# Multiple choice test 9 Answers

[1.](ii) TF . $\bigtriangledown f = (-2,3,6)^T \neq \bar{0}$. So the extrema lie on the boundary. We have to maximize :

$$\begin{cases} -2x+3y+6z+42 \\ \text{Subject to} \quad x^2+\frac{3y^2}{2}+3z^2+2x-6z=\frac{3}{2} \end{cases}$$

$$\bigtriangledown g = \begin{pmatrix} 2x+2 \\ 3y \\ 6z-6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

But $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is not in $M = \{\bar{x} : a(\bar{x}) = \frac{3}{2}\}$ . So the extrema satisfy :

$$\bigtriangledown f + \lambda \bigtriangledown g = \bar{0} = \begin{pmatrix} -2 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2x+2 \\ 3y \\ 6z-6 \end{pmatrix}$$

$$\Rightarrow \lambda = \frac{1}{x+1} = -\frac{1}{y} = -\frac{1}{z-1} \Rightarrow \begin{cases} x = \frac{1}{\lambda}-1 \\ y = -\frac{1}{\lambda} \\ z = -\frac{1}{\lambda}+1 \end{cases}$$

$$g(x,y,z) = \frac{1}{\lambda^2} - \frac{1}{\lambda} + 1 + \frac{3}{2\lambda^2} + 3(\frac{1}{\lambda^2} - \frac{2}{\lambda} + 1) + \frac{2}{\lambda} - 2 + \frac{6}{\lambda} - 6 = \frac{\sqrt{5}}{\lambda^2} - 4 = \frac{3}{2} \Rightarrow \lambda = \pm 1$$

$$\text{The extrema are : } \begin{cases} \lambda = 1 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \Rightarrow f = 39 \ \text{Minimum} \\ \lambda = -1 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow f = 61 \ \text{Maximum} \end{cases}$$

[2.](iv) $\bigtriangledown f = (ye^{(x-1)y}, (x-1)e^{(x-1)y})^T = \bar{0}$ only if $y=0, x=1$ so the extrema are on $\partial E$. Using the parametrization $x = cos(\theta), y = sin(\theta)$ we reduce the problem to maximizing/minimizing $(cos(\theta)-1)sin(\theta) = g(\theta)$.

$g'(\theta) = -sin^2(\theta) + cos^2(\theta) - cos(\theta) = 2cos^2(\theta) - cos(\theta) - 1 = (2cos(\theta)+1)(cos(\theta)-1)$

$cos(\theta) = 1 \Rightarrow e^{(x-1)y} = e^0 = 1$

$cos(\theta) = -\frac{1}{2} \Rightarrow x = -\frac{1}{2}, y = \pm\frac{\sqrt{3}}{2}$

$$\begin{cases} f(-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = e^{\frac{3\sqrt{3}}{4}} > 1 \\ f(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = e^{-\frac{3\sqrt{3}}{4}} < 1 \end{cases} \Rightarrow \text{(iv)}.$$

[3.] (iv) $\bigtriangledown f = (\frac{x}{2} - y, \frac{y}{2} - x)^T) = \bar{0}$ only if $(x, y) = (0, 0)$. Here, $f = 0$ on the boundary.
Let's apply the change $(x, y) = (cos(\theta), sin(\theta))$ :
$f(x, y) = \frac{1}{4} - cos(\theta)sin(\theta) = g(\theta)$
$g'(\theta) = sin^2(\theta) - cos^2(\theta) = 0$ for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
For $\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \quad cos(\theta)sin(\theta) = \frac{1}{2} \Rightarrow f = -\frac{1}{4} \Rightarrow$ Minimum.
For $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \quad cos(\theta)sin(\theta) = -\frac{1}{2} \Rightarrow f = \frac{1}{4} \Rightarrow$ Maximum.

[4.](i) $g(x, y) = e^{x^2} + e^{y^2} - 4 \Rightarrow \bigtriangledown g(x, y) = (2xe^{x^2}, 2ye^{y^2}) = \bar{0}$ only if $(x, y) = (0, 0)$where the constraint is not satisfied. The Lagrange approach is then justified :

$$\bigtriangledown f + \lambda \bigtriangledown g = \begin{pmatrix} -xe^{-x^2} \\ ye^{-y^2} \end{pmatrix} + \lambda \begin{pmatrix} 2xe^{x^2} \\ 2ye^{y^2} \end{pmatrix} = \bar{0}$$

$$\begin{cases} \lambda = -\frac{1}{2}e^{-2x^2} \text{ or } x = 0 \\ \lambda = -\frac{1}{2}e^{-2y^2} \text{ or } y = 0 \end{cases} \Rightarrow \text{Only 2 cases which are : } \begin{cases} x = 0 \text{ and } \lambda = -\frac{1}{2}e^{-2y^2} \\ y = 0 \text{ and } \lambda = -\frac{1}{2}e^{-2x^2} \end{cases}$$

