Multiple Choice Questions 7 : Solutions

1. Answer (i). Since
$$\nabla f(x,y) = (2xe^{x^2}, \cos(y)),$$
we have that $\nabla f(x,y) \neq 0$ for all $x, y \in U$. Indeed, $\nabla f(x,y) = \mathbf{0}$ only if $x = 0$ and $y = \frac{\pi}{2} + k\pi$. But since $\left(0, \frac{\pi}{2} + k\pi\right) \notin \mathcal{B}(\mathbf{0}, \frac{\pi}{4})$ the gradient is non zero in U. It's not injective because $f(x,y) = f(-x,y)$ for all $(x,y) \in U$.

2. Answer (i). Let denote
$$v_1(x,y) = \cos\left(\frac{x}{\sqrt{x^2+y^2}} + \cos\left(\sqrt{x^2+y^2}\right)\right) \quad \text{and} \quad v_2(x,y) = e^{\frac{1}{\sqrt{x^2+y^2}}}.$$
We have that
$$\begin{aligned}
\frac{\partial v_1}{\partial x} &= -\sin\left(\frac{x}{\sqrt{x^2+y^2}} + \cos(\sqrt{x^2+y^2})\right)\left(\frac{y^2}{(x^2+y^2)^{3/2}} - \sin\left(\sqrt{x^2+y^2}\right)\cdot\frac{x}{\sqrt{x^2+y^2}}\right),\\
\frac{\partial v_1}{\partial y} &= \sin\left(\frac{x}{\sqrt{x^2+y^2}} + \cos(\sqrt{x^2+y^2})\right)\left(\frac{xy}{(x^2+y^2)^{3/2}} + \sin\left(\sqrt{x^2+y^2}\right)\cdot\frac{y}{\sqrt{x^2+y^2}}\right),\\
\frac{\partial v_2}{\partial x} &= -\frac{x}{(x^2+y^2)^{3/2}}e^{\frac{1}{\sqrt{x^2+y^2}}},\\
\frac{\partial v_2}{\partial y} &= -\frac{y}{(x^2+y^2)^{3/2}}e^{\frac{1}{\sqrt{x^2+y^2}}},
\end{aligned}$$
and thus
$$\det \mathcal{J}_{\bar{v}}(1,1) =$$
$$\det\begin{pmatrix} -\sin\left(\frac{1}{\sqrt{2}} + \cos(\sqrt{2})\right)\left(\frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{2}}\sin(\sqrt{2})\right) & \sin\left(\frac{1}{\sqrt{2}} + \cos(\sqrt{2})\right)\left(\frac{1}{2\sqrt{2}} + \frac{1}{\sqrt{2}}\sin(\sqrt{2})\right) \\ -\frac{e^{1/\sqrt{2}}}{\sqrt{2}} & -\frac{e^{1/2\sqrt{2}}}{2\sqrt{2}} \end{pmatrix}$$
$$= \frac{e^{1/\sqrt{2}}}{4}\sin\left(\frac{1}{\sqrt{2}} + \cos(\sqrt{2})\right) \neq 0,$$
because $\sqrt{2} \in \left(1, \frac{\pi}{2}\right)$ (since $\sqrt{2} \approx 1,4$ and $\frac{\pi}{2} \approx 1,57$) and
$$\frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{2}} + \cos(\sqrt{2}) \leq 2 < \frac{3\pi}{4}.$$
It's not invertible avec all $\mathbb{R}^2\backslash\{(0,0)\}$ since for example $f(0,1) = f(0,-1)$, and thus not one-to-one over all $\mathbb{R}^2\backslash\{(0,0)\}$.

3. Answer (i). Let denote $\bar{v}(x,y) = (v_1(x,y), v_2(x,y))$. Then $\bar{v}^{-1} = (v_1^{-1}, v_2^{-1})$. Finally,

$$g(s,t) = f(\bar{v}^{-1}(s,t)) = v_1^{-1}(s,t)v_2^{-1}(s,t).$$

Therefore

$$\frac{\partial g}{\partial s} = v_2^{-1}(s,t)\frac{\partial v_1^{-1}}{\partial s} + v_1^{-1}\frac{\partial v_2^{-1}}{\partial s},$$
$$\frac{\partial g}{\partial t} = v_2^{-1}(s,t)\frac{\partial v_1^{-1}}{\partial t} + v_1^{-1}\frac{\partial v_2^{-1}}{\partial t}.$$

We know have to compute de Jacobian matrix of $\bar{v}^{-1}$ at $(e,3)$. First remark that $\bar{v}^{-1}(e,3) = (1,1)$. We compute the Jacobian of $\bar{v}$ at $(1,1)$. It's given by

$$\mathcal{J}_{\bar{v}}(1,1) = \begin{pmatrix} ye^{xy} & xe^{xy} \\ 1+2x & 2y \end{pmatrix}\Bigg|_{(1,1)} = \begin{pmatrix} e & e \\ 3 & 2 \end{pmatrix},$$

and thus

$$\mathcal{J}_{\bar{v}^{-1}}(e,3) = \left(\mathcal{J}_{\bar{v}}(1,1)\right)^{-1} = -\frac{1}{e}\begin{pmatrix} 2 & -e \\ -3 & e \end{pmatrix}.$$

