Analysis II
Prof. Jan Hesthaven
Spring Semester 2015–2016
Posted February 26, 2016

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Solutions to Session of February 29, 2016

1. **A property of the Euclidean norm.** By the properties of the scalar product, we get

$$\begin{aligned}||\mathbf{x} \pm \mathbf{y}||_2^2 &= \langle \mathbf{x} \pm \mathbf{y}, \mathbf{x} \pm \mathbf{y} \rangle \\ &= \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle \pm \langle \mathbf{x}, \mathbf{y} \rangle \pm \langle \mathbf{y}, \mathbf{x} \rangle \\ &= ||\mathbf{x}||_2^2 + ||\mathbf{y}||_2^2 \pm \langle \mathbf{x}, \mathbf{y} \rangle \pm \langle \mathbf{y}, \mathbf{x} \rangle\end{aligned}$$

and hence

$$||\mathbf{x} + \mathbf{y}||_2^2 + ||\mathbf{x} - \mathbf{y}||_2^2 - 2||\mathbf{x}||_2^2 - 2||\mathbf{y}||_2^2 = 0$$

We also call this identity *the parallelogram identity.* Why?

2. **Cauchy-Schwarz's inequality in a Euclidean space.** Let $(E, \langle \cdot, \cdot \rangle)$ be a Euclidean space. Show that for all $\mathbf{x}, \mathbf{y} \in E$:

$$\left| \langle \mathbf{x}, \mathbf{y} \rangle \right| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \cdot \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

Solution. For all $\mathbf{x}, \mathbf{y} \in E$ and λ real:

$$0 \leq \langle \mathbf{x} - \lambda \mathbf{y}, \mathbf{x} - \lambda \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - 2\lambda \langle \mathbf{x}, \mathbf{y} \rangle + \lambda^2 \langle \mathbf{y}, \mathbf{y} \rangle.$$

We minimize compared to λ: if $\mathbf{y} = \mathbf{0}$ there is nothing to prove (the two members of Cauchy-Schwarz's inequality are equal to zero). If $\mathbf{y} \neq \mathbf{0}$, then $\langle \mathbf{y}, \mathbf{y} \rangle > 0$ by positivity of the scalar product, and the minimum of this polynomial of degree 2 in λ is obtained in $\lambda = \dfrac{\langle \mathbf{x}, \mathbf{y} \rangle}{\langle \mathbf{y}, \mathbf{y} \rangle}$. We get

$$0 \leq \langle \mathbf{x}, \mathbf{x} \rangle - \frac{\langle \mathbf{x}, \mathbf{y} \rangle^2}{\langle \mathbf{y}, \mathbf{y} \rangle}$$

hence Cauchy-Schwarz's inequality.

3. **Hölder's inequality and norms on $\mathbb{R}^\mathbf{n}$.** For $\mathbf{x} \in \mathbb{R}^n$ and $p \geq 1$ let

$$||\mathbf{x}||_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}.$$

Moreover, let

$$||\mathbf{x}||_\infty = \max_{1 \leq k \leq n} |x_k|.$$

(a) Show Hölder's inequality: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\dfrac{1}{p} + \dfrac{1}{p'} = 1$ (with the convention that if $p = 1$, then $p' = \infty$ and vice versa):

$$\left| \langle \mathbf{x}, \mathbf{y} \rangle \right| \leq ||\mathbf{x}||_p ||\mathbf{y}||_{p'}.$$

(Hint: use Young's inequality: for p and p' satisfying $1/p + 1/p' = 1$

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \text{for all } a, b \in \mathbb{R}$$

to show that for all $t > 0$:

$$\big|\langle \mathbf{x}, \mathbf{y}\rangle\big| \le \frac{t^p||\mathbf{x}||_p^p}{p} + \frac{t^{-p'}||\mathbf{y}||_{p'}^{p'}}{p'}$$

and deduce Hölder's inequality from it.)

Solution. Note first that we can assume $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$ since otherwise the inequality is trivial (the two members are equal to zero). By the triangular inequality for the absolute value, we derive the basic inequality:

$$\big|\langle \mathbf{x}, \mathbf{y}\rangle\big| = \left|\sum_{k=1}^{n} x_k y_k\right| \le \sum_{k=1}^{n} |x_k| \cdot |y_k|.$$

Hölder's inequality follows directly for $p = 1$:

$$\big|\langle \mathbf{x}, \mathbf{y}\rangle\big| \le ||\mathbf{x}||_1 ||\mathbf{y}||_\infty.$$

