Multiple choice test 8 Answers

[1.](iii) is the right answer. First of all, (i) and (ii) are easiy eliminated because f is 2π-periodic with respect to x and y and hence cannot have a unique golbal minimum,and cannot have a saddle point. Furthermore, $\bigtriangledown f(x,y) = \begin{pmatrix} cos(x) \\ sin(x) \end{pmatrix}$

So, $\bigtriangledown f(x,y) = 0 \iff \begin{cases} x = \pi/2 + k_1\pi & k_1 \in \mathbb{Z} \\ y = k_2\pi & k_2 \in \mathbb{Z} \end{cases}$

$(\pi/2, 0)$ being a stationary point, (iv) is eliminated and we can conclude.

[2.] (i) A : $\forall (x_0, y_0) \in B(0, \frac{1}{2}), (x_0, y_0)$ is a strict (and global) minimum. Indeed, $\forall (x,y) \neq (x_0, y_0)$, $(x,y) \in [-1,1]^2$,

$$f(x,y) - f(x_0,y_0) = (x-x_0)^2 + (y-y_0)^2 = [d((x,y),(x_0,y_0))]^2 > 0$$

B : The four corners of the square are local maxima. One has to notice that the further (x,y) is from (x_0, y_0) the bigger $f(x,y)$ is. As $(x_0, y_0) \in B(0, \frac{1}{2})$, every corner is, locally, the furthest point. We can calculate, for $\epsilon_1, \epsilon_2 \geq 0$,

$$\begin{aligned} f(1,1) - f(1-\epsilon_1, 1-\epsilon_2) &= (1-x_0)^2 + (1-x_0-\epsilon_1)^2 + (1-y_0)^2 + (1-y_0-\epsilon_2)^2 \\ &= \epsilon_1(2(1-x_0)-\epsilon_1) + \epsilon_2(2(1-y_0)-\epsilon_2) \\ &\geq \epsilon_1(1-\epsilon_1) + \epsilon_2(1-\epsilon_2) \end{aligned}$$

Which is positive when $\epsilon_1, \epsilon_2 \leq 1$.
We have proven that the corner $(1,1)$ is a local maximum $\forall (x_0, y_0) \in B(0, \frac{1}{2})$. By symmety, this implies the result for the 3 other corners.

[3.](iii) local minimum in $(4, -3)$

$$\begin{aligned} \bigtriangledown f(x,y) &= \begin{pmatrix} 2x-8 \\ 8y+24 \end{pmatrix} \\ \bigtriangledown f(x,y) = 0 &\iff (x,y) = (4,-3) \\ Hess(f)(4,-3) &= \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \end{aligned}$$

The two eigenvalues of $Hess(f)$ are $2 > 0$ and $8 > 0$.

[4.](iv) f has a unique saddle-point.

$$\bigtriangledown f(x,y) = \begin{pmatrix} 2x-3y^2 \\ 3y^2-6xy \end{pmatrix}$$

We notice that (0,0) is a stationary point. But $f(0,y) = y^3$ and $f(x,0) = x^2$. Hence, (0,0) is neither an extremum, or a saddle-point. Let's resolve $\bigtriangledown f(x,y) = 0$ for $(x,y) \neq (0,0)$:

$$\begin{cases} x = \frac{3}{2}y^2 \\ 3y^2 - 9y^3 = 0 \end{cases} \iff \begin{cases} x = \frac{1}{6} \\ y = \frac{1}{3} \end{cases}$$

So f admits only one stationary point other than $(0,0)$ so (iv) is obviously the right answer. Let's check that $(\frac{1}{6}, \frac{1}{3})$ is a saddle-point.

$$Hess(f)(\tfrac{1}{6}, \tfrac{1}{3}) = \begin{pmatrix} 2 & -2 \\ -2 & 1 \end{pmatrix}$$
$$Trace(Hess(f)(\tfrac{1}{6}, \tfrac{1}{3})) = 3$$
$$\mid Hess(f)(\tfrac{1}{6}, \tfrac{1}{3}) \mid = -2 < 0 \text{ so } (\tfrac{1}{6}, \tfrac{1}{3}) \text{ is a saddle point}$$

[5.](ii) Local maximum in $(-2^{\frac{1}{3}}, 2^{\frac{1}{3}})$

$$\bigtriangledown f(x,y) = \begin{pmatrix} y - \frac{2}{x^2} \\ x + \frac{2}{y^2} \end{pmatrix}$$
$$\bigtriangledown f(x,y) = 0 \iff \begin{cases} y = \frac{2}{x^2} \\ x + \frac{x^4}{2} = 0 \end{cases} \iff \begin{cases} y = 2^{\frac{1}{3}} \\ x = -2^{\frac{1}{3}} \end{cases} \quad (\implies \text{(ii) or (iii)})$$
$$Hess(f)(2^{\frac{1}{3}}, -2^{\frac{1}{3}}) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$
$$\mid Hess(f)(2^{\frac{1}{3}}, -2^{\frac{1}{3}}) \mid = 3 > 0 \text{ so } (2^{\frac{1}{3}}, -2^{\frac{1}{3}}) \text{ can't be a saddle point}$$

