Analysis II
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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Solutions to Exercise Session, May 9, 2016

1. Compute the integral

$$\lim_{y\to 0}\int_0^y \frac{(y-x)e^{-x^2}}{\sin y^2}\,dx.$$

Solution. Define $g(y)=\int_0^y (y-x)e^{-x^2}\,dx$, then (recall that the upper integration extrema depends upon y)

$$g'(y)=\int_0^y e^{-x^2}\,dx \quad , \quad g''(y)=e^{-y^2}$$

Recall that $(\sin y^2)''=(2y\cos y^2)'=2\cos y^2-4y^2\sin y^2$, so that

$$\lim_{y\to 0}\frac{g''(y)}{(\sin y^2)''}=\frac{1}{2}$$

Finally, by l'Hôpital's rule:

$$\lim_{y\to 0}\int_0^y \frac{(y-x)e^{-x^2}}{\sin y^2}\,dx \equiv \lim_{y\to 0}\frac{g(y)}{\sin y^2}=\lim_{y\to 0}\frac{g''(y)}{(\sin y^2)''}=\frac{1}{2}.$$

2. Let $D=[-1,1]\times[-1,1]$ and $f(x,y)=\max(1-|x|,1-|y|)$. Calculate the volume of

$$E=\{(x,y,z)\in\mathbb{R}^3 : (x,y)\in D \text{ and } 0\le z\le f(x,y)\}.$$

Draw the level sets of $f(x,y)$.

Solution. By symmetry of the problem, we have

$$\mathrm{Vol}(E)=4\iint_{[0,1]\times[0,1]} f(x,y)\,dxdy,$$

so

$$\begin{aligned}\mathrm{Vol}(E)&=4\int_0^1\left(\int_0^y(1-x)\,dx+\int_y^1(1-y)\,dx\right)dy\\&=4\int_0^1\left(y-\frac{y^2}{2}+(1-y)^2\right)dy\\&=4\int_0^1\left(y+\frac{y^2}{2}\right)dy\\&=\frac{8}{3}.\end{aligned}$$

3. Calculate

$$\iint_{\mathbb{R}^2}\frac{e^y}{(1+x^2+e^y)^2}\,dxdy.$$

Solution.

$$\iint_{\mathbb{R}^2} \frac{e^y}{(1+x^2+e^y)^2}\,dxdy = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{e^y}{(1+x^2+e^y)^2}\,dy\right)dx$$
$$= \int_{-\infty}^{\infty}\left[\frac{-1}{1+x^2+e^y}\right]_{y=-\infty}^{y=\infty} dx$$
$$= \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$
$$= \Big[\arctan x\Big]_{x=-\infty}^{x=\infty}$$
$$= \pi.$$

4. Let $B_R = \{(x,y)\in \mathbf{R}^2 : x^2+y^2 < R^2\}$. Calculate

$$\iint_{B_R} xy\,dxdy, \quad \iint_{B_R} x^2\,dxdy, \quad \iint_{B_R} y^2\,dxdy$$

Solution. Using polar coordinates $x = r\cos\theta, y = r\sin\theta$ we get

$$\iint_{B_R} xy\,dxdy = \int_0^R \left(\int_0^{2\pi} r^2\cos\theta\sin\theta\,d\theta\right) rdr$$
$$= \int_0^R r^2\left(\int_0^{2\pi} (\sin^2\theta)'/2\,d\theta\right) rdr$$
$$= 0.$$

and

$$\iint_{B_R} x^2\,dxdy = \int_0^R \left(\int_0^{2\pi} r^2\cos^2\theta\,d\theta\right) rdr$$
$$= \int_0^R r^2\left(\frac{\sin\theta\cos\theta+\theta}{2}\Big|_0^{2\pi}\right) rdr$$
$$= \pi\int_0^R r^3\,dr$$
$$= \frac{\pi R^4}{4}.$$

or by symmetry

$$\iint_{B_R} x^2\,dxdy = \iint_{B_R} y^2\,dxdy = \frac{1}{2}\iint_{B_R} x^2+y^2\,dxdy = \frac{\pi R^4}{4}.$$

5. Check that

$$\frac{d}{dx}\left(2\arctan\left(e^x\right)\right) = \frac{1}{\cosh x}$$

Calculate then

$$\iint_{\mathbb{R}^2} \frac{1}{\cosh(x^2+y^2)}\,dxdy.$$

Solution. Using polar coordinates $x = r\cos\theta, y = r\sin\theta$ we get

$$\iint_{\mathbb{R}^2} \frac{1}{\cosh(x^2+y^2)}\, dxdy = 2\pi \int_0^\infty \frac{r}{\cosh r^2}\, dr$$
$$= 2\pi \int_0^\infty \frac{d}{dr}\Big(\arctan\left(e^{r^2}\right)\Big)\, dr$$
$$= \frac{\pi^2}{2}.$$

