Analysis II
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Solutions to Exercise Session, May 2, 2016

1. Let $D = [0,1] \times [0, \pi/2]$. Calculate

$$\iint_D \frac{x \sin y}{1+x^2}\, dxdy.$$

Solution.

$$\begin{aligned}\iint_D \frac{x \sin y}{1+x^2}\, dxdy &= \int_0^1 \frac{x}{1+x^2}\, dx \cdot \int_0^{\pi/2} \sin y\, dy \\ &= \ln(1+x^2)\Big|_0^1 \cdot (-\cos y)\Big|_0^{\pi/2} \\ &= \frac{\ln 2}{2}.\end{aligned}$$

2. Let $D = [0,1] \times [1,2]$. Calculate

$$\iint_D \frac{x}{x^2+y^2}\, dxdy.$$

Solution.

$$\int_0^1 \frac{x}{x^2+y^2}\, dx = \int_0^1 \frac{1}{2}\frac{d}{dx} \ln(x^2+y^2)\, dx = \frac{1}{2}(\ln(1+y^2) - \ln y^2).$$

and

$$\frac{1}{2}\int_1^2 \ln(1+y^2)\, dy = \frac{y\ln(1+y^2)}{2}\Big|_{y=1}^{y=2} - \int_1^2 \frac{y^2}{1+y^2}\, dy = \frac{y\ln(1+y^2)}{2} - y + \arctan y\Big|_{y=1}^{y=2},$$

$$-\frac{1}{2}\int_1^2 \ln(y^2)\, dy = y - y\ln y\Big|_{y=1}^{y=2}$$

where using $\arctan 1 = \pi/4$:

$$\iint_D \frac{x}{x^2+y^2}\, dxdy = \arctan 2 + \ln 5 - \frac{5\ln 2}{2} - \frac{\pi}{4}$$

3. Let $D = [0,\pi] \times [0,1]$. Calculate

$$\iint_D x \sin xy\, dxdy.$$

Solution.

$$\begin{aligned}
\iint_D x\sin xy\,dxdy &= \int_0^\pi \left(\int_0^1 x\sin xy\,dy\right) dx \\
&= \int_0^\pi (-\cos xy)\Big|_{y=0}^{y=1} dx \\
&= \int_0^\pi (1-\cos x)\,dx \\
&= (x-\sin x)\Big|_0^\pi = \pi
\end{aligned}$$

4. Let D be the interior of the triangle of summits $A=(0,0)$, $B=(\pi,0)$ and $C=(\pi,\pi)$. Calculate

$$\iint_D x\cos(x+y)\,dxdy.$$

Solution. Trivially $D=\{(x,y):0\le y\le x\le\pi\}$. So

$$\begin{aligned}
\iint_D x\cos(x+y)\,dxdy &= \int_0^\pi \left(\int_0^x x\cos(x+y)\,dy\right) dx \\
&= \int_0^\pi (x\sin(x+y))\Big|_{y=0}^{y=x} dx \\
&= \int_0^\pi x\sin 2x - x\sin x\,dx \\
&= \int_0^\pi \frac{x}{2}(-\cos 2x)' + x(\cos x)'\,dx \\
&= -\frac{3\pi}{2} + \int_0^\pi \frac{\cos 2x}{2} - \cos x\,dx \\
&= -\frac{3\pi}{2}.
\end{aligned}$$

5. Let $D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\}$. Calculate the volume of

$$E=\{(x,y,z)\in\mathbb{R}^3 : (x,y)\in D \text{ and } 0\le z\le\sqrt{1-x^2-y^2}\}.$$

Deduce the volume of the unit ball $B_1(\mathbf{0})$ in $\mathbb{R}^3$.