[6.](ii)

$$\bigtriangledown f(x,y) = \begin{pmatrix} 26x^2 + y \\ 6x^2 + x \end{pmatrix} \implies (0,0) \text{ stationary point.}$$

If $(x,y) \neq (0,0)$,

$$\bigtriangledown f(x,y) = 0 \iff \begin{cases} 26 * 34y^4 + y = 0 \\ x = -6y^2 \end{cases}$$
$$(\text{as } y \neq 0) \iff \begin{cases} y^3 = -\frac{1}{4*6^3} \\ x = -6y^2 \end{cases} \iff \begin{cases} y = -\frac{1}{6*4^{\frac{1}{3}}} \\ x = -\frac{1}{6*4^{\frac{2}{3}}} \end{cases}$$

f only admits 2 stationary points. Furthermore, $f(0,y) = 2y^3$ so $(0,0)$ is neither a maximum or a minimum.

[7.](iv) $\max_{x,y \in T} f(x,y) = 6$. f attains its maximum either by a stationary point on $\dot{T}$, or on the boundary of T.

<u>In $\dot{T}$:</u>

$$\bigtriangledown f(x,y) = \begin{pmatrix} y - 1 \\ x - 3 \end{pmatrix}$$

So $(3,1)$ is the only stationary point of f in $\dot{T}$ and $Hess(f)(3,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has a negative determinant.

<u>In δT :</u> $\delta T = \{ \ (x,0) : 1 \leq x \leq 6 \ \} \bigcup \{ \ (1,y) : 0 \leq y \leq 5 \ \} \bigcup \{ \ (x, 6-x) : 1 \leq x \leq 6 \ \}$

1. On $A := \{ \ (x,0) : 1 \leq x \leq 6 \ \}$
   $\max_A f(x,0) = \max_A 7 - x = 6$

2. On $B := \{ \ (1,y) : 0 \leq y \leq 5 \ \}$
   $\max_B f(1,y) = \max_A 6 - 2y = 6$

3. On $C := \{ \ (x, 6-x) : 1 \leq x \leq 6 \ \}$
   $\max_C f(x, 6-x) = \max_C 7 + x(6-x) - x - 18 - 3x = \max_C -11 + 8x - x^2 = 5$ (attained for x=4)

[8.](iii) Saddle-point, $\forall a \in \mathbb{R}$,

$$Hess(f)(1,1) = \begin{pmatrix} 2 & 4a \\ 4a & -2 \end{pmatrix}$$
$$| \ Hess(f)(1,1) \ | = -4 - 16a^2 < 0 \forall a \in \mathbb{R},$$

Hence $(1,1)$ is always a saddle point. [9.](iv) $(0,0,8)$ and$(0,0,-8)$
We're looking for the minimum of the function $f(x,y) = x^2 + y^2 + xy + 64$ (Which represents the disctance between the origin and the point (x,y,z) for (x,y,z) on the surface $z = xy + 64$.

$$\bigtriangledown f(x,y) = \begin{pmatrix} 2x + y \\ 2y + x \end{pmatrix}$$
$$\bigtriangledown f(x,y) = 0 \iff \begin{cases} 2x + y = 0 \\ 2y + x = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$$

The closest points to the origin on the surface are obviously stationary points for f (on $\mathbb{R}$). Hence, (iv) is the right answer.

[10.](i) 10x10x10

We denote : $\begin{cases} x = length \\ y = width \\ z = height \\ s = surface \\ v = Volume \end{cases}$

$$\begin{cases} s = 2(xy + xz + yz) \\ v = xyz \end{cases} \quad \text{But } s = 600 \implies z = \frac{300-xy}{x+y}$$

$$v(x,y) = \frac{xy(300-xy}{xy} \iff \begin{cases} \delta_x v(x,y) = \frac{(300y-2xy^2)(x+y)-xy(300-xy)}{(x+y^2} = \frac{-x^2y^2+300y^2-2xy^3}{(x+y)^2} = \frac{y^2}{(x+y)^2}(300 - 2xy - \\ \delta_y v(x,y) = \frac{x^2}{(x+y)^2}(300 - 2xy - y^2) \quad \text{by symmetry} \end{cases}$$

We can exclude the stationary point $x = y = 0$ which is not a maximum.

For $x, y > 0$:

$$\nabla f(x,y) = 0 \iff \begin{cases} 300 - 2xy = x^2 \\ x^2 - y^2 = 0 \end{cases} \iff \begin{cases} x^2 = 100 \\ x = y \end{cases} \iff \begin{cases} x = 10 \\ y = 10 \end{cases}$$

And we can conclude.