\begin{pmatrix} \mathbf{h} \\ \mathbf{k} \end{pmatrix} \in \mathbb{R}^{2n}$ of Euclidean norm smaller than δ we get

$$|b(\mathbf{x} + \mathbf{h}, \mathbf{y} + \mathbf{k}) - b(\mathbf{x}, \mathbf{y})| < \epsilon.$$

Note that for the Euclidean norm in $\mathbb{R}^{2n}$:

$$|| \begin{pmatrix} \mathbf{h} \\ \mathbf{k} \end{pmatrix} ||_2^2 = ||\mathbf{h}||_2^2 + ||\mathbf{k}||_2^2$$

with the norms on the right taken in $\mathbb{R}^n$. By estimation of the solution 1 above, for all $\mathbf{x}, \mathbf{y}, \mathbf{h}, \mathbf{k} \in \mathbb{R}^n$:

$$|b(\mathbf{x} + \mathbf{h}, \mathbf{y} + \mathbf{k}) - b(\mathbf{x}, \mathbf{y})| \leq ||A||_2||\mathbf{y} + \mathbf{k}||_2||\mathbf{h}||_2 + ||A||_2||\mathbf{x}||_2||\mathbf{k}||_2.$$

We can assume that $||\mathbf{y} + \mathbf{k}||_2 < C$, $||\mathbf{x}||_2 < C$ for a constant $C > 0$. Then,

$$|b(\mathbf{x} + \mathbf{h}, \mathbf{y} + \mathbf{k}) - b(\mathbf{x}, \mathbf{y})| \leq C||A||_2(||\mathbf{h}||_2 + ||\mathbf{k}||_2) \leq C||A||_2\sqrt{2||\mathbf{h}||_2^2 + 2||\mathbf{k}||_2^2}$$

by the inequality $a + b \leq \sqrt{2a^2 + 2b^2}$ for all $a, b \geq 0$. We choose $\delta = \dfrac{\epsilon}{\sqrt{2}C||A||_2}$.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = \sum_{k=1}^{n} g(x_k)$, where x_k denotes the kth component of the vector $\mathbf{x}$, $x_k = \langle \mathbf{e}_k, \mathbf{x} \rangle$, is a continuous function for all $\mathbf{x} \in \mathbb{R}^n$.

Solution. Either by sequences or by

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{k=1}^{n} g(x_k + h_k) - g(x_k)$$

and $|h_k| \leq ||\mathbf{h}||_2$, applying the continuity of g:

$$\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{k=1}^{n} \lim_{h_k\to 0} g(x_k + h_k) - g(x_k) = 0.$$

Alternatively we can argue that the functions $h_k : \mathbb{R}^n \to \mathbb{R}$ defined by $h_k(\mathbf{x}) = g(\langle \mathbf{e}_k, \mathbf{x} \rangle)$ are continuous on $\mathbb{R}^n$ (it is the composition of a continuous function with a continuous linear form- see exercise under) and f is a finite sum of continuous functions.

(d) Let $g : \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ be continuous functions. Show that $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = g(h(\mathbf{x}))$ is a continuous function for all $\mathbf{x} \in \mathbb{R}^n$.

Solution. Let $\mathbf{x}, \mathbf{x}_j \in \mathbb{R}^n$ s.t. $\lim_{j\to\infty} \mathbf{x}_j = \mathbf{x}$. Then, by continuity of h, the numerical sequence $a_j := (h(\mathbf{x}_j))_j$ is convergent and has for limit $a := h(\mathbf{x})$. By the continuity of g: $\lim_{j\to\infty} g(a_j) = g(a)$, ie

$$\lim_{j\to\infty} f(\mathbf{x}_j) = f(\mathbf{x}).$$

3. **Limits of real functions.**

(a) Calculate

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

Solution. The limit doesn't exist since $f(0, y) = -1$ for $y \neq 0$ and $f(x, 0) = 1$ if $x \neq 0$.

(b) Calculate

$$\lim_{(x,y)\to(0,0)} xy\,\frac{x^2 - y^2}{x^2 + y^2}$$

Solution. Note that

$$\left| xy\,\frac{x^2 - y^2}{x^2 + y^2} \right| \leq |xy| \leq x^2 + y^2$$

Hence

$$\lim_{(x,y)\to(0,0)} xy\,\frac{x^2 - y^2}{x^2 + y^2} = 0$$

(c) Calculate

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{x^2 + y^2}$$

Solution. We use polar coordinates. We substitute $x = r\cos\theta$ and $y = r\sin\theta$ and investigate the limit of resulting expression as $r \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{x^2 + y^2} = \lim_{r\to 0} \frac{e^{-\frac{1}{r}}}{r^2} = 0$$

(d) Show that the function

$$f(x, y) = \frac{2x^2 y}{x^4 + y^2}$$

has no limit as (x, y) approaches $(0, 0)$. In particular show the value of the limit take varies between -1 and 1 along curves $y = kx^2$.

