Multiple Choice Questions 6 : Solutions

1. Answer (*iv*). The Jacobian is given by

$$\mathcal{J}_{\boldsymbol{v}}(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial}{\partial x}xy^2 & \frac{\partial}{\partial y}xy^2 \\ \frac{\partial}{\partial x}xy^3 & \frac{\partial}{\partial y}xy^3 \\ \frac{\partial}{\partial x}x & \frac{\partial}{\partial y}x \end{pmatrix} = \begin{pmatrix} y^2 & 2xy \\ y^3 & 3xy \\ 1 & 0 \end{pmatrix}.$$

2. Answer (*i*). A simple computation give

$$\nabla \times F(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial y}z^3 - \frac{\partial}{\partial z}y^2 \\ \frac{\partial}{\partial z}x - \frac{\partial}{\partial x}z^3 \\ \frac{\partial}{\partial x}y^2 - \frac{\partial}{\partial y}x \end{pmatrix} = \mathbf{0}.$$

3. Answer (*iv*). The gradient is defined for function with real value only.

4. Answers (*i*), (*ii*) and (*iii*). Let's prove it !

(i) Since $\mathrm{Hess}(f)_{ii} = \frac{\partial^2 f}{\partial x_i^2}$, the claim follow.

(ii)

$$\nabla \cdot \nabla f(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x_1} f(\boldsymbol{x}) \\ \vdots \\ \frac{\partial}{\partial x_n} f(\boldsymbol{x}) \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{x}).$$

(iii) It's the definition of the Laplacian.

5. Answer (*iv*). Let $\boldsymbol{v} = (v_1, v_2, v_3)$ and $\boldsymbol{w} = (w_1, w_2, w_3)$. We have that

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}.$$

We can see that (*ii*) can't be correct since there is no term of the form $D_2 v_3 \cdot D_2 w_3$. Also, (*iii*) is not defined since $\nabla \cdot \boldsymbol{v}$ is a scalar and not a vector. Then, the only possibility is (*i*). Let show that it's not correct.

$$\begin{aligned} \mathrm{div}(\boldsymbol{v} \times \boldsymbol{w}) &= w_1(D_2 v_3 - D_3 v_2) - v_1(D_2 w_3 - D_3 w_2) \\ &+ w_2(D_3 v_1 - D_1 v_3) - v_2(D_3 w_1 - D_1 w_3) \\ &+ w_3(D_1 v_2 - D_2 v_1) - v_3(D_1 w_2 - D_2 w_1) \\ &= \langle \nabla \times \boldsymbol{v}, \boldsymbol{w} \rangle - \langle \boldsymbol{v}, \nabla \times \boldsymbol{w} \rangle . \end{aligned}$$

6. Answer (iv). We have that

$$\begin{aligned}
\nabla(\boldsymbol{v}\cdot\boldsymbol{w}) &= \nabla\left(\sum_{i=1}^n v_i w_i\right)\\
&= \begin{pmatrix}\sum_{i=1}^n w_i D_1 v_i\\ \vdots \\ \sum_{i=1}^n w_i D_n v_i\end{pmatrix} + \begin{pmatrix}\sum_{i=1}^n v_i D_1 w_i\\ \vdots \\ \sum_{i=1}^n v_i D_n w_i\end{pmatrix}\\
&= \mathcal{J}_{\boldsymbol{v}}^T\boldsymbol{w} + \mathcal{J}_{\boldsymbol{w}}^T\boldsymbol{v}\\
&= (\boldsymbol{w}^T\mathcal{J}_{\boldsymbol{v}})^T + (\boldsymbol{v}^T\mathcal{J}_{\boldsymbol{w}})^T\\
&= (\boldsymbol{w}^T\mathcal{J}_{\boldsymbol{v}} + \boldsymbol{v}^T\mathcal{J}_{\boldsymbol{w}})^T
\end{aligned}$$

7. Answer (iv). Recall that

$$\mathcal{J}_{\boldsymbol{w}}(0,-2) = \Big(\mathcal{J}_{\boldsymbol{v}}(1,-1)\Big)^{-1}.$$

However

$$\mathcal{J}_{\boldsymbol{v}}(x,y)\big|_{(x,y)=(1,-1)} = \begin{pmatrix}3x^2 & -2y\\ -2x & 3y^2\end{pmatrix}\Bigg|_{(x,y)=(1,-1)} = \begin{pmatrix}3 & 2\\ -2 & 3\end{pmatrix},$$

and thus

$$\mathcal{J}_{\boldsymbol{w}}(0,-2) = \frac{1}{13}\begin{pmatrix}3 & -2\\ 2 & 3\end{pmatrix}.$$

8. Answer (ii). For the first assertion,

$$\mathcal{J}_{\boldsymbol{v}}(x,y) = \begin{pmatrix}2x-2y & -2x\\ 2y & 2x-2y\end{pmatrix}\Bigg|_{(x,y)=(1,1)} = \begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix},$$

and thus $\det\mathcal{J}_{\boldsymbol{v}}(1,1) = 4$. For the second assertion, let $\boldsymbol{v}\in\mathcal{C}^1(\mathbb{R}^2)$ locally invertible at $\boldsymbol{x}_0$. Let denote $\boldsymbol{w}$ it's inverse in a neighborhood of $\boldsymbol{x}_0$. If $\boldsymbol{w}\in\mathcal{C}^1$, then $\det\mathcal{J}_{\boldsymbol{v}}(\boldsymbol{x}_0)\neq 0$ (see lecture). On the other hand, if $\boldsymbol{w}\notin\mathcal{C}^1$, we can't say anything. Indeed, $v(x,y)=(x^3,y^3)$ is in $\mathcal{C}^1(\mathbb{R}^2)$, it's inverse is $\boldsymbol{w}(s,t)=(s^{1/3},t^{1/3})$ but $\det\mathcal{J}_{\boldsymbol{v}}(0,0)=0$.

Remark : If $\boldsymbol{v}$ is an invertible vector field that is $\mathcal{C}^1$ and if it's inverse is also $\mathcal{C}^1$, then $\boldsymbol{v}$ is called a diifeomorphism.

9. Answer (iv). We use the formula

$$Q(x,y) = f(0,0)+xD_xf(0,0)+yD_yf(0,0)+\frac{x^2}{2}D_{xx}f(0,0)+xyD_{xy}f(0,0)+\frac{y^2}{2}D_{yy}f(0,0).$$

We find easily

$$\nabla f(x,y) = (-\sin(x-y)+2\cos(x-y), \sin(x-y)-2\cos(x-y)$$

and thus

$$\nabla f(0,0) = (2, -2).$$

Also

$$\mathrm{Hess}(f)(x,y) = \begin{pmatrix} -\cos(x-y) - 2\sin(x-y) & \cos(x-y) + 2\sin(x-y) \\ \cos(x-y) + 2\sin(x-y) & -\cos(x-y) - 2\sin(x-y) \end{pmatrix}$$

and thus

$$\mathrm{Hess}(f)(0,0) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Replacing all those values in the formula above allow us to conclude.

10. Answer (i). We use the same formula as the previous exercise and by computation, we find

$$\nabla f(0,0) = (-2, -1)$$

and

$$\mathrm{Hess}(f)(0,0) = \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}.$$

Details of calculation are left to the reader.