$\underline{x = 0}$ : $e^{y^2} = 4 - 1 = 3 \Rightarrow y = \pm\sqrt{(log(3)}$
$\underline{y = 0}$ : $e^{x^2} = 4 - 1 = 3 \Rightarrow x = \pm\sqrt{(log(3)}$
If $x = 0, \quad f(0, \pm\sqrt{(log(3)}) = \pm\frac{1}{2}(1 - e^{-log(3)}) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$
If $x = 0, \quad f(\pm\sqrt{(log(3)}, 0) = -\frac{1}{2}(1 - e^{log(3)}) = -\frac{1}{3}$

[5.](iii) Let $g_1(x, y, z) = x^2 + y^2 + 2z^2 - 4$ and $g_2(x, y, z) = xyz - 1$.
We want to find the extrema of $f(x, y, z) = x^2 + y^2 + z^2$ subject to $g_1 = g_2 = 0$
Remark : If $(x, y, z)$ is an extremum, then so are $(-x, -y, z), (x, -y, -z), (-x, y, -z)$ because $f, g_1, g_2$ remain unchanged. So then for an extremum, we can suppose that $x$ and $y$ are positive which implies that $z$ is positive as well (because $xyz = 1 > 0$). We are now looking for positive x, y and z :

First list : $\bigtriangledown g_1 = \begin{pmatrix} 2x \\ 2y \\ 4z \end{pmatrix}, \bigtriangledown g_2 = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$

If $\bigtriangledown g_1$and $\bigtriangledown g_2$ are linearly independant (as it is impossible that either one of them is equal to 0), we have :

$$\begin{pmatrix} 2x \\ 2y \\ 4z \end{pmatrix} = c \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} \Rightarrow x = \frac{2x}{yz} = \frac{2y}{xz} \Rightarrow \frac{x}{y} = \frac{y}{x} \Rightarrow x^2 = y^2$$

$\Rightarrow x = y$ because $x, y > 0$
$4z = cxy = cx^2 = \frac{2}{z}x^2 \Rightarrow 2z^2 = x^2$
$g_2 = 0 \Rightarrow x^2 + y^2 + 2z^2 = 3x^2 = 4 \Rightarrow x = y = \frac{2}{\sqrt{3}}$ and $z = \frac{\sqrt{2}}{\sqrt{3}} \Rightarrow xyz = \frac{4\sqrt{2}}{3\sqrt{3}} \neq 1$ .
So $\nexists(x, y, z)$ such that $g_1(\bar{x}) = 0$ and $g_2(\bar{x}) = 0$ and at the same time, $\bigtriangledown g_1$and $\bigtriangledown g_2$ are linearly dependant, whch means that the first list is empty.

Second list : We're looking for $x, y, z > 0$ with $\bigtriangledown f = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 4z \end{pmatrix} + \sigma \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$

Then $\begin{cases} 2x - 2\lambda x = \sigma zy \\ 2y - 2\lambda y = \sigma xz \end{cases} \Rightarrow \begin{cases} \sigma xyz = (2x - 2\lambda x)x = (2 - 2\lambda)x^2 \\ \sigma xyz = (2y - 2\lambda y)y = (2 - 2\lambda)y^2 \end{cases}$

We notice that $\lambda = 1$ is impossible because it would imply that $\sigma = 0$ and hence that $z = 0$ which violates the condition $g_2 = 0$ . Which means that $x^2 = y^2 \Rightarrow x = y$ as they are both positive.

So $x^2 + y^2 + 2z^2 - 4 = 0 \iff 2x^2 + 2z^2 - 4 = 0 \iff \frac{2}{z} + 2z^2 - 4 = 0 \iff z^3 - 2z + 1 = 0 \iff (z - 1)(z^2 + z - 1) = 0 \Rightarrow z = 1$ or $z = \frac{-1 \pm \sqrt{5}}{2}$

We know that $z$ is positive so 2 choices :

$z = 1, x = y = z = 1$ or

$z = \frac{-1+\sqrt{5}}{2} \Rightarrow x^2 = y^2 = \frac{1}{\frac{\sqrt{5}-1}{2}} = \frac{\sqrt{5}+1}{2}$

$\Rightarrow x^2 + y^2 + z^2 = 3$ in the first case and $x^2 + y^2 + z^2 = \sqrt{5} + 1 + (\frac{-1+\sqrt{5}}{2})^2 = \frac{5+\sqrt{5}}{2} > 3$ in the second case.

So the first case is a minimum and the second case is a maximum.

[6.](iv) We have to maximize $f = \prod_{k=1}^{n} x_k^2$ subject to $g = \sum_{k=1}^{n} x_k^2 = 1$.

