Analysis II
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ÉCOLE POLYTECHNIQUE
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1. **A property of the Euclidean norm.** Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $||\cdot||_2$ be the Euclidean norm, i.e.
$$||\mathbf{x}||_2 = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \left(\sum_{k=1}^{n} x_k^2 \right)^{\frac{1}{2}}$$
for all $\mathbf{x} \in \mathbb{R}^n$. Calculate
$$||\mathbf{x}+\mathbf{y}||_2^2 + ||\mathbf{x}-\mathbf{y}||_2^2 - 2||\mathbf{x}||_2^2 - 2||\mathbf{y}||_2^2.$$

2. **Cauchy-Schwarz's inequality in the Euclidean space.** Let $(E, \langle \cdot, \cdot \rangle)$ be a Euclidean space. Show that for all $\mathbf{x}, \mathbf{y} \in E$:
$$\big|\langle \mathbf{x}, \mathbf{y} \rangle\big| \leq \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \cdot \sqrt{\langle \mathbf{y}, \mathbf{y} \rangle}.$$

3. **Hölder's inequality and norms on $\mathbb{R}^\mathbf{n}$.**[1] For $\mathbf{x} \in \mathbb{R}^n$ and $p \geq 1$ let
$$||\mathbf{x}||_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}.$$
Moreover, let
$$||\mathbf{x}||_\infty = \max_{1 \leq k \leq n} |x_k|.$$
   (a) Show Hölder's inequality: for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\frac{1}{p} + \frac{1}{p'} = 1$ (with the convention that if $p = 1$, then $p' = \infty$ and vice versa):
$$\big|\langle \mathbf{x}, \mathbf{y} \rangle\big| \leq ||\mathbf{x}||_p ||\mathbf{y}||_{p'}.$$
   (Hint: use Young's inequality: for p and p' satisfying $1/p + 1/p' = 1$
$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \text{for all } a, b \in \mathbb{R}$$
   to show that for all $t > 0$:
$$\big|\langle \mathbf{x}, \mathbf{y} \rangle\big| \leq \frac{t^p ||\mathbf{x}||_p^p}{p} + \frac{t^{-p'} ||\mathbf{y}||_{p'}^{p'}}{p'}$$
   and deduce Hölder's inequality from it.)

   (b) Show that $||\mathbf{x}||_\infty$ defines a norm on $\mathbb{R}^n$.

   (c) Show that $||\mathbf{x}||_1$ defines a norm on $\mathbb{R}^n$.

[1] This is a challenging exercise were you prove a cornerstone result.

(d) Let $1 < p < \infty$. Show that $||\mathbf{x}||_p$ defines a norm on $\mathbb{R}^n$. To show the triangular inequality use the convexity of the following function $u \mapsto |u|^p$. First show that for all $t \in]0,1[$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$||\mathbf{x}+\mathbf{y}||_p^p \leq t^{1-p}||\mathbf{x}||_p^p + (1-t)^{1-p}||\mathbf{y}||_p^p.$$

Deduce from it the triangular inequality by finding the optimal t.

(e) Second proof of the triangular inequality. Let $1 < p < \infty$. Show that

$$||\mathbf{x}+\mathbf{y}||_p^p \leq \sum_{k=1}^{n} |x_k||x_k+y_k|^{p-1} + |y_k||x_k+y_k|^{p-1}$$

and use Hölder's inequality.

(f) For all $\mathbf{x} \in \mathbb{R}^n$, give $\lim_{p\to\infty} ||\mathbf{x}||_p$.

4. **Subsets of $\mathbb{R}^n$**

(a) Let $S = \{(x,y) \in \mathbb{R}^2 : 0 < y < (1+x^2)e^{-|x|}\}$. Give $\mathring{S}, \bar{S}$ and ∂S. Then calculate the area of S.

(b) Let $T = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + 4y^2 < 4\}$. Give $\mathring{T}, \bar{T}$ and ∂T. Then calculate the area of T.

(c) Consider the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$. Give $\mathring{\mathbb{Q}}, \bar{\mathbb{Q}}$ and $\partial\mathbb{Q}$.

Useful formulas.

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad \Gamma(n+1) = n!$$

$$\int^x \sqrt{1-t^2}\,dt = \frac{x\sqrt{1-x^2}+\arcsin x}{2}$$

$$E_{a,b} = \{(x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}, \quad a,b > 0 \quad \text{Area}(E_{a,b}) = \pi ab.$$

5. Let $f : X \to \mathbb{R}$ be a continuous function on a metric space (X, d_X). Show that for all $c \in \mathbb{R}$:

(a) $E = \{\mathbf{x} \in X : f(\mathbf{x}) = c\}$ is closed.

(b) $F = \{\mathbf{x} \in X : f(\mathbf{x}) \leq c\}$ is closed.

(c) $G = \{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ is open.

6. Let $(E, \langle\cdot,\cdot\rangle)$ be a Euclidean space. Let $\mathbf{v} \in E$, $\langle\mathbf{v},\mathbf{v}\rangle = 1$. Then

$$P\mathbf{x} = \langle\mathbf{v},\mathbf{x}\rangle\mathbf{v} \tag{1}$$

defines an orthogonal projector (it is the orthogonal projection on $\mathbf{v}$). Show that P is continuous.