Analysis II
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ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Solutions to Exercise Session, March 7, 2016

1. **Two Formulas.**

   (a) Let $\mathbf{f}, \mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}^n$ be two functions of class C^1. Show that

   $$\frac{d}{dt}\langle \mathbf{f}(t), \mathbf{g}(t)\rangle = \langle \mathbf{f}'(t), \mathbf{g}(t)\rangle + \langle \mathbf{f}(t), \mathbf{g}'(t)\rangle.$$

   Solution. By the definition of the scalar product and the properties of the derivative (linearity and the product rule), we get

   $$\begin{aligned}\frac{d}{dt}\langle \mathbf{f}(t), \mathbf{g}(t)\rangle &= \frac{d}{dt}\sum_{k=1}^{n} f_k(t)g_k(t)\\ &= \sum_{k=1}^{n} f_k'(t)g_k(t) + f_k(t)g_k'(t)\\ &= \langle \mathbf{f}'(t), \mathbf{g}(t)\rangle + \langle \mathbf{f}(t), \mathbf{g}'(t)\rangle.\end{aligned}$$

   (b) Let $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The cross product $\mathbf{a} \times \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b}$ is defined by

   $$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

   Calculate

   $$\langle \mathbf{a} \times \mathbf{b}, \mathbf{a}\rangle,\ \langle \mathbf{a} \times \mathbf{b}, \mathbf{b}\rangle \text{ and } \langle \mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b}\rangle.$$

   Let $\mathbf{f}, \mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}^3$ be two functions of class C^1. Show that

   $$\frac{d}{dt}\big(\mathbf{f}(t) \times \mathbf{g}(t)\big) = \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t).$$

   Solutions.

   $$\langle \mathbf{a} \times \mathbf{b}, \mathbf{a}\rangle = 0, \quad \langle \mathbf{a} \times \mathbf{b}, \mathbf{b}\rangle = 0$$

   and

   $$\langle \mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b}\rangle = \langle \mathbf{a}, \mathbf{a}\rangle\langle \mathbf{b}, \mathbf{b}\rangle - \langle \mathbf{a}, \mathbf{b}\rangle^2.$$

   Let $\mathbf{f} = (f_1, f_2, f_3)$ and $\mathbf{g} = (g_1, g_2, g_3)$. We only write the computation for the first component:

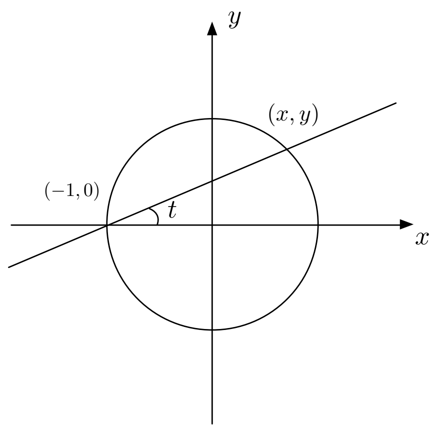
   $$\left(\frac{d}{dt}\mathbf{f}(t) \times \mathbf{g}(t)\right)_1 = \frac{d}{dt}(f_2(t)g_3(t) - f_3(t)g_2(t)) =$$
   $$= f_2'(t)g_3'(t) + f_2(t)g_3'(t) - f_3'(t)g_2'(t) - f_3(t)g_2'(t) = (\mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t))_1.$$

2. **parameterization of a circle.** Consider the unit circle in the 2D plane:

   $$x^2 + y^2 = 1. \qquad (1)$$

(a) Find a trigonometric parameterization $\theta \to (x(\theta), y(\theta))$ of the unit circle.

(b) **Rational Parameterization.** Consider the line that meets the circle at the point $(-1, 0)$ and another point (x, y) that has the slope t. Find the parameterization of the circle $t \to (x(t), y(t))$. What is the computational difference between this parameterization and the one in (a)?



Solution.

(a) The trigonometric parameterization is given by $(\cos(\theta), \sin(\theta))$ where $\theta \in [0, 2\pi]$.

(b) the line passes through the point $(-1, 0)$ that has the slope t. The equation of the line then can be written as $y = tx + t$. We find the coordinates of the points that the line and circle meet by solving the following system of equations

$$\begin{cases} x^2 + y^2 = 1 \\ y = tx + t \end{cases}$$

By substituting y into the first equation we get

$$(1 + t^2)x^2 + 2t^2x + t^2 - 1 = 0 \quad \Longrightarrow \quad (x + 1)((1 + t^2)x + t^2 - 1) = 0$$

So for $t \in [-\pi/2, \pi/2]$

$$x(t) = \frac{1 - t^2}{1 + t^2}, \quad y(t) = \frac{2t}{1 + t^2}$$

The benefit of this parameterization is that for any rational number t we receive a rational number for x and y so there is no need for approximating irrational values when dealing with sin and cos.

