MULTIPLE CHOICE QUESTIONS 5 : SOLUTIONS

1. Answer (iv).

- By a direct calculation :

$$w(x,y,z) = w(t^2, 1-t, 1+3t) = t^2 e^{\frac{1-t}{1+3t}}.$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}t} &= 2te^{\frac{1-t}{1+3t}} + t^2 e^{\frac{1-t}{1+3t}} \left(\frac{1-t}{1+3t}\right)' \\ &= 2te^{\frac{1-t}{1+3t}} + t^2 e^{\frac{1-t}{1+3t}} \left(\frac{-(1+3t)-3(1-t)}{(1+3t)^2}\right) \\ &= e^{\frac{y}{z}}\left(2t + x\left(\frac{-z-3y}{z^2}\right)\right) \\ &= e^{\frac{y}{z}}\left(2t - \frac{x}{z} - \frac{3yx}{z^2}\right).\end{aligned}$$

- We can also use the chain rule as following : let $\kappa(t) = (t^2, 1-t, 1+3t)$. Then $w(t) = w(\kappa(t))$ and thus

$$\begin{aligned}w'(t) &= \langle \nabla w(\kappa(t)), \kappa'(t)\rangle \\ &= \left\langle \left(e^{\frac{y}{z}}, \frac{x}{z}e^{\frac{y}{z}}, \frac{-xy}{z^2}e^{\frac{y}{z}}\right)^T, (2t,-1,3)^T \right\rangle \\ &= e^{\frac{y}{z}}2t + \frac{x}{z}e^{\frac{y}{z}}(-1) - \frac{xy}{z^2}e^{\frac{y}{z}}3 \\ &= e^{\frac{y}{z}}\left(2t - \frac{x}{z} - \frac{3xy}{z^2}\right).\end{aligned}$$

2. Answer (iv). The equation of the hyperplane is given at $\boldsymbol{a}$ by

$$z = \nabla f(\boldsymbol{a}) \cdot (\boldsymbol{x} - \boldsymbol{a}) + f(\boldsymbol{a}).$$

Applying this formula for $\boldsymbol{a} = (-3, 2)$, we get

$$\begin{aligned}z = 2 + (1,-2)\cdot(x-(-3), y-2) = 2 &\iff z = 2 + (1,-2)\cdot(x+3, y-2) \\ &\iff z = 2 + x + 3 - 2(y-2) \\ &\iff z = 9 + x - 2y.\end{aligned}$$

Remark : It may be a good thing to check that $(-3, 2, 2)$ verify the equation of the hyperplane that we have found to be sure we didn't do any mistakes in our calculation.

3. Answer (*ii*). The linear approximation at $\boldsymbol{a}$ is given by

$$f(\boldsymbol{x}) = \nabla f(\boldsymbol{a}) \cdot (\boldsymbol{x} - \boldsymbol{a}) + f(\boldsymbol{a}).$$

We have that

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \frac{\partial}{\partial x}\sqrt{7 - x^2 - 2y^2} \\ &= \frac{-x}{\sqrt{7 - x^2 - y - 2}} \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y}(x, y) &= \frac{\partial}{\partial y}\sqrt{7 - x^2 - 2y^2} \\ &= \frac{-2y}{\sqrt{7 - x^2 - 2y^2}}. \end{aligned}$$

Therefore,

$$\nabla f(2, -1) = (-2, 2).$$

Applying the formula above with $\boldsymbol{a} = (2, -1)$ we get

$$z = 7 + 2y - 2x.$$

4. Answer (*iii*). Let first observe that

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r}.$$

Therefore

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{-x}{r^3}$$

and thus

$$\frac{\partial^2}{\partial x^2}\frac{1}{r} = \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) = \frac{-r^3 + x \cdot 3r^2 \cdot \frac{x}{r}}{r^6} = \frac{3x^2 - r^2}{r^5}$$

and by an argument of symmetry, we also have

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = \frac{3y^2 - r^2}{r^5}.$$

Finally

$$\frac{\partial^2}{\partial x \partial y}\left(\frac{1}{r}\right) = \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{3xy}{r^5} = \frac{\partial^2}{\partial y \partial x}\left(\frac{1}{r}\right),$$

and thus

$$\operatorname{Hess}\left(\frac{1}{r}\right)(x, y) = \frac{1}{r^5}\begin{pmatrix} 3x^2 - r^2 & 3xy \\ 3xy & 3y^2 - r^2 \end{pmatrix}.$$

