Analysis II
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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Exercise Session, March 7, 2016

1. **Two Formulas.**

   (a) Let $\mathbf{f}, \mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}^n$ be two functions of class C^1. Show that

   $$\frac{d}{dt}\langle \mathbf{f}(t), \mathbf{g}(t)\rangle = \langle \mathbf{f}'(t), \mathbf{g}(t)\rangle + \langle \mathbf{f}(t), \mathbf{g}'(t)\rangle.$$

   (b) Let $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The cross product $\mathbf{a} \times \mathbf{b}$ of $\mathbf{a}$ and $\mathbf{b}$ is defined by

   $$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}.$$

   Calculate

   $$\langle \mathbf{a} \times \mathbf{b}, \mathbf{a}\rangle,\ \langle \mathbf{a} \times \mathbf{b}, \mathbf{b}\rangle \text{ and } \langle \mathbf{a} \times \mathbf{b}, \mathbf{a} \times \mathbf{b}\rangle.$$

   Let $\mathbf{f}, \mathbf{g} : \mathbb{R} \longrightarrow \mathbb{R}^3$ be two functions of class C^1. Show that

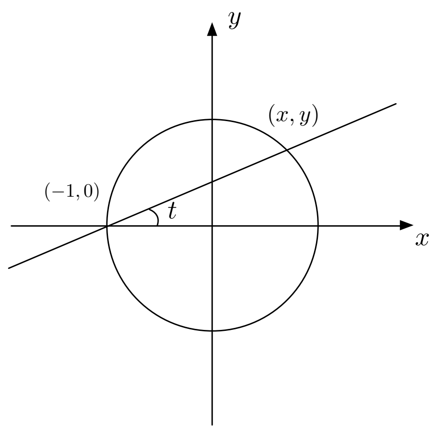
   $$\frac{d}{dt}\big(\mathbf{f}(t) \times \mathbf{g}(t)\big) = \mathbf{f}'(t) \times \mathbf{g}(t) + \mathbf{f}(t) \times \mathbf{g}'(t).$$

2. **parameterization of a circle.** Consider the unit circle in the 2D plane:

   $$x^2 + y^2 = 1. \tag{1}$$

   (a) Find a trigonometric parameterization $\theta \to (x(\theta), y(\theta))$ of the unit circle.

   (b) **Rational Parameterization.** Consider the line that meets the circle at the point $(-1, 0)$ and another point (x, y) that has the slope t. Find the parameterization of the circle $t \to (x(t), y(t))$. What is the computational difference between this parameterization and the one in (a)?



3. **Harmonic oscillator in three dimensions.** Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ be a curve of class C^2 such that $\ddot{\mathbf{r}}(t) = -\omega^2 \mathbf{r}(t)$, $\omega > 0$. For $m > 0$, we introduce the momentum $\mathbf{p}(t) = m\dot{\mathbf{r}}(t)$ and the angular momentum $\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}(t)$.

   (a) Show that $\mathbf{L}(t)$ is constant.

   (b) Show that the energy $E(t) := \dfrac{\langle \mathbf{p}(t), \mathbf{p}(t)\rangle}{2m} + \dfrac{m\omega^2 \langle \mathbf{r}(t), \mathbf{r}(t)\rangle}{2}$ is constant.

4. Find the derivative of,

$$f(t) = \langle \mathbf{u}(t), \mathbf{v}(t)\rangle.$$

   When $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are given as,

$$\mathbf{u}(t) = \begin{pmatrix} 1 \\ -3t^2 \\ 4t^3 \end{pmatrix}, \quad \mathbf{v}(t) = \begin{pmatrix} t \\ \cos t \\ \sin t \end{pmatrix}.$$

   (a) $f'(t) = 1 - 6t\cos t + 15t^2 \sin t + 4t^3 \cos t$

   (b) $f'(t) = 6t\cos t + 15t^2 \sin t + 4t^3 \cos t$

   (c) $f'(t) = 3t\cos t + 7t^2 \sin t + 4t^3 \cos t$

   (d) $f'(t) = 1 - 3t\cos t + 11t^2 \sin t + 4t^3 \cos t$

5. Which of the following integrals gives the length of the curve,

$$\mathbf{c}(t) = \begin{pmatrix} 2t^2 \\ t \end{pmatrix}, \qquad 0 \le t \le 4.$$

   (a) $I = \int_0^4 \sqrt{16t^2 + 1}\ dt$

   (b) $I = 2\int_0^4 \left|16t^2 + 1\right|\ dt$

   (c) $I = \int_0^2 \sqrt{16t^2 + 1}\ dt$

   (d) $I = 2\int_0^2 \left|16t^2 + 1\right|\ dt$

6. Find the unit tangent vector $\mathbf{T}(t)$ to the graph of the vector function

$$\mathbf{r}(t) = \begin{pmatrix} 3\sin t \\ 4t \\ 3\cos t \end{pmatrix}.$$

   (a) $\mathbf{T}(t) = \begin{pmatrix} \frac{3}{5}\cos t \\ \frac{4}{5} \\ \frac{3}{5}\sin t \end{pmatrix}$

   (b) $\mathbf{T}(t) = \begin{pmatrix} 3\sin t \\ 4t \\ 3\cos t \end{pmatrix}$

   (c) $\mathbf{T}(t) = \begin{pmatrix} 3\sin t \\ -4 \\ 3\cos t \end{pmatrix}$

   (d) $\mathbf{T}(t) = \begin{pmatrix} \frac{3}{5}\cos t \\ \frac{4}{5} \\ -\frac{3}{5}\sin t \end{pmatrix}$