6. Calculate

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2-2y}\, dxdy.$$

Solution. By completing the square for y we get

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2-2y}\, dxdy = e^1 \iint_{\mathbb{R}^2} e^{-x^2-(y+1)^2}\, dxdy$$

and by change of variable $(x, y+1) \longrightarrow (x, y)$ we get (invariance of $\mathbb{R}^2$ under translations)

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2-2y}\, dxdy = e^1 \iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dxdy = e\pi$$

following the calculation done in class.

7. Let $B_R(\mathbf{0}) = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le R^2\}$. Calculate

$$\iiint_{B_R(\mathbf{0})} xy\, dxdydz,$$

$$\iiint_{B_R(\mathbf{0})} z^2\, dxdydz.$$

Solution. By a change of variable in spherical coordinates

$$\iiint_{B_R(\mathbf{0})} xy\, dxdydz = \iiint_{[0,R]\times[0,\pi]\times[0,2\pi]} xy\, r^2 \sin\theta\, drd\theta d\phi$$
$$= \iiint_{[0,R]\times[0,\pi]\times[0,2\pi]} r^4 \sin^3\theta \cos\phi \sin\phi\, drd\theta d\phi = 0$$

since

$$\int_0^{2\pi} \cos\phi \sin\phi\, d\phi = 0.$$

Then

$$\iiint_{B_R(\mathbf{0})} z^2\, dxdydz = \iiint_{[0,R]\times[0,\pi]\times[0,2\pi]} z^2\, r^2 \sin\theta\, drd\theta d\phi$$
$$= \iiint_{[0,R]\times[0,\pi]\times[0,2\pi]} r^4 \cos^2\theta \sin\theta\, drd\theta d\phi$$
$$= 2\pi \iint_{[0,R]\times[0,\pi]} r^4 \cos^2\theta \sin\theta\, drd\theta$$
$$= 2\pi \int_0^R r^4 \Big(-\frac{\cos^3\theta}{3}\Big)\Big|_0^\pi\, dr$$
$$= \frac{4\pi}{3} \int_0^R r^4\, dr = \frac{4\pi R^5}{15}$$

Or quicker: by symmetry

$$\iiint_{B_R(\mathbf{0})} z^2 \, dxdydz = \frac{1}{3} \iiint_{B_R(\mathbf{0})} r^2 \, dxdydz = \frac{4\pi}{3} \int_0^R r^4 \, dr = \frac{4\pi R^5}{15}.$$

8. Compute the integral

$$I = \int_D \left(3\pi - 2 \arctan \frac{y}{x} \right) dxdy$$

where D is the portion of the first quadrant delimited by the circle $x^2 + y^2 = 16$.

Solution. By switching to polar coordinates, $D = \{(r, \theta) \, : \, 0 \le r \le 4 \, , \, 0 \le \theta \le \pi/2\}$. Then:

$$I = \int_0^4 \int_0^{\pi/2} (3\pi - 2\theta) r \, d\theta dr = 10\pi^2$$

9. By means of polar coordinates, compute the integral

$$I = \int_R 4e^{-x^2 - y^2} \, dxdy$$

where R is the region of the (x, y)-plane bounded by the graph of $x = \sqrt{4 - y^2}$ and the y-axis.

Solution. In terms of polar coordinates, $R = \{(r, \theta) \, : \, 0 \le r \le 2 \, , \, -\pi/2 \le \theta \le \theta/2\}$ and

$$I = 4 \int_0^2 \int_{-\pi/2}^{\pi/2} re^{-r^2} \, rdrd\theta = 4\pi \int_0^2 re^{-r^2} \, dr = 4\pi \int_0^2 \left(-\frac{e^{-r^2}}{2} \right)' dr = 2\pi(1 - e^{-4})$$

10. Use cylindrical coordinate to evaluate the integral

$$I = \int_W y \, dV$$

where W is the solid bounded by the (infinite!) cylinders $x^2 + y^2 = 4$, $x^2 + y^2 = 6$, and the planes $z = 0$, $z = x + 3$.

Solution. The integral vanishes because of symmetry. In fact, W is symmetric with respect to the (x, z)-plane, as is the integrand.