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*Analysis II*
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# Analysis I Course Review Solutions

1. (a) x→ 0lim $\frac{\sin x}{\ln\left(\frac{1}{1+x}\right)} = x \to 0\lim \frac{\sin x}{-\ln(1+x)} = x \to 0\lim \frac{\cos x}{-\frac{1}{1+x}} = -1$

   (b) For $\alpha > 0$ using l'Hospital rule

   $$\lim_{x\to 0^+} x^\alpha \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^\alpha}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\alpha x^{\alpha-1}}{x^{2\alpha}}} = \lim_{x\to 0^+} -\frac{1}{\alpha}x^\alpha = 0$$

   (c) By a (long and) direct calculation using l'Hospital's rule x→ 1lim$\frac{\arctan\left(\frac{1-x}{1+x}\right)}{x-1} = x \to 1\lim\frac{-\frac{1}{1+x^2}}{1} = -\frac{1}{2}$. Alternatively, we write $y = \frac{1-x}{1+x}$, so $x-1 = \frac{-2y}{1+y}$ and x→ 1lim$\frac{\arctan\left(\frac{1-x}{1+x}\right)}{x-1} = y \to 0\lim\frac{-(y+1)\arctan y}{2y} = y \to 0\lim\frac{-\arctan y - \frac{y+1}{1+y^2}}{2} = -\frac{1}{2}$

   (d) One has truncated Taylor expansions $\sinh x = x + O(x^3)$ and $\ln(1+y) = y + O(y^2)$ so x→ 0lim $\frac{\ln(1+x^2)}{\sinh^2 x} = 1$

   (e) $x \to 1\lim \frac{\cos(\frac{\pi x}{2})\sin(x-1)}{\ln((x-1)^2)} = 0$

   (f) x→ 0+lim $\frac{\cos x - \cos\frac{1}{x}}{e^x - e^{\frac{1}{x}}} = 0$ since $e^{\frac{1}{x}}$ goes to infinity and the other terms are bounded.

2. (a) $f(x) = xe^{-x^2}$ is of class $C^\infty$ with f'(x) =(1-2x$^2$)$e^{-x^2} f''(x) = -2x(3-2x^2)e^{-x^2}$. Stationary points: $x_1 = -\frac{\sqrt{2}}{2}$ (strict local minimum, $f(x_1) = -\frac{\sqrt{2}}{2}e^{-\frac{1}{2}}$) and $x_2 = -\frac{\sqrt{2}}{2}$ (strict local maximum, $f(x_2) = \frac{\sqrt{2}}{2}e^{-\frac{1}{2}}$). These are in fact global extrema since $\lim_{x\to\pm\infty} f(x) = 0$. Inflexion points: $x = 0$ $x = \pm\frac{\sqrt{6}}{2}$.

   (b) On the boundary $f(-1) = f(3) = 0$ and $\lim_{x\to\pm\infty} f(x) = \infty$. More precisely, one observes that $\lim_{n\to-\infty} f(x) - (1-x) = 0$ $\lim_{n\to\infty} f(x) - (x-1) = 0$ so that there are asymptotes $y = 1-x$ when $x \to -\infty$ and $y = x-1$ when $x \to +\infty$. $f$ is of class $C^\infty$ on the open set $]-\infty, -1[\cup]3, \infty[$ with f'(x) =x-1$\frac{}{\sqrt{(x+1)(x-3)} f''(x) = \frac{-4}{(\sqrt{(x+1)(x-3)})^3}}$. $f$ is strictly monotonic and strictly concave on its domain.

3. (a) We note that $f(x) = \exp(x\ln x - x)$ and so $f'(x) = (\ln x)f(x)$. Hence $x = 1$ is the unique stationary point of the function $f$ as $f(x) > 0$ and $\ln x$ is strictly increasing.

   (b) $f$ has a strict local minimum at $x = 1$ since $f'(x) < 0$ if $x < 1$ and $f'(x) > 0$ if $x > 1$. The truncated Taylor expansion of order 4 at this point is given by f(x)=e$^{-1}\left(1 + 12(x-1)^2 - 16(x-1)^3 + 524(x-1)^4\right)$.

