Analysis II
Prof. Jan Hesthaven
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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Exercise Session, May 2, 2016

1. Let $D = [0,1] \times [0, \pi/2]$. Calculate
$$\iint_D \frac{x \sin y}{1+x^2}\, dxdy.$$

2. Let $D = [0,1] \times [1,2]$. Calculate
$$\iint_D \frac{x}{x^2+y^2}\, dxdy.$$

3. Let $D = [0,\pi] \times [0,1]$. Calculate
$$\iint_D x \sin xy \, dxdy.$$

4. Let D be the interior of the triangle of summits $A = (0,0)$, $B = (\pi, 0)$ and $C = (\pi, \pi)$. Calculate
$$\iint_D x \cos(x+y)\, dxdy.$$

5. Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. Calculate the volume of
$$E = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in D \text{ and } 0 \le z \le \sqrt{1-x^2-y^2}\}.$$
Deduce the volume of the unit ball $B_1(\mathbf{0})$ in $\mathbb{R}^3$.

   Useful formulas.
$$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a},\ a > 0,\ |x| < |a|,$$

6. Calculate
$$\iint_{\mathbb{R}^2} e^{-|x-1|-|y|}\, dxdy.$$

7. **True/False**

   (a) There exists a function $f(x,y)$ which is differentiable at $(x_0, y_0) \in \mathbb{R}^2$ and its directional derivatives at each direction $\vec{v} = (\cos\theta, \sin\theta)$ for $0 \le \theta < 2\pi$ equals $\cos^2\theta + 2\sin\theta$.

   ☐ True ☐ False

   (b) Consider $f(x,y) = x^2 - 2xy + \frac{4}{3}y^2 - 4y$. The local minimum of f is a strict local minimum.

   ☐ True ☐ False

   (c) Assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and differentiable at $a \in \mathbb{R}^n$. Also assume that $g : \mathbb{R}^m \to \mathbb{R}$ is differentiable at $f(a)$. Then $\nabla(gof)(a) = \nabla(g(f(a))) \times J_f(a)$. Note that here the gradient of a function is a row vector.

   ☐ True ☐ False

(d) Assume $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and differentiable on a bounded closed set $D \subset \mathbb{R}^n$. Then f attains its global minimum on the boundary if and only if whenever $\nabla f(x_0) = 0$ for some $x \in \mathbb{R}^n$ then $H_f(x_0)$ has both positive and negative eigenvalues.

☐ True ☐ False

(e) Let $f(x, y, z) = x + y^2 + z^3$. Then at the point (0,1,1) the function decrease fastest in the direction of $(\frac{-e}{\sqrt{2}}, \frac{-2e}{\sqrt{2}}, \frac{-3e}{\sqrt{2}})$.

☐ True ☐ False

(f) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the piecewise function defined as:

$$f = \begin{cases} (x^2 + y^2)\sin(1/\sqrt{x^2 + y^2}) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

Then:

i. Function f is differentiable at the origin. ☐ True ☐ False

ii. At the origin, the differential of f is equal to $\nabla f(0, 0) \cdot (x, y)$. ☐ True ☐ False

iii. The partial derivatives of f vanish at the origin. ☐ True ☐ False

iv. The partial derivatives of f are continuous at the origin. ☐ True ☐ False