Analysis II
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ÉCOLE POLYTECHNIQUE
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Solutions to Exercise Session, March 14, 2016

1. **Level curves.** Find the equation of the level curve of the function $f(x,y)$ that passes through the given point.

   (a) $f(x,y) = 16 - x^2 - y^2$, $(2\sqrt{2}, \sqrt{2})$

   (b) $f(x,y) = \sqrt{x^2-1}$, $(1,0)$

   (c) $f(x,y) = \int_x^y \frac{d\theta}{\sqrt{1-\theta^2}}$, $(0,1)$

2. **Continuous functions.**

   (a) Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$. Show that the function $f(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle \cdot \langle \mathbf{b}, \mathbf{x} \rangle$ is continuous for all $\mathbf{x} \in \mathbb{R}^n$.

   (b) For $A \in M_{n,n}(\mathbb{R})$ let $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form given by $b(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, A\mathbf{y} \rangle$. Show that b is continuous for all $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{2n}$.

   (c) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = \sum_{k=1}^{n} g(x_k)$, where x_k denotes the kth component of the vector $\mathbf{x}$, $x_k = \langle \mathbf{e}_k, \mathbf{x} \rangle$, is a continuous function for all $\mathbf{x} \in \mathbb{R}^n$.

3. **Limits of real functions.**

   (a) Calculate
   $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

   (b) Calculate
   $$\lim_{(x,y)\to(0,0)} xy\,\frac{x^2-y^2}{x^2+y^2}$$

   (c) Calculate
   $$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{\sqrt{x^2+y^2}}}}{x^2+y^2}$$

   (d) Show that the function
   $$f(x,y) = \frac{2x^2y}{x^4+y^2}$$
   has no limit as (x,y) approaches $(0,0)$. In particular show the value of the limit take varies between -1 and 1 along curves $y = kx^2$.

   (e) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the continuous function defined by
   $$f(x,y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } xy \neq 0, \\ 1 & \text{if } xy = 0. \end{cases}$$
   Show that f is partially differentiable and give its partial derivatives.

4. **Continuity.** Study continuity of following functions as a function of $\alpha > 0$.

(a)

$$f(x,y) = \begin{cases} \dfrac{x^{2\alpha}}{x^2+y^2}, & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(b)

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^\alpha}, & \text{if } (x,y) \neq 0 \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

5. **Partial derivatives.**

(a) Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$. Show that the function $f(\mathbf{x}) = \langle \mathbf{a}, \mathbf{x} \rangle \cdot \langle \mathbf{b}, \mathbf{x} \rangle$ is partially differentiable for all $\mathbf{x} \in \mathbb{R}^n$ and give its gradient.

(b) For $A \in M_{n,n}(\mathbb{R})$, let $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the bilinear form given by $b(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, A\mathbf{y} \rangle$. Show that b is partially differentiable for all $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{2n}$ and give its gradient.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = \sum_{k=1}^{n} g(x_k)$, where x_k denotes the kth component of the vector $\mathbf{x}$, $x_k = \langle \mathbf{e}_k, \mathbf{x} \rangle$, is a partially differentiable function for all $\mathbf{x} \in \mathbb{R}^n$. Give its gradient.

(d) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable for all $t \in \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ partially differentiable for all $\mathbf{x} \in \mathbb{R}^n$. Show that $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(\mathbf{x}) = g(h(\mathbf{x}))$ is a partially differentiable function for all $\mathbf{x} \in \mathbb{R}^n$. Give its gradient.

6. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}}) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that f is differentiable at $(0,0)$ but is not of class C^1 at this point.

7. For $x \in \mathbb{R}$ and $t > 0$ we consider the function $f(x,t)$ defined by

$$f(x,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{x^2}{4t}).$$

(a) Show that f verifies the heat equation, i.e.

$$\frac{\partial f}{\partial t}(x,t) - \frac{\partial^2 f}{\partial x^2}(x,t) = 0$$

(b) Calculate

$$\int_{\mathbb{R}} f(x,t)\, dx$$

(c) Let $g(x,y,t)$ given by $g(x,y,t) = f(x,t)f(y,t)$. Calculate

$$\frac{\partial g}{\partial t}(x,y,t) - \frac{\partial^2 g}{\partial x^2}(x,y,t) - \frac{\partial^2 g}{\partial y^2}(x,y,t).$$

Remark: $\frac{\partial^2}{\partial x^2} = D_{xx}$ etc.

8. True of False.

   (a) A continuous function is partially differentiable.

   ☐ True ☐ False

   (b) If all the directional derivatives of f exist, then all the partial derivatives also exist.

   ☐ True ☐ False

   (c) If all the partial derivatives of f exist, then all the directional derivative also exit.

   ☐ True ☐ False

   (d) If all the partial derivatives of f exist, then f is continuous.

   ☐ True ☐ False