$\bigtriangledown g = 2\bar{x} = \bar{0} \iff \bar{x} = 0$ which is not the maximum and doesn't satisfy the condition because $g(\bar{0}) = 0 \neq 1$. Then we can use the Lagrange approach :We are looking for $\bar{x}, \lambda$ such that :

$\bigtriangledown f + \lambda \bigtriangledown g = \bar{0}$ which means $\forall k, \quad 2x_n \prod_{j \neq k} x_j^2 + \lambda 2x_n = \bar{0}$ . As we can't have $x_n = 0$ for the maximum, these equations imply :

$\lambda = -\frac{1}{\prod_{j \neq k} x_j^2} = -\frac{x_n^2}{f(\bar{x})}$ , which means that $x_k^2 = x_j^2 \quad \forall j, k$ hence $\forall n, x_n^2 = \frac{1}{n}$

[7.] (iii) We can see that $\forall x, y, z$ on the ellipsoid $| x |, | y |, | z | \leq 10^5$ so $| x + y + z | \leq 3 \times 10^5 < c10^8$ so the hyperplane doesn't intersect the ellipsoid and generally there is only one minimum. The distance of a point (x,y,z) on the hyperplane to the ellipsoid is given by : $| \frac{x+y+z}{\sqrt{3}} - 10^8 |$. As we have that $x + y + z \leq 3.10^5 < c10^8$, we have to minimize $10^8 - \frac{x+y+z}{\sqrt{3}}$ which is just about maximizing $f(x, y, z) := x + y + z$ subject to $g(x, y, z) := x^2 + y^2 + \frac{10}{9} z^2 = 10^{10}$

$\bigtriangledown g = \begin{pmatrix} 2x \\ 2y \\ \frac{20}{9} z \end{pmatrix} = \bar{0} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \bar{0}$ Which is impossible on the ellipsoid. Hence, the Lagrange approach works.

$\bigtriangledown f + \lambda \bigtriangledown g = \bar{0} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2x \\ 2y \\ \frac{20}{9} z \end{pmatrix} \iff \lambda = -\frac{1}{2x} = -\frac{1}{2y} = -\frac{9}{20z} \iff x = y = \frac{10}{9} z$

And $x^2 + y^2 + \frac{10}{9} z^2 = 2x^2 + \frac{10}{9} x^2 = 10^{10} \Rightarrow x = 10^5 \sqrt{\frac{10}{29}} = y, z = 9.10^4 \sqrt{\frac{10}{29}}$

Which means $x + y + z = 10^5 \sqrt{\frac{10}{29}} (1 + 1 + \frac{9}{10}) = 10^4 \sqrt{290}$ .

The distance is : $\frac{10^8-10^4\sqrt{290}}{\sqrt{3}} = \frac{10^4(10^4-\sqrt{290})}{\sqrt{3}}$

[8.](i) We will rather look for the minimum of $f^2(x,y) = x^2+3y^2$ subject to $x+y=3$.
<u>Remark :</u> For $g = x+y-3, \bigtriangledown g = \begin{pmatrix}1\\1\end{pmatrix} \neq \begin{pmatrix}0\\0\end{pmatrix}$. Which means the first list is empty, we will then consider the second list.

$\bigtriangledown(f^2) + \lambda \bigtriangledown g = \bar{0} \iff \begin{pmatrix}2x\\6y\end{pmatrix} + \lambda\begin{pmatrix}1\\1\end{pmatrix} = \bar{0} \iff \begin{cases}2x+\lambda=0\\6y+\lambda=0\end{cases} \Rightarrow x = 3y$

$x+y = 4y = 3 \Rightarrow y = \frac{3}{4}, x = \frac{9}{4}$

$f(x,y) = \sqrt{x^2+3y^2} = \frac{1}{4}\sqrt{81+27} = \frac{3}{4}\sqrt{12} = \frac{3\sqrt{3}}{2}$

Remark : When $x,y \to \infty, f \to \infty$ so as f is continuous, the minimum exists and without calculation we can already rule out (iv). We can also consider $x=3$ (and $y=0$) to see that the minimum is $\leq 3$ so (ii) isn't possible .

[9.](ii) Just like [8.], the list 1 is empty and we look for $x, y, \lambda$ with $\begin{pmatrix}3y\\3x\end{pmatrix} + \lambda\begin{pmatrix}1\\1\end{pmatrix} = \bar{0}$ $\Rightarrow x = y = 4$. And the maximum (it's obviously a maximum) is equal to 48.

[10.](iv) We have to maximize $xy$ subject to $g(x,y) := \frac{x^2}{36} + \frac{y^2}{9} - 1 = 0$

$\bigtriangledown g = (\frac{x}{18}, \frac{2y}{9})^T = \bar{0} \iff (x,y) = (0,0)$

This violates the condition $g(x,y) = 0$ so the list 1 is empty. We then look for $x, y, \lambda$ such that $\begin{pmatrix}y\\x\end{pmatrix} + \lambda\begin{pmatrix}\frac{x}{18}\\\frac{2y}{9}\end{pmatrix} = \bar{0} \iff \lambda = \frac{18y}{x} = \frac{9x}{2y} \Rightarrow 2y = x$

And $\frac{x^2}{36} + \frac{y^2}{9} = 1 \Rightarrow y = \frac{3}{\sqrt{2}}, x = 3\sqrt{2}$ and $xy = 9$