Let $p > 1$. By Young's inequality, for all $t > 0$ and all k:

$$|x_k| \cdot |y_k| = |tx_k| \cdot |t^{-1}y_k| \le \frac{t^p|x_k|^p}{p} + \frac{t^{-p'}|y_k|^{p'}}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

Hence, replacing the sums by the norms:

$$\big|\langle \mathbf{x}, \mathbf{y}\rangle\big| \le \frac{t^p||\mathbf{x}||_p^p}{p} + \frac{t^{-p'}||\mathbf{y}||_{p'}^{p'}}{p'}.$$

If we define

$$f(t) := \frac{t^p||\mathbf{x}||_p^p}{p} + \frac{t^{-p'}||\mathbf{y}||_{p'}^{p'}}{p'}$$

then $f\ :]0,\infty[\to]0,\infty[$ is a strictly convex function with a unique global minimum. Indeed

$$f'(t) := t^{p-1}||\mathbf{x}||_p^p - t^{-p'-1}||\mathbf{y}||_{p'}^{p'}, \quad f''(t) > 0.$$

The unique stationary point is given by

$$t_0^{p+p'} = \frac{||\mathbf{y}||_{p'}^{p'}}{||\mathbf{x}||_p^p}$$

and

$$f(t_0) = \Big(\frac{1}{p} + \frac{1}{p'}\Big)\left(||\mathbf{x}||_p^{\frac{pp'}{p+p'}}||\mathbf{y}||_{p'}^{\frac{pp'}{p+p'}}\right) = ||\mathbf{x}||_p ||\mathbf{y}||_{p'}$$

Noting that $f(t) \ge \big|\langle \mathbf{x}, \mathbf{y}\rangle\big|$ (the first inequality), we proved Hölder's inequality.

(b) Show that $||\mathbf{x}||_\infty$ defines norm on $\mathbb{R}^n$.

Solution. 1. Positivity. $||\mathbf{x}||_\infty = \max_{1\le k\le n} |x_k| = 0$ if and only if $|x_k| = 0$ for all k which is equivalent to $\mathbf{x} = \mathbf{0}$.

2. Homogeneity. For all $\lambda \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^n$ by homogeneity of the absolute value:

$$||\lambda\mathbf{x}||_\infty = \max_{1\le k\le n} |\lambda x_k| = \max_{1\le k\le n} |\lambda||x_k| = |\lambda| \max_{1\le k\le n} |x_k| = |\lambda|\,||\mathbf{x}||_\infty$$

3. Triangular inequality. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

$$\begin{aligned} ||\mathbf{x}+\mathbf{y}||_\infty &= \max_{1\le k\le n} |x_k + y_k| \le \max_{1\le k\le n} |x_k| + |y_k| \\ &\le \max_{1\le k\le n} |x_k| + \max_{1\le k\le n} |y_k| \\ &= ||\mathbf{x}||_\infty + ||\mathbf{y}||_\infty. \end{aligned}$$

(c) Show that $||\mathbf{x}||_1$ defines a norm on $\mathbb{R}^n$.

Solution. **1. Positivity.** $||\mathbf{x}||_1 = \sum_{k=1}^{n} |x_k| = 0$ if and only if $|x_k| = 0$ for all k which is equivalent to $\mathbf{x} = \mathbf{0}$.
2. Homogeneity. For all $\lambda \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^n$ by homogeneity of the absolute value:

$$||\lambda \mathbf{x}||_1 = \sum_{k=1}^{n} |\lambda x_k| = |\lambda| \sum_{k=1}^{n} |x_k| = |\lambda| \, ||\mathbf{x}||_1.$$

3. Triangular inequality. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

$$\begin{aligned} ||\mathbf{x} + \mathbf{y}||_1 = \sum_{k=1}^{n} |x_k + y_k| &\leq \sum_{k=1}^{n} |x_k| + |y_k| \\ &= ||\mathbf{x}||_1 + ||\mathbf{y}||_1. \end{aligned}$$

(d) Let $1 < p < \infty$. Show that $||\mathbf{x}||_p$ defines a norm on $\mathbb{R}^n$. To prove the triangular inequality, use the convexity of the following function $u \mapsto |u|^p$. First show that for all $t \in]0, 1[$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

$$||\mathbf{x} + \mathbf{y}||_p^p \leq t^{1-p} ||\mathbf{x}||_p^p + (1-t)^{1-p} ||\mathbf{y}||_p^p.$$

Deduce the triangular inequality by finding the optimal t.