   (c) Note that we can rewrite $a^b$ as $e^{b\ln a}$ using this trick we can write $f(x)$ as

   $$f(x) = e^{x\ln x - x}$$

   Since $g(x) = e^x$ is a continuous function, we have

   $$\lim_{x\to 0^+} e^{x\ln x - x} = e^{\lim_{x\to 0^+} x\ln x - x}$$

   We saw in exercise 1.b that $\lim_{x\to 0^+} x\ln x = 0$ so

   $$\lim_{x\to 0^+} f(x) = e^0 = 1$$

   By l'Hospital's rule (one may verify the required conditions are satisfied) we get x→ $0_+\lim\frac{f(x)-1}{x\ln x} = x \to 0_+\lim\frac{f'(x)}{1+\ln x} = x \to 0_+\lim\frac{(\ln x)f(x)}{1+\ln x} = \left(x \to 0_+\lim \frac{\ln x}{1+\ln x}\right)(x \to 0_+\lim f(x))$ $= \left(x \to 0_+\lim\frac{\frac{1}{x}}{\frac{1}{x}}\right)(x \to 0_+\lim f(x)) = 1.$

4. (a) By the change of variable $s = \sqrt{t}$, i.e., $t = s^2$, $\int_0^{+\infty} e^{-\sqrt{t}} dt = \int_0^{+\infty} 2se^{-s} ds = 2\Gamma(2) = 2$
   (b) $\int_1^{+\infty} \frac{\ln t}{t^3} dt = \frac{1}{2} \int_1^{+\infty} \ln t (-t^{-2})' dt = \frac{1}{2} \int_1^{+\infty} \frac{1}{t^3} \, dt = \frac{1}{4}$
   (c) $\int_0^{+\infty} \frac{\arctan t}{1+t^2} dt = \frac{\arctan^2 t}{2} \Big|_0^{\infty} = \frac{\pi^2}{8}$

5. (a) The integral converges if $\alpha < 1$ and diverges otherwise.
   (b) The integral converges if $\alpha > 1$ and diverges otherwise.

6. (a) True. This is the statement of l'Hospital's rule.
   (b) False. Take, for example, $f(x) = x + \sin x$ and $g(x) = x$. We have $\frac{f'(x)}{g'(x)} = 1 + \cos x$ which has no limit at infinity. However, $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} 1 + \frac{\sin x}{x} = 1$.
   (c) False. Take the counterexample for the preceding question.
   (d) True. $f$ and $g$ are differentiable on $[x, y]$, so by the mean value theorem, there exists $c \in ]x, y[$ such that $\frac{f'(c)}{g'(c)} = \frac{f(y)-f(x)}{g(y)-g(x)} = 1$.
   (e) True. Since $g$ is differentiable on $R$, the function $\sin g(x)$ is differentiable on $R$. If $g(a) = 0$, then $\sin g(a) = 0$ and we may apply l'Hospital's rule which guarantees the existence of the limit. If $g(a) \neq 0$, then by continuity, the limit is simply $\frac{\sin g(a)}{g(a)}$.
   (f) False. Take, for example, $g(x) = x$ and $a = 1$. We have $g(a) = 1 \neq 0$ (note that in this case l'Hospital's rule does not apply). By continuity, the limit is $\frac{\sinh 1}{1} \neq \cosh 1$.

7. (a) False. Since 7 is odd, $f$ admits an inflexion point at $a$.
   (b) True. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$ be the Taylor expansion of $f$ to order $n$ at 0. We then have $f(-x) = a_0 - a_1 x + \cdots + (-1)^n a_n x^n + o(x^n)$. Since $f$ is odd and the Taylor expansion is unique, we conclude that $a_{2m} = 0$ for all $0 \leq 2m \leq n$. As $a_k = \frac{f^{(k)}(0)}{k!}$, we obtain the result.
   (c) True. This was shown in class.