3. **Helix.** Consider the curve

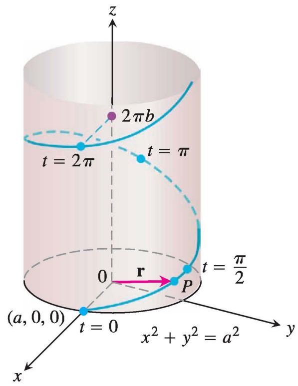
$$\mathbf{r}(t) = \begin{pmatrix} a\cos(t) \\ a\sin(t) \\ bt \end{pmatrix}$$

for $a, b > 0$.

(a) Draw the curve for $t \in [0, 2\pi]$. Use a commercial software if you can e.g. MATLAB, Mathematica or etc.

(b) Find the curvature κ for the helix.

(c) What is the largest value κ can have for a given value b.

(a) 

(d) What is the length of the curve for $t \in [0, 2\pi]$.

Solution.

(b) The curvature κ of $\mathbf{r}(t)$ is given by,

$$\kappa(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}.$$

and we have

$$\mathbf{r}'(t) = \begin{pmatrix} -a\sin(t) \\ a\cos(t) \\ b \end{pmatrix}, \quad \mathbf{r}''(t) = \begin{pmatrix} -a\cos(t) \\ -a\sin(t) \\ 0 \end{pmatrix}.$$

Thus

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{pmatrix} ab\sin(t) \\ -ab\cos(t) \\ a^2 \end{pmatrix}$$

and

$$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{a^2b^2 + a^4}, \quad \|\mathbf{r}'\| = \sqrt{a^2 + b^2}$$

Hence

$$\kappa = \frac{a}{a^2 + b^2}$$

(c) We set the derivative with respect to a to be zero

$$\kappa'(a) = \frac{-a^2 + b^2}{(a^2 + b^2)^2} = 0 \Longrightarrow a = b.$$

It is easy to check that for $a = b$, κ attains its maximum.

(d) The formula for the length of a curve is given by

$$L = \int_a^b |r'(t)| \, dt$$

So

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{a^2\sin^2(t) + a^2\cos^2(t) + b^2} \, dt \\ &= \int_0^{2\pi} \sqrt{a^2 + b^2} \, dt \\ &= 2\pi\sqrt{a^2 + b^2} \end{aligned}$$

4. **Free movement.** Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ be a curve of class C^2 such that $\ddot{\mathbf{r}}(t) = \mathbf{0}$. For $m > 0$, we introduce the momentum $\mathbf{p}(t) = m\dot{\mathbf{r}}(t)$ and the angular momentum $\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}(t)$.

   (a) Show that $\mathbf{L}(t)$ is constant.

   (b) Show that the energy $E(t) := \dfrac{\langle \mathbf{p}(t), \mathbf{p}(t)\rangle}{2m}$ is constant.

   Solutions. $\mathbf{L}(t)$ is of class C^1 and, by exercise 3, $\dot{\mathbf{L}}(t) = m\dot{\mathbf{r}}(t) \times \dot{\mathbf{r}}(t) + m\mathbf{r}(t) \times \ddot{\mathbf{r}}(t) = \mathbf{0}$. By the mean value theorem (see Analyse 1) each component of $\mathbf{L}(t)$ is constant so $\mathbf{L}(t)$ is constant. The energy is of class C^1. It is constant since $\dot{\mathbf{p}}(t) = m\ddot{\mathbf{r}}(t) = \mathbf{0}$ and, by exercise 3,

$$\dot{E}(t) = \frac{\langle \dot{\mathbf{p}}(t), \mathbf{p}(t)\rangle}{2m} + \frac{\langle \mathbf{p}(t), \dot{\mathbf{p}}(t)\rangle}{2m} = 0,$$

   and we conclude again with the mean value theorem.

   Solution - b. Alternately we can apply the mean value theorem to solve for $\mathbf{r}(t)$. The equation $\dot{\mathbf{p}}(t) = m\ddot{\mathbf{r}}(t) = \mathbf{0}$ implies that $\mathbf{p}(t)$ is constant, ie $\mathbf{p}(t) = \mathbf{p}_0 = m\mathbf{v}_0$ for a $\mathbf{p}_0 \in \mathbb{R}^3$. It follows that $\dot{\mathbf{r}} - \dot{(\mathbf{v}_0 t)} = \mathbf{0}$ and so $\mathbf{r}(t) = \mathbf{v}_0 t + \mathbf{r}_0$ for a $\mathbf{r}_0 \in \mathbb{R}^3$. By a direct calculation, it follows that

$$\mathbf{L}(t) = m\mathbf{r}_0 \times \mathbf{v}_0, \quad E(t) := \frac{\langle \mathbf{p}_0, \mathbf{p}_0\rangle}{2m} = \frac{m\langle \mathbf{v}_0, \mathbf{v}_0\rangle}{2}.$$

5. **Harmonic oscillator in three dimensions.** Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ be a curve of class C^2 such that $\ddot{\mathbf{r}}(t) = -\omega^2\mathbf{r}(t)$, $\omega > 0$. For $m > 0$, we introduce the momentum $\mathbf{p}(t) = m\dot{\mathbf{r}}(t)$ and the angular momentum $\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}(t)$.