Solution. **1. Positivity.** $||\mathbf{x}||_p^p = \sum_{k=1}^{n} |x_k|^p = 0$ if and only if $|x_k| = 0$ for all k which is equivalent to $\mathbf{x} = \mathbf{0}$.
2. Homogeneity. For all $\lambda \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^n$ by homogeneity of the absolute value:

$$||\lambda \mathbf{x}||_p = \left(\sum_{k=1}^{n} |\lambda x_k|^p \right)^{\frac{1}{p}} = |\lambda| \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} = |\lambda| \, ||\mathbf{x}||_p.$$

3. Triangular inequality. By convexity of the function $u \mapsto |u|^p$ we have for all $x_k, y_k \in \mathbb{R}$ and $0 < t < 1$:

$$\begin{aligned} |x_k + y_k|^p &= |tt^{-1} x_k + (1-t)(1-t)^{-1} y_k|^p \\ &\leq t |t^{-1} x_k|^p + (1-t) |(1-t)^{-1} y_k|^p = t^{1-p} |x_k|^p + (1-t)^{1-p} |y_k|^p \end{aligned}$$

hence, by taking the sum on k:

$$||\mathbf{x} + \mathbf{y}||_p^p \leq t^{1-p} ||\mathbf{x}||_p^p + (1-t)^{1-p} ||\mathbf{y}||_p^p.$$

The function $f :]0, 1[\to]0, \infty[$ defined by

$$f(t) := t^{1-p} ||\mathbf{x}||_p^p + (1-t)^{1-p} ||\mathbf{y}||_p^p$$

is a strictly convex function with a unique global minimun. Indeed

$$f'(t) := (p-1)\big(- t^{-p} ||\mathbf{x}||_p^p + (1-t)^{-p} ||\mathbf{y}||_p^p \big), \quad f''(t) > 0.$$

The unique stationary point is given by

$$\frac{1 - t_0}{t_0} = \frac{||\mathbf{y}||_p}{||\mathbf{x}||_p}$$

ie

$$t_0 = \frac{||\mathbf{x}||_p}{||\mathbf{x}||_p + ||\mathbf{y}||_p}, \quad 1 - t_0 = \frac{||\mathbf{y}||_p}{||\mathbf{x}||_p + ||\mathbf{y}||_p}$$

and

$$f(t_0) = \big(||\mathbf{x}||_p + ||\mathbf{y}||_p \big)^p \geq ||\mathbf{x} + \mathbf{y}||_p^p.$$

(e) Second proof of the triangular inequality. Let $1 < p < \infty$. Show that

$$||\mathbf{x}+\mathbf{y}||_p^p \leq \sum_{k=1}^{n} |x_k||x_k+y_k|^{p-1} + |y_k||x_k+y_k|^{p-1}$$

and apply Hölder's inequality.

Solution. Using Hölder's inequality with p and p' such that $p' = \dfrac{p}{p-1}$:

$$\sum_{k=1}^{n} |x_k||x_k+y_k|^{p-1} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |x_k+y_k|^p\right)^{\frac{p-1}{p}} = ||\mathbf{x}||_p ||\mathbf{x}+\mathbf{y}||_p^{p-1}$$

and

$$\sum_{k=1}^{n} |y_k||x_k+y_k|^{p-1} \leq \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |x_k+y_k|^p\right)^{\frac{p-1}{p}} = ||\mathbf{y}||_p ||\mathbf{x}+\mathbf{y}||_p^{p-1}$$

hence

$$||\mathbf{x}+\mathbf{y}||_p^p \leq (||\mathbf{x}||_p + ||\mathbf{y}||_p)||\mathbf{x}+\mathbf{y}||_p^{p-1}$$

and so we get the triangular inequality.

(f) For all $\mathbf{x} \in \mathbb{R}^n$ give $\lim_{p\to\infty} ||\mathbf{x}||_p$.

Solution . Note that for all $p \geq 1$ and all $\mathbf{x} \in \mathbb{R}^n$:

$$||\mathbf{x}||_\infty \leq ||\mathbf{x}||_p \leq n^{\frac{1}{p}} ||\mathbf{x}||_\infty$$

hence by the squeeze theorem

$$\lim_{p\to\infty} ||\mathbf{x}||_p = ||\mathbf{x}||_\infty.$$

4. **Subsets of $\mathbb{R}^n$**

(a) Let $S = \{(x,y) \in \mathbb{R}^2 : 0 < y < (1+x^2)e^{-|x|}\}$. Give $\overset{\circ}{S}, \bar{S}$ and ∂S. Then calculate the area of S.

(b) Let $T = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + 4y^2 < 4\}$. Give $\overset{\circ}{T}, \bar{T}$ et ∂T. Then calculate the area of T.

(c) Consider the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$. Give $\overset{\circ}{\mathbb{Q}}, \bar{\mathbb{Q}}$ and $\partial\mathbb{Q}$.