Solution. Using the formula

$$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a},\ a>0,\ |x|<|a|$$

with $a = \sqrt{1-y^2}$ we get

$$\begin{aligned}
\mathrm{Vol}(E) &= \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{1-y^2-x^2}\, dx \right) dy \\
&= \int_{-1}^{1} \frac{x}{2}\sqrt{1-y^2-x^2} + \frac{1-y^2}{2} \arcsin \frac{x}{\sqrt{1-y^2}} \Bigg|_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} dy \\
&= \int_{-1}^{1} \frac{1-y^2}{2} (\arcsin 1 - \arcsin(-1))\, dy \\
&= \frac{\pi}{2} \int_{-1}^{1} 1 - y^2\, dy \\
&= \frac{2\pi}{3}.
\end{aligned}$$

The set E represents the half unit ball. So

$$\mathrm{Vol}(B_1(\mathbf{0})) = \frac{4\pi}{3}.$$

6. Calculate

$$\iint_{\mathbb{R}^2} e^{-|x-1|-|y|}\, dxdy.$$

Solution.

$$\begin{aligned}
\iint_{\mathbb{R}^2} e^{-|x-1|-|y|}\, dxdy &= \int_{-\infty}^{\infty} e^{-|x-1|}\, dx \cdot \int_{-\infty}^{\infty} e^{-|y|}\, dy \\
&= \int_{-\infty}^{\infty} e^{-|t|}\, dt \cdot \int_{-\infty}^{\infty} e^{-|y|}\, dy \\
&= 4 \int_{0}^{\infty} e^{-t}\, dt \cdot \int_{0}^{\infty} e^{-y}\, dy \\
&= 4.
\end{aligned}$$

7. **True/False**

(a) **False.** Without loss of generality assume that $(x_0, y_0) = (0, 0)$. Directional derivative of a function is given by $\nabla f|_{(0,0)} \cdot \vec{v}$. Now consider,

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(0) \\ \sin(0) \end{pmatrix}$$

By assumption we have,

$$\nabla f|_{(0,0)} \cdot \vec{v} = 1$$

and

$$\nabla f|_{(0,0)} \cdot (-\vec{v}) = \nabla f|_{(0,0)} \cdot (\cos \pi, \sin \pi) = 1 \neq -\nabla f|_{(0,0)} \cdot \vec{v}$$

Which is a contradiction. Note that if $\vec{v} = (\cos\theta, \sin\theta)$ then $\nabla f|_{(0,0)} \cdot \vec{v}$ has to be a linear function of $\cos\theta$ and $\sin\theta$ and in this case it is not.

(b) **True.** The Hessian is positive-definite.

(c) **True.** Because $J_{vow} = J_v|_w \times J_W$.

(d) **False.** It is possible for a function to have the global minimum on the boundary of D, but also a local minimum in the interior of D, where then the hessian matrix is definite.

(e) **True.** Since this vector is in direction of $-\nabla f$.

(f) **Solution.**

i. **True.** Let us recall that a function f is differentiable at x_0 if there exists a linear functional L, which depends on x_0, such that $f(x_0+h) = f(x_0) + Lh + \mathrm{o}(|h|)$ in a neighborhood of x_0. If such functional were to exist, then

$$Lv = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{|h|}$$

where $v = h/|h|$ is a unit vector. Since f is a radial function, namely it depends upon $r = \sqrt{x^2+y^2}$ only, it is immediate to show that the previous limit exists and is equal to zero. Thus, f is differentiable at the origin and $L = 0$.

ii. **True.** Since f is differentiable, then its differential can be expressed *via* the gradient.

iii. **True.** Indeed:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(1/|h|)}{h} = 0$$

and analogously for the other partial derivative.

iv. **False.** Indeed, away from the origin:

$$\frac{\partial f}{\partial x} = 2x \sin(1/\sqrt{x^2+y^2}) - \frac{x\cos(1/\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$$
$$\frac{\partial f}{\partial y} = 2y \sin(1/\sqrt{x^2+y^2}) - \frac{y\cos(1/\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$$

The limits of those functions, as we approach the origin, clearly do not exist.

Notice that f is an example of a function that, although being differentiable in a neighborhood of the origin, does not have continuous partial derivatives at the origin.