Solution. We take the limit along the curve $y = kx$. If $x \neq 0$

$$f(x, y)\Big|_{y=kx^2} = \frac{2kx^4}{(1 + k^2)x^4} = \frac{2k}{1 + k^2}$$

So

$$\lim_{\text{along } y=kx^2} f(x, y) = \lim_{(x,y)\to(0,0)} f(x, y)\Big|_{y=kx^2} = \frac{2k}{1 + k^2}$$

This limit varies with the path of approach. Now take $k = \tan\theta$ then

$$\frac{2k}{1 + k^2} = \frac{2\tan\theta}{1 + \tan^2\theta} = \sin 2\theta$$

And $\sin 2\theta$ varies between -1 and 1.

(e) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the continuous function defined by

$$f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } xy \neq 0, \\ 1 & \text{if } xy = 0. \end{cases}$$

Show that f is partially differentiable and give its partial derivatives.

Solution. If $xy \neq 0$, then

$$D_x \frac{\sin(xy)}{xy} = \frac{xy^2\cos(xy) - y\sin(xy)}{x^2y^2}$$

$$D_y \frac{\sin(xy)}{xy} = \frac{x^2y\cos(xy) - x\sin(xy)}{x^2y^2}$$

If $xy = 0$, there are three cases: $x = 0, y \neq 0$ or $x \neq 0, y = 0$ or also $x = 0, y = 0$. For example, for the first case:

$$D_x f(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h} = 0$$

and

$$D_y f(0,y) = \lim_{h\to 0} \frac{f(0,y+h) - f(0,y)}{h} = 0$$

4. **Continuity.** Study continuity of following functions as a function of $\alpha > 0$.

(a)

$$f(x,y) = \begin{cases} \dfrac{x^{2\alpha}}{x^2+y^2}, & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution. For $(x,y) \neq (0,0)$ the denominator is non-zero and f is a combination of continuous functions. Therefor for all $\alpha > 0$, $f(x,y)$ is continuous $\forall (x,y) \neq (0,0)$. We check the continuity at $(x,y) = (0,0)$. Using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$\lim_{(x,y)\to(0,0)} \frac{x^{2\alpha}}{x^2+y^2} = \lim_{r\to 0} \frac{r^{2\alpha}\cos^{2\alpha}\theta}{r^2}$$

The value of the limit depends on α:

- case $\alpha > 1$: The limit is 0 because $|r^{2\alpha}\cos^{2\alpha}\theta| \leq |r^{2\alpha}| \to 0$
- case $\alpha = 1$: The value of the limit is $1 \cdot \cos\theta$
- case $0 < \alpha < 1$: The limit is $+\infty$ if $\cos\theta \neq 0$ and the limit is 0 if $\cos\theta = 0$.

So f is continuous on $\mathbb{R}^2$ if $\alpha > 1$ and is continuous on $\mathbb{R}^2\backslash(0,0)$ when $0 < \alpha \leq 1$

(b)

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^\alpha}, & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution. For $(x,y) \neq (0,0)$ the denominator is non-zero and f is a combination of continuous functions. Therefor for all $\alpha > 0$, $f(x,y)$ is continuous $\forall (x,y) \neq (0,0)$. We check the continuity at $(x,y) = (0,0)$. Using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ we have

$$\lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^\alpha} = \lim_{r\to 0} r^{2(1-\alpha)}\cos\theta\sin\theta$$

The value of the limit depends on α:

- $\alpha = 1$: the limit is $\cos\theta\sin\theta$.
- $0 < \alpha < 1$: the limit is 0.
- $\alpha > 1$: depending on θ it can be 0, $+\infty$ and $-\infty$.

So f is continuous on $\mathbb{R}^2$ if $0 < \alpha < 1$ and is continuous on $\mathbb{R}^2\backslash(0,0)$ when $\alpha \geq 1$

5. **Partial derivatives.**

(a) Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$. Show that the function $f(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x}\rangle \cdot \langle \mathbf{b}, \mathbf{x}\rangle$ is partially differentiable for all $\mathbf{x} \in \mathbb{R}^n$ and give its gradient.