For $(x, y) = (3, 4)$ we have $r = 5$ and thus

$$\operatorname{Hess}\left(\frac{1}{r}\right)(3, 4) = \frac{1}{5^5}\begin{pmatrix} 2 & 36 \\ 36 & 23 \end{pmatrix}.$$

5. Answer (iv). As the question 1, we'll use the formula

$$\frac{\mathrm{d}f(\kappa(t))}{\mathrm{d}t} = \langle \nabla f(\kappa(t)), \kappa'(t) \rangle \,,$$

for $f(x,y) = x\ln(x+11y)$ and $\kappa(t) = (\sin(t), \cos(t))$. We have that

$$\nabla f(x,y) = \left(\ln(x+11y) + \frac{x}{x+11y}, \frac{11x}{x+11y} \right),$$

$$\kappa'(t) = (\cos(t), -\sin(t))$$

and thus

$$\begin{aligned} h'(t) &= \langle \nabla f(\kappa(t)), \kappa'(t) \rangle \\ &= \left(\ln(x+11y) + \frac{x}{x+11y} \right) \cos(t) + \frac{11x}{x+11y}(-\sin(t)) \\ &= \ln(x+11y)\cos(t) + \frac{x}{x+11y}(\cos(t) - 11\sin(t)). \end{aligned}$$

6. Answer (iii). For the first assertion,

$$D_x f(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{f(x,0)}{x} = \lim_{x\to 0} \frac{x^3 \cdot 0}{x^2 + 0^2} = 0,$$

and

$$D_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0} \frac{0y}{0+y^2} = 0.$$

Therefore

$$D_x f(0,0) = D_y f(0,0) = 0.$$

For the second assertion, we have that

$$\nabla f(x,y) = (D_x f, D_y f) = \left(\frac{3x^2y}{x^2+y^2} - \frac{2x^4y}{(x^2+y^2)^2}, \frac{x^3}{x^2+y^2} - \frac{2x^3y^2}{(x^2+y^2)^2} \right).$$

Therefore

$$D_{xy} f(0,0) = \lim_{x\to 0} \frac{D_y(x,0) - D_y(0,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1,$$

and

$$D_{yx} f(0,0) = \lim_{y\to 0} \frac{D_x(0,y) - D_x(0,0)}{y} = 0.$$

Therefore

$$D_{xy} f(0,0) \neq D_{yx} f(0,0).$$

Remark : The fact that $D_{xy}f(0,0) \neq D_{yx}f(0,0)$ tell us that f is not $\mathcal{C}^2$ at $\mathbf{0}$.

7. Answer (*iv*). In the previous question, we compute $D_x f(0,0) = D_y f(0,0) = 0$. Since the partial derivative are continuous at $\mathbf{0}$, we have that

$$\nabla f(\mathbf{0}) = \mathbf{0}.$$

By the way, the normal orientation is $(0,0,1)$ and thus, the equation of the hyperplane is given by $z = 0$.

8. Answer (*ii*). We have that

$$\begin{aligned}\frac{\partial^2}{\partial x_i \partial x_j}(fg) &= \frac{\partial}{\partial x_i}\left(g\frac{\partial f}{\partial x_j} + f\frac{\partial g}{\partial x_j}\right) \\ &= \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j} + g\frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} + f\frac{\partial^2 g}{\partial x_i \partial x_j}.\end{aligned}$$

Therefore,

$$\text{Hess}(fg)(\bar{x})_{ij} = g\text{Hess}(f)(\bar{x})_{ij} + f\text{Hess}(f)(\bar{x})_{ij} + \nabla f(\bar{x})\nabla g(\bar{x})^T_{ij} + \nabla g(\bar{x})\nabla f(\bar{x})^T_{ij},$$

and the claim follow.