We finally conclude that

$$\frac{\partial g}{\partial s}(e,3) = 1\cdot\frac{2}{-e} + 1\cdot\left(\frac{-3}{-e}\right) = e^{-1},$$
$$\frac{\partial g}{\partial t}(e,3) = 1\cdot\left(\frac{-e}{-e}\right) + 1\cdot\left(\frac{e}{-e}\right) = 0,$$

and thus

$$\nabla_{st}g(e,3) = \left(e^{-1}, 0\right).$$

4. Answer (ii). The function f is $\mathcal{C}^1$. We have that

$$D_x f(0,0) = (2x + y\cos(xy))|_{(0,0)} = 0,$$

and

$$D_y f(0,0) = (2e^y + x\cos(xy))|_{(0,0)} = 2 \neq 0.$$

The implicit function theorem allow us to conclude on the existence. Moreover

$$\varphi'(0) = -\frac{D_x f(0,\varphi(0))}{D_y f(0,f(0))} = -\frac{0}{2} = 0.$$

5. Answer (i). We have that $f(0,0,1) = 0$ and that f is $\mathcal{C}^1$. We have that

$$\frac{\partial f}{\partial z}f(0,0,1) = \left(xe^y + 4z^3e^{xy} + z^4xe^{xz}\right)\Big|_{(0,0,1)} = 4 \neq 0.$$

Therefore, by the implicit function theorem, there is an implicit function $z = \varphi(x,y)$ in a neighborhood of $(0,0,1)$ with $\varphi(0,0) = 1$. Moreover

$$\begin{aligned}\frac{\partial\varphi}{\partial x}(0,0,1) &= \frac{D_x f(0,0,1)}{D_z(0,0,1)} \\ &= -\frac{1}{4}\left(2x-3+e^y+ze^y+z^5e^x\right)\Big|_{(0,0,1)} \\ &= 0.\end{aligned}$$

Also,

$$\frac{\partial\varphi}{\partial y}(0,0) = -\frac{1}{4}\left(2y+3y^2+xe^y+xze^y\right)\Big|_{(0,0,1)} = 0.$$

We conclude that the equation of the tangent plane at $(0,0,1)$ is given by $z=1$.

6. Answer (iii).

$$\begin{aligned}2x^3-y^3+9xy+1=0 &\implies \frac{\mathrm{d}}{\mathrm{d}x}\Big(2x^3-y^3+9xy+1\Big)=0 \\ &\implies 6x^2-3y^2\frac{\mathrm{d}y}{\mathrm{d}x}+9y+9x\frac{\mathrm{d}y}{\mathrm{d}x}=0 \\ &\implies \frac{\mathrm{d}y}{\mathrm{d}x}(9x-3y^2)=-9y-6x^2 \\ &\implies \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x^2+3y}{y^2-3x}.\end{aligned}$$

7. Answer (ii). As previously

$$\begin{aligned}xe^{2y}-yz+ze^{3x}=0 &\implies \frac{\partial}{\partial x}(xe^{2y}-yz+ze^{3x})=0 \\ &\implies e^{2y}-y\frac{\partial z}{\partial x}+\frac{\partial z}{\partial x}e^{3x}+3ye^{3x}=0 \\ &\implies \frac{\partial z}{\partial x}(e^{3x}-y)=-(e^{2y}-3ze^{3x}) \\ &\implies \frac{\partial z}{\partial x}=-\frac{e^{2y+3ye^{3x}}}{e^{3x}-y}.\end{aligned}$$

8. Answer (i). We use the rule

$$\frac{\partial z}{\partial\theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial\theta},$$

where

$$\frac{\partial x}{\partial\theta}=-2r\sin(2\theta)\quad\text{and}\quad\frac{\partial y}{\partial\theta}=2r\cos(\theta).$$

The claim follow.

9. Answer (iv). Even if we can't use the implicit function theorem (since $D_x g(0,0)=D_y g(0,0)=0$) the function defined by $\varphi_2(y)=y^3$ verify the wanted properties.

10. Answer (iv). Since

$$D_x f(0,0) = (e^{x-\alpha y^2})\Big|_{(0,0)} = 1 \neq 0,$$

for all $\alpha \in \mathbb{R}$ and

$$D_y f(0,0) = 0$$

for all $\alpha \in \mathbb{R}$ (because if $\alpha \neq 0$ we also have that $(-2\alpha y^2 e^{x-\alpha y^2})\Big|_{(0,0)} = 0$). Since f is $\mathcal{C}^1$, we can use the implicit function theorem and conclude.