Solution-1. $f = g \cdot h$ is the product of two partially differentiable functions $g(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x}\rangle$ and $h(\mathbf{x}) = \langle \mathbf{b}, \mathbf{x}\rangle$. By the product rule:

$$\nabla f(\mathbf{x}) = \nabla(gh)(\mathbf{x}) = h(\mathbf{x})\nabla g(\mathbf{x}) + g(\mathbf{x})\nabla h(\mathbf{x}). \tag{1}$$

It follows that $\nabla g(\mathbf{x}) = \mathbf{a}$, $\nabla h(\mathbf{x}) = \mathbf{b}$:

$$\nabla f(\mathbf{x}) = \langle \mathbf{b}, \mathbf{x}\rangle\mathbf{a} + \langle \mathbf{a}, \mathbf{x}\rangle\mathbf{b}.$$

Solution-2. For all $k = 1, \ldots, n$, $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$:

$$f(\mathbf{x} + t\mathbf{e}_k) - f(\mathbf{x}) = t\langle \mathbf{a}, \mathbf{e}_k\rangle\langle \mathbf{b}, \mathbf{x} + t\mathbf{e}_k\rangle + t\langle \mathbf{a}, \mathbf{x}\rangle\langle \mathbf{b}, \mathbf{e}_k\rangle$$

hence

$$t^{-1}(f(\mathbf{x} + t\mathbf{e}_k) - f(\mathbf{x})) = a_k\langle \mathbf{b}, \mathbf{x}\rangle + b_k\langle \mathbf{a}, \mathbf{x}\rangle + ta_kb_k.$$

By letting t go to zero, we get the result.

(b) For $A \in M_{n,n}(\mathbb{R})$, let $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the bilinear form given by $b(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, A\mathbf{y}\rangle$. Show that b is partially differentiable for all $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{2n}$ and give its gradient.

Solution . For the partial derivatives of x_k, the argument $\mathbf{y}$ is constant, hence it is the study of the linear form $\mathbf{x} \mapsto \langle \mathbf{x}, A\mathbf{y}\rangle$. We find $\nabla_{\mathbf{x}} b(\mathbf{x}, \mathbf{y}) = A\mathbf{y}$. For the partial derivatives of y_k the argument $\mathbf{x}$ is constant, hence it is the study of the linear form $\mathbf{y} \mapsto \langle A^T\mathbf{x}, \mathbf{y}\rangle$ (we have to put the matrix in the constant argument). We find $\nabla_{\mathbf{y}} b(\mathbf{x}, \mathbf{y}) = A^T\mathbf{x}$. The gradient of b is the vector in $\mathbb{R}^{2n}$ given by

$$\nabla b(\mathbf{x}, \mathbf{y}) = \nabla_{\mathbf{x},\mathbf{y}} b(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} A\mathbf{y} \\ A^T\mathbf{x} \end{pmatrix}.$$

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = \sum_{k=1}^{n} g(x_k)$, where x_k denotes the kth component of the vector $\mathbf{x}$, $x_k = \langle \mathbf{e}_k, \mathbf{x}\rangle$, is a partially differentiable function for all $\mathbf{x} \in \mathbb{R}^n$. Give its gradient.

Solution. By the definition of partial derivatives:

$$\frac{\partial f(\mathbf{x})}{\partial x_j} = \lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h} = \lim_{h\to 0} \frac{g(x_j + h) - g(x_j)}{h} = g'(x_j)$$

for all $\mathbf{x} \in \mathbb{R}^n$ and g' denotes the derivative function of g. Hence,

$$\nabla f(\mathbf{x}) = \sum_{k=1}^{n} g'(x_k)\mathbf{e}_k.$$

(d) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable for all $t \in \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ partially differentiable for all $\mathbf{x} \in \mathbb{R}^n$. Show that $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = g(h(\mathbf{x}))$ is a partially differentiable function for all $\mathbf{x} \in \mathbb{R}^n$. Give its gradient.

