ÉCOLE POLYTECHNIQUE
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Solutions to Exercise Session, May 23, 2016

1. Calculate
$$\iint_{\mathbb{R}^2} \frac{1}{(1+x^2+(y-x)^2)^2}\,dxdy.$$

   Solution. By invariance under translations
$$\iint_{\mathbb{R}^2} \frac{1}{(1+x^2+(y-x)^2)^2}\,dxdy = \iint_{\mathbb{R}^2} \frac{1}{(1+x^2+y^2)^2}\,dxdy$$
   Using polar coordinates $x = r\cos\theta, y = r\sin\theta$ we get
$$\iint_{\mathbb{R}^2} \frac{1}{(1+x^2+(y-x)^2)^2}\,dxdy = 2\pi\int_0^\infty \frac{r}{(1+r^2)^2}\,dr = \pi\int_0^\infty -\frac{d}{dr}\frac{1}{1+r^2}\,dr = \pi.$$

2. Calculate
$$\iiint_{\mathbb{R}^3} e^{-x^2-2y^2-3z^2}\,dxdydz.$$

   Solution.
$$\begin{aligned}\iiint_{\mathbb{R}^3} e^{-x^2-2y^2-3z^2}\,dxdydz &= \int_{\mathbb{R}} e^{-x^2}\,dx \int_{\mathbb{R}} e^{-2y^2}\,dy \int_{\mathbb{R}} e^{-3z^2}\,dz \\ &= \frac{1}{\sqrt{2}}\int_{\mathbb{R}} e^{-\frac{x^2}{2}}\,dx\ \frac{1}{2}\int_{\mathbb{R}} e^{-\frac{y^2}{2}}\,dy\ \frac{1}{\sqrt{6}}\int_{\mathbb{R}} e^{-\frac{z^2}{2}}\,dz \\ &= \frac{(2\pi)^{\frac{3}{2}}}{4\sqrt{3}} = \frac{\pi^{\frac{3}{2}}}{\sqrt{6}} = \frac{\pi^{\frac{3}{2}}\sqrt{6}}{6}.\end{aligned}$$

3. Let
$$E = \{(x,y,z)\in\mathbf{R}^3 : x\in[0,1], y^2+z^2\le x^2\}$$
   Describe E and give $|E| = \mathrm{Vol}(E)$.

   Solution. The set E represents a cone around the axis x. The summit is $(0,0,0)$.
$$|E| = \pi\int_0^1 x^2\,dx = \frac{\pi}{3}.$$

4. For the differential equation:
$$dy/dx = 9x^2y$$
   find the general solution.
   (a) $y(x) = Ae^{3x^3}$
   (b) $y(x) = Ae^{3x^4}$
   (c) $y(x) = Ae^{x^2}$
   (d) $y(x) = Ae^{x^3}$

Solution. The correct answer is (a). By separating the variables, the equation becomes:

$$\frac{dy}{y} = 9x^2 dx$$

and after integrating both sides we get:

$$\ln y = 3x^3 + C$$

and so $y = Ae^{3x^3}$.

5. Suppose that y_0 satisfies:

$$(x^2 + 9)dy/dx = xy, \qquad y(0) = 3.$$

Find the value of $y_0(9)$.

(a) $y_0(9) = 4\sqrt{10}$
(b) $y_0(9) = 3\sqrt{10}$
(c) $y_0(9) = 40$
(d) $y_0(9) = 3\sqrt{17}$

Solution. The correct answer is (b). The general solution to the equation can be found by separation of variables:

$$\frac{dy}{y} = \frac{dx}{x^2 + 9}$$

and after integrating both sides we get $y_0 = A(x^2+9)^{1/2}$. If we require $y_0(0) = 3$ then $A = 1$ and hence $y_0(9) = 3\sqrt{10}$.

6. Suppose that y_0 satisfies:

$$(x + 3)dy/dx = y - 1, \qquad y(1) = 2.$$

Find the value of $y_0(4)$.

(a) $y_0(4) = 7/2$
(b) $y_0(4) = -1$
(c) $y_0(4) = 3$
(d) $y_0(4) = 11/4$

Solution. The correct answer is (d). The general solution to the equation can be found by separation of variables:

$$\frac{dy}{y-1} = \frac{1}{x+3}dx$$

and after integrating both sides we get $y_0 = 1 + A(x + 3)$. If we require $y_0(1) = 2$ then $A = 1/4$ and hence $y_0(4) = 11/4$.

7. For the differential equation:

$$dy/dx = \frac{e^{5x}}{6y^5}$$

find the general solution.

(a) $y(x) = \pm\sqrt[5]{e^{5x}/5 + C}$
(b) $y(x) = \pm\sqrt[5]{e^{5x}/5} + C$
(c) $y(x) = \pm\sqrt[6]{e^{5x}/5 + C}$
(d) $y(x) = \pm\sqrt[6]{e^{5x}/5} + C$

Solution. The correct answer is (c). By separating the variables, the equation becomes:

$$6y^5\,dy = e^{5x}dx$$

and after integrating both sides we get:

$$y^6 = e^{5x}/5 + C$$

and so $y(x) = \pm\sqrt[6]{e^{5x}/5 + C}$.