   (a) Show that $\mathbf{L}(t)$ is constant.

   (b) Show that the energy $E(t) := \dfrac{\langle \mathbf{p}(t), \mathbf{p}(t)\rangle}{2m} + \dfrac{m\omega^2\langle \mathbf{r}(t), \mathbf{r}(t)\rangle}{2}$ is constant.

   Solutions. Using the equation $\ddot{\mathbf{r}}(t)$ we find as above

$$\dot{\mathbf{L}}(t) = m\dot{\mathbf{r}}(t) \times \dot{\mathbf{r}}(t) + m\mathbf{r}(t) \times \ddot{\mathbf{r}}(t) = \mathbf{0} + m\mathbf{r}(t) \times (-\omega^2\mathbf{r}(t)) = \mathbf{0}$$

   and we conclude by the mean value theorem (see exercise above). Similarly, the energy is of class C^1, and using the equation for $\ddot{\mathbf{r}}(t)$ and the properties of the scalar product (symmetry, linearity in each component) we get

$$\begin{aligned}\dot{E}(t) &= \frac{\langle \dot{\mathbf{p}}(t), \mathbf{p}(t)\rangle}{2m} + \frac{\langle \mathbf{p}(t), \dot{\mathbf{p}}(t)\rangle}{2m} + \frac{m\omega^2\langle \dot{\mathbf{r}}(t), \mathbf{r}(t)\rangle}{2} + \frac{m\omega^2\langle \mathbf{r}(t), \dot{\mathbf{r}}(t)\rangle}{2}\\ &= m\langle \ddot{\mathbf{r}}(t), \dot{\mathbf{r}}(t)\rangle + m\omega^2\langle \mathbf{r}(t), \dot{\mathbf{r}}(t)\rangle\\ &= m\langle \ddot{\mathbf{r}}(t) + \omega^2\mathbf{r}(t), \dot{\mathbf{r}}(t)\rangle\\ &= m\langle \mathbf{0}, \dot{\mathbf{r}}(t)\rangle = 0.\end{aligned}$$

   Remark. Usually, we don't know which quantities are constant (we say "conserved"). To find them, we multiply the equation by some appropriate functions. For example, if we take the cross product of the equation $\ddot{\mathbf{r}}(t) = -\omega^2\mathbf{r}(t)$ with $\mathbf{r}(t)$, we find

$$\ddot{\mathbf{r}}(t) \times \mathbf{r}(t) = -\omega^2\mathbf{r}(t) \times \mathbf{r}(t) = \mathbf{0}$$

   which brings us to $\dot{\mathbf{L}}(t) = \mathbf{0}$. If we take the scalar product of $\ddot{\mathbf{r}}(t) = -\omega^2\mathbf{r}(t)$ with $\dot{\mathbf{r}}(t)$, we find

$$\langle \ddot{\mathbf{r}}(t), \dot{\mathbf{r}}(t)\rangle = -\langle \omega^2\mathbf{r}(t), \dot{\mathbf{r}}(t)\rangle$$

which is equivalent to

$$\frac{d}{d\,t}\langle \dot{\mathbf{r}}(t), \dot{\mathbf{r}}(t)\rangle = -\omega^2 \frac{d}{d\,t}\langle \mathbf{r}(t)\mathbf{r}(t)\rangle$$

hence the conservation of the energy.

6. Find the derivative of,

$$f(t) = \langle \mathbf{u}(t), \mathbf{v}(t)\rangle.$$

When $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are given as,

$$\mathbf{u}(t) = \begin{pmatrix} 1 \\ -3t^2 \\ 4t^3 \end{pmatrix}, \quad \mathbf{v}(t) = \begin{pmatrix} t \\ \cos t \\ \sin t \end{pmatrix}.$$

(a) $f'(t) = 1 - 6t\cos t + 15t^2 \sin t + 4t^3 \cos t$

(b) $f'(t) = 6t\cos t + 15t^2 \sin t + 4t^3 \cos t$

(c) $f'(t) = 3t\cos t + 7t^2 \sin t + 4t^3 \cos t$

(d) $f'(t) = 1 - 3t\cos t + 11t^2 \sin t + 4t^3 \cos t$

Solutions. (a) is correct.