Solution a. $\overset{\circ}{S} = S$. The reason is essentially the strict inequalities in the definition of S and the continuity of the boundaries given by the functions $y = f(x) = (1+x^2)e^{-|x|}$ and $y = 0$. The rigourous proof consists in proving that for all point $(x,y) \in S$ there exists a ball B_ϵ of center (x,y) and of radius $\epsilon > 0$ such that $B_\epsilon \subset S$. Let then $(x_0, y_0) \in S$ given.

- There exists $h > 0$ such that $]y_0 - h, y_0 + h[\subset]0, f(x_0)[$. Hence, the segment $\{x_0\} \times]y_0 - h, y_0 + h[$ is in S.
- By continuity of $f(x) = (1+x^2)e^{-|x|}$ there exists $\delta > 0$ such that $f(x) > y_0 + h$ for all $x \in]x_0 - \delta, x_0 + \delta[$.
- Hence, the rectangle $]x_0 - \delta, x_0 + \delta[\times]y_0 - h, y_0 + h[$ is in S.
- Choose $\epsilon = \min(h, \delta)$ for the radius of the ball (Euclidean).

Then we have $\partial S = \{(x,y) \in \mathbb{R}^2 : 0 = y, \text{ or } y = (1+x^2)e^{-|x|}\}$ and

$$\bar{S} = S \cup \partial S = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq (1+x^2)e^{-|x|}\}$$

Calculation of the area:

$$\text{Area}(S) = \int_{-\infty}^{\infty} (1+x^2)e^{-|x|}\, dx = 2\int_{0}^{\infty} (1+x^2)e^{-|x|}\, dx = 2\Gamma(1) + 2\Gamma(3) = 6$$

The domain S - Spiked Helmet.

Solution b. $\overset{\circ}{T} = T$, $\partial T = \{(x,y) \in \mathbb{R}^2 : 1 = x^2 + 4y^2, \text{ or } x^2 + 4y^2 = 4\}$ and

$$\bar{T} = T \cup \partial T = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + 4y^2 \leq 4\}$$

Calculation of the area: The boundary of T is given by the two ellipses $E(1,1/2)$ and $E(2,1)$. Note that $E(1,1/2) \subset E(2,1)$. So

$$\text{Area}(T) = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}.$$

Solution c. By a result from the course Analyse I, the set $\mathbb{Q}$ is dense in $\mathbb{R}$. Between two real numbers there always exists a rational number and vice versa (see also exercises Analyse I, chapter 1). Hence, all point of $\mathbb{Q}$ is a boundary point. So

$$\overset{\circ}{\mathbb{Q}} = \emptyset, \quad \partial\mathbb{Q} = \bar{\mathbb{Q}} = \mathbb{R}.$$

5. Let $f : X \to \mathbb{R}$ be a continuous function on a metric space (X, d_X). Show that for all $c \in \mathbb{R}$:
   (a) $E = \{\mathbf{x} \in X : f(\mathbf{x}) = c\}$ is closed.
   (b) $F = \{\mathbf{x} \in X : f(\mathbf{x}) \leq c\}$ is closed.
   (c) $G = \{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ is open.

   Solution. If E is empty, then E is closed. If E is not empty, then for all adherent points $\mathbf{x}$ of E and for all sequences $(\mathbf{x}_n)_n$ of elements of E that converges to $\mathbf{x}$: $f(\mathbf{x}_n) = c$ for all n and by continuity of f

   $$c = \lim_{n\to\infty} f(\mathbf{x}_n) = f(\mathbf{x})$$

   hence $\mathbf{x} \in E$. For F it is the same idea (replace "$= c$" by "$\leq c$"). The set G is the complementary set of the closed set $\{\mathbf{x} \in X : f(\mathbf{x}) \geq c\}$, hence it is open.

6. Let $(E, \langle\cdot,\cdot\rangle)$ be an Euclidean space. Let $\mathbf{v} \in E$, $\langle\mathbf{v},\mathbf{v}\rangle = 1$. Then

   $$P\mathbf{x} = \langle\mathbf{v},\mathbf{x}\rangle\mathbf{v} \tag{1}$$

   defines an orthogonal projector (it is the orthogonal projection on $\mathbf{v}$). Show that P is continuous.

   Solution.

   $$||P\mathbf{x}||^2 = \langle P\mathbf{x}, P\mathbf{x}\rangle = \langle\mathbf{v},\mathbf{x}\rangle^2 \leq ||\mathbf{x}||^2$$

   by Cauchy-Schwarz's inequality.