8. Find the general solution of the equation:

$$2ydy/dx = 9x.$$

(a) $y = \pm\sqrt{\frac{9}{2}x^2 + C}$

(b) $y = \pm\sqrt{\frac{9}{2}x^2} + C$

(c) $y = \pm\sqrt{\frac{9}{2}x^2}$

(d) $y = \pm\sqrt{\frac{2}{9}x^2 + C}$

Solution. The correct answer is (a). If we separate the variables we get:

$$2ydy = 9xdx$$

which integrates to $2y^2 = 9x^2 + C$ from which we get (a).

9. The solution $y(x)$ of the differential equation $(x^2+9)y' + xy - xy^2 = 0$ for $x \in \mathbb{R}$ with the initial condition $y(0) = 1/4$ also satisfies:

(a) $y(4) = 1/6$

(b) $y(4) = -1/4$

(c) $y(4) = 6$

(d) $y(4) = 1$

Solution. The correct answer is (b). If we separate the variables we get:

$$\frac{dy}{y^2 - y} = \frac{x\ dx}{x^2+9}$$

We have

$$\int \frac{dy}{y^2-y} = \int \frac{1}{y-1} - \frac{1}{y}\ dy = \ln(\frac{y-1}{y}) + C_1, \quad \frac{y-1}{y} > 0$$

and

$$\int \frac{x}{x^2+9}\ dx = \frac{1}{2}\ln(x^2+9) + C_2$$

If we put everything together we get

$$\ln(\frac{y-1}{y}) = \frac{1}{2}\ln(x^2+9) + C \Longrightarrow (\frac{y-1}{y})^2 = A(x^2+9)$$

If we use the initial condition $y(0) = 1/4$ we get that $A = 1$ and finally for we can compute $y(4)$,

$$(\frac{y-1}{y})^2 = 25 \Longrightarrow y(4) = 1/6 \text{ or } y(4) = -1/4$$

where $y = -1/4$ is the acceptable solution.

10. Find the general solution of the following equations

(a) $y' - \frac{3y}{x+1} = (x+1)^4$

(b) $\cos(x)y' + \sin(x)y = 2\cos^3(x)\sin(x) - 1$

Solution.

(a) The differential equation is of the form $y' + P(x)y = Q(x)$. We first find the integrating factor

$$I = e^{\int P\ dx} = e^{\int \frac{-3}{x+1}\ dx} = e^{-3\ln(x+1)} = e^{\ln(x+1)^{-3}} = \frac{1}{(x+1)^3}$$

We multiply both sides of differential equation with I to get

$$\frac{1}{(x+1)^3}y' - \frac{3y}{(x+1)^4} = (x+1)$$

by integrating both sides we get

$$\frac{y}{(x+1)^3} = \frac{1}{2}x^2 + x + C$$

So the general solution is

$$y = (x+1)^3(\frac{1}{2}x^2 + x + C)$$

(b) We first write the differential equations in the form of $y' + P(x)y = Q(x)$:

$$y' + \frac{\sin(x)}{\cos(x)}y = 2\cos^2(x)\sin(s) - \frac{1}{\cos(x)}$$

Now we find the integral factor

$$I = e^{\int P(x)\ dx} = e^{\int \frac{\sin(x)}{\cos(x)}\ dx} = e^{-\ln|\cos(x)|} = \frac{1}{\cos(x)}$$

Now we multiply both sides of the differential equation with I

$$\frac{y'}{\cos(x)} + \frac{\sin(x)}{\cos^2(x)} = 2\sin(x)\cos(x) - \frac{1}{\cos^2(x)}$$

Taking the integral of both sides yields

$$\frac{y}{\cos(x)} = -\frac{1}{2}\cos(x) - \tan(x) + C$$

So the general solution is

$$y = -\frac{1}{2}\cos(x)\cos(2x) - \sin(x) + C\cos(x)$$

11. For each of the following differential equations check if the solution exists and is unique.

(a) $y' = 1 + y^2$, $y(0) = 0$

(b) $y' = \frac{2y}{x}$, $y(a) = b$
solve the differential equation (b) and sketch the family of solutions for some initial conditions $y(a) = b$. What happens when $a = 0$ or $b = 0$? Compare this with the existence-uniqueness theorem.

Solution.

(a) Let $F(x, y) = 1+y^2$. Then both $F(x, y)$ and $\frac{\partial}{\partial y}F(x, y) = 2y$ are defined and continuous at all points (x, y), so by the theorem we can conclude that a solution exists in some open interval centered at 0, and is unique in some (possibly smaller) interval centered at 0.

(b) In this example, $F(x, y) = 2y/x$ and $\frac{\partial}{\partial y} F(x, y) = 2/x$. Both of these functions are defined for all $x \neq 0$ so the existence-uniqueness theorem tells us that for each $a \neq 0$ there exists a unique solution defined in a open interval around a. By separating variables and integrating, we derive solutions to this equation of the form

$$y = Cx^2$$

for any constant C. Notice that all of these solutions pass through the point $(0, 0)$, and that none of them pass through any point $(0, b)$ with $b \neq 0$. So the initial value problem

$$y' = 2y/x,\ y(0) = 0$$

has infinitely many solutions, but the initial value problem

$$y' = 2y/x,\ y(0) = b,\ b \neq 0$$

has no solutions.