The product rule for derivatives of dot products says that,

$$f'(t) = \langle \mathbf{u}(t), \mathbf{v}'(t)\rangle + \langle \mathbf{u}'(t), \mathbf{v}(t)\rangle$$

Now for the given $\mathbf{u}(t)$ and $\mathbf{v}(t)$,

$$\mathbf{u}'(t) = \begin{pmatrix} 0 \\ -6t \\ 12t^2 \end{pmatrix}, \quad \mathbf{v}'(t) = \begin{pmatrix} 1 \\ -\sin t \\ \cos t \end{pmatrix}$$

Thus,

$$f'(t) = (1 + 3t^2 \sin t + 4t^3 \cos t) + (-6t\cos t + 12t^2 \sin t)$$

And so,

$$f'(t) = 1 - 6t\cos t + 15t^2 \sin t + 4t^3 \cos t.$$

7. Which of the following integrals gives the length of the curve,

$$\mathbf{c}(t) = \begin{pmatrix} 2t^2 \\ t \end{pmatrix}, \qquad 0 \le t \le 4.$$

(a) $I = \int_0^4 \sqrt{16t^2 + 1}\ dt$

(b) $I = 2\int_0^4 \left|16t^2 + 1\right|\ dt$

(c) $I = \int_0^2 \sqrt{16t^2 + 1}\ dt$

(d) $I = 2\int_0^2 \left|16t^2 + 1\right|\ dt$

Solutions. **(a) is correct.**

The arc length for the curve $\mathbf{c}(t)$, $a \leq t \leq b$, is given by the integral expression,

$$I = \int_a^b ||\mathbf{c}'(t)||\, dt.$$

For the given $\mathbf{c}(t)$ we have,

$$\mathbf{c}(t) = \begin{pmatrix} 4t \\ 1 \end{pmatrix}$$

Consequently, the curve has

$$\text{arc length} = \int_0^4 \sqrt{16t^2 + 1}\, dt.$$

8. Find the unit tangent vector $\mathbf{T}(t)$ to the graph of the vector function

$$\mathbf{r}(t) = \begin{pmatrix} 3\sin t \\ 4t \\ 3\cos t \end{pmatrix}.$$

(a) $\mathbf{T}(t) = \begin{pmatrix} \frac{3}{5}\cos t \\ \frac{4}{5} \\ \frac{3}{5}\sin t \end{pmatrix}$

(b) $\mathbf{T}(t) = \begin{pmatrix} 3\sin t \\ 4t \\ 3\cos t \end{pmatrix}$

(c) $\mathbf{T}(t) = \begin{pmatrix} 3\sin t \\ -4 \\ 3\cos t \end{pmatrix}$

(d) $\mathbf{T}(t) = \begin{pmatrix} \frac{3}{5}\cos t \\ \frac{4}{5} \\ -\frac{3}{5}\sin t \end{pmatrix}$

Solutions. **(d) is correct.**

The unit tangent vector $\mathbf{T}(t)$ to $\mathbf{r}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

For the given $\mathbf{r}(t)$ we have,

$$\mathbf{r}'(t) = \begin{pmatrix} 3\cos t \\ 4 \\ -3\sin t \end{pmatrix}$$

while,

$$|\mathbf{r}'(t)| = \sqrt{3^2(\cos^2 t + \sin^2 t) + (4)^2} = 5.$$

Consequently,

$$\mathbf{T}(t) = \begin{pmatrix} \frac{3}{5}\cos t \\ \frac{4}{5} \\ -\frac{3}{5}\sin t \end{pmatrix}$$

9. Determine the curvature, κ, of the curve

$$\mathbf{r}(t) = \begin{pmatrix} t^2 \\ 0 \\ 4t \end{pmatrix}$$

(a) $\kappa(t) = \frac{8t}{4t^2+16}$

(b) $\kappa(t) = \frac{8}{(4t^2+16)^{1/2}}$

(c) $\kappa(t) = \frac{8t}{(4t^2+16)^{3/2}}$

(d) $\kappa(t) = \frac{8}{(4t^2+16)^{3/2}}$

Solutions. (d) is correct

The curvature κ of $\mathbf{r}(t)$ is given by,

$$\kappa(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}.$$

So for the given $\mathbf{r}(t)$ we have,

$$\mathbf{r}'(t) = \begin{pmatrix} 2t \\ 0 \\ 4 \end{pmatrix}, \quad \mathbf{r}''(t) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

Thus

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}$$

and

$$||\mathbf{r}'(t) \times \mathbf{r}''(t)|| = \sqrt{8^2} = 8, \quad ||\mathbf{r}'(t)|| = (4t^2 + 16)^{1/2}.$$

Hence,

$$\kappa(t) = \frac{8}{(4t^2 + 16)^{3/2}}.$$