Solution . The function $\rho(t) := h(\mathbf{x}+t\mathbf{e}_k)$ is differentiable in $t=0$ and $\rho'(0) = \dfrac{\partial h(\mathbf{x})}{\partial x_k}$. The composite function $g(\rho(t))$ is differentiable at $t=0$ and

$$\frac{\partial f(\mathbf{x})}{\partial x_k} = \frac{d}{dt}\Big|_{t=0} g(\rho(t)) = g'(\rho(0))\rho'(0) = g'(h(\mathbf{x}))\frac{\partial h(\mathbf{x})}{\partial x_k},$$

hence $\nabla f(\mathbf{x}) = g'(h(\mathbf{x}))\nabla h(\mathbf{x})$.

6. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}}) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is differentiable at $(0,0)$ but is not of class C^1 at this point.

Solution. Let $r = \sqrt{x^2+y^2}$, $r \geq 0$. f is differentiable at $(0,0)$ and $d_0 f(x,y) = 0$ since

$$\lim_{r\to 0_+} \frac{f(x,y) - f(0,0)}{r} = \underset{r\to 0}{r} \sin(\frac{1}{r}) = 0$$

If $(x,y) \neq (0,0)$ the function f is partially differentiable (even differentiable) and noting that f is radially symmetric:

$$\frac{\partial f(x,y)}{\partial x} = \frac{x}{r}\big(2r\sin r^{-1} - \cos r^{-1}\big),$$
$$\frac{\partial f(x,y)}{\partial y} = \frac{y}{r}\big(2r\sin r^{-1} - \cos r^{-1}\big)$$

These functions don't have any limits when $(x,y) \to (0,0)$ (because of $\cos r^{-1}$).

7. For $x \in \mathbb{R}$ and $t > 0$ we consider the function $f(x,t)$ defined by

$$f(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t}).$$

(a) Show that f verifies the heat equation, i.e.

$$\frac{\partial f}{\partial t}(x,t) - \frac{\partial^2 f}{\partial x^2}(x,t) = 0$$

Solution.

$$\frac{\partial f}{\partial x}(x,t) = -\frac{x}{2t} f(x,t)$$

and

$$\frac{\partial^2 f}{\partial x^2}(x,t) = -\frac{1}{2t} f(x,t) - \frac{x}{2t}\frac{\partial f}{\partial x}(x,t) = (-\frac{1}{2t} + \frac{x^2}{4t^2})f(x,t) = \frac{\partial f}{\partial t}(x,t)$$

(b) Calculate

$$\int_{\mathbb{R}} f(x,t)\, dx$$

Solution. By the change of variable $y = x/\sqrt{2t}$ i.e. $dx/dy = \sqrt{2t}$, we get the Gauss integral :

$$\int_{\mathbb{R}} f(x,t)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\,dy = 1$$

(c) Let $g(x,y,t)$ given by $g(x,y,t) = f(x,t)f(y,t)$. Calculate

$$\frac{\partial g}{\partial t}(x,y,t) - \frac{\partial^2 g}{\partial x^2}(x,y,t) - \frac{\partial^2 g}{\partial y^2}(x,y,t).$$

Remark: $\frac{\partial^2}{\partial x^2} = D_{xx}$ etc.

Solution. By the product rule and the result in (a) we get

$$\frac{\partial g}{\partial t}(x,y,t) - \frac{\partial^2 g}{\partial x^2}(x,y,t) - \frac{\partial^2 g}{\partial y^2}(x,y,t) =$$
$$\frac{\partial f}{\partial t}(x,t)f(y,t) + f(x,t)\frac{\partial f}{\partial t}(y,t) - \frac{\partial^2 f}{\partial x^2}(x,t)f(y,t) - f(x,t)\frac{\partial^2 f}{\partial y^2}(y,t) =$$
$$f(x,t)\left(\frac{\partial f}{\partial t}(y,t) - \frac{\partial^2 f}{\partial y^2}(y,t)\right) + f(y,t)\left(\frac{\partial f}{\partial t}(x,t) - \frac{\partial^2 f}{\partial x^2}(x,t)\right) = 0.$$

8. True of False.

(a) A continuous function is partially differentiable.

☐ True ☐ False

Solution. False, for example take $f(x) = \|x\|_2$ which is a continuous function and not differentiable.

(b) If all the directional derivatives of f exist, then all the partial derivatives also exist.

☐ True ☐ False

Solution. True, Just take the directions to be the basis of the space.

(c) If all the partial derivatives of f exist, then all the directional derivative also exit.

☐ True ☐ False

Solution. True, write any vector v as a linear combination of basis vectors then the statement follows immediately.

(d) If all the partial derivatives of f exist, then f is continuous.

☐ True ☐ False

Solution. False, take the following function for example

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

all partial derivatives exist but is not continuous.