Multiple choice test 12 answers

[1.] a. The solution is : $y(t) = \int\limits_0^t f(x)dx + y_0 \ \forall t \in I$.

[2.] b. Recall the counter-example from class : $y' = 2\sqrt{|y|}, y_0 = 0$ has 2 solutions on [-1,1], which are $y_1(t) = 0$ and $y_2(t) = t^2$

[3.]b. f_2 continuous and f_0 Lipschitz.
An equation of order 3 in dimension 1 = An equation of order 1 in dimension 3. So we consider $\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$ instead of y .
Let $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and the equation becomes :

$$Y' = \begin{pmatrix} y_2 \\ y_3 \\ y_3.f_2(x) + f_0(y_1) \end{pmatrix} := F(Y) \tag{1}$$

And $Y(0) = \begin{pmatrix} y_0 \\ y'_0 \\ y''_0 \end{pmatrix}$ On a bounded interval I. If F is Lipschitz, then the problem admits a unique solution. Let $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ and $\tilde{Y} = \begin{pmatrix} \tilde{y_1} \\ \tilde{y_2} \\ \tilde{y_3} \end{pmatrix}$ and we compute :
$\| F(Y) - F(\tilde{Y}) \|^2 \leq (y_2 - \tilde{y_2})^2 + (y_3 - \tilde{y_3})^2 + 2(f_2(x))^2(y_1 - \tilde{y_2})^2 + 2(f_0(y_1) - f_0(\tilde{y_1})^2$
(Here we used the fact that $(a + b)^2 \leq 2a^2 + 2b^2$.)

- If f_0 is Lipschitz, $\exists L_0 \geq 0$ a constant such that $\forall y, \tilde{y}, (f_0(y) - f_0(\tilde{y}))^2 \leq L_0(y - \tilde{y})^2$. Then we get :
$\| F(Y) - F(\tilde{Y}) \|^2 \leq 2L_0(y - \tilde{y})^2 + (1 + 2(f_2(x)^2)(y_2 - \tilde{y_2})^2 + (y_3 - \tilde{y_3})^2 \leq \max(2L_0, 1 + 2(f_2(x))^2, 1) * \| Y - \tilde{Y} \|^2$
- Now, if f_2 is continuous on bounded I , then for every closed interval $I' \subseteq I$ (which is also bounded). The Cauchy problem admits a unique solution on I' . That means that the problem admits a unique solution on $I(\forall x \in I, \exists I' \subseteq I$, closed, such that $x \in I')$.

[4.] (II) A unique golbal solution.
$f \in C^1$ so f is easily Lipschitz. Hence we have the existence and unicity of local solutions (on every closed intervalof given length). We can then put together these solutions to get a global solution, which is unique.

[5.] (4.) $y(x) = Ae^{3x^3}$
You just have to check.

[6.] (2) $y_0(9) = 3\sqrt{10}$
$(x^2+9)\frac{dy}{dx} = xy \iff \frac{dy}{dx} = \frac{x}{x^2+9}y \iff y(x) = Ae^{\frac{1}{2}\ln(x^2+9)} = A\sqrt{x^2+9}$
But $y(0) = 3 \iff A = 1 \Rightarrow y(x) = \sqrt{x^2+9}$ hence $y(9) = \sqrt{90} = 3\sqrt{10}$.

[7.](3) $y_4 = \frac{11}{4}$
$(x+3)y' = y - 1, \quad y(1) = 2$

$$y' = \frac{1}{x+3}y - \frac{1}{x+3} \tag{2}$$

1. General solution of the homogeneous equation : $y_0(x) = Ae^{\ln(x+3}, A \in \mathbb{R}$
2. We let A vary in terms of x : $A \to A(x), y(x) = A(x)(x+3)$.
   $y'(x) = A'(x)(x+3) + A(x)$ and we want (2) so
   $\Rightarrow A'(x)(x+3) = -\frac{1}{x+3} \iff A(x) = \frac{1}{x+3} + c, c \in \mathbb{R}$.
3. Finally, we find c to obtain $y_1(1) = 2$ where $y_1(x) = (\frac{1}{x+3} + c)(x+3) = 1 + c(x+3)$
   $2 = y_1(1) = 1 + 4c \iff c = \frac{1}{4}$.

Hence : $y_1(x) = 1 +$ Âă$\frac{x+3}{4}$ and $y_1(4) = \frac{11}{4}$

[8.](2) $y = e^{\pm\sqrt{2x^5+c}}$.
We have to change the equation a little bit : $y' = \frac{5x^4y}{\ln(y)} \iff \frac{y'}{y}\ln(y) = 5x^4 \iff (\ln(y))'\ln(y) = 5x^4$
Let $Y := \ln(y)$ and the system becomes : $Y'Y = 5x^4$
$\iff (\frac{1}{2}Y)^2)' = 5x^4 \iff \frac{1}{2}Y^2 = x^5 + c, c \in \mathbb{R}$.
$\iff Y = \pm\sqrt{2x^5+c}, c \in \mathbb{R}$ but we know that $y = e^Y$
$\iff y = e^{\pm\sqrt{2x^5+c}}$.

[9.](1) $u = -7 + Ce^{\frac{1}{2}t^2+6t}$

$$u' = (6+t)u + 7t + 42 \tag{3}$$

1. We solve the homogeneous equation to find the solution u_0 : $u_0(t) = Ae^{\frac{1}{2}t^2+6t}$

2. We vary the constant A in terms of t : $u_1(t) = A(t)e^{\frac{1}{2}t^2+6t}$
   $u_1'(t) = A'(t)e^{\frac{1}{2}t^2+6t} + (6+t)u_1(t)$
   By (3) : $A'(t) = (7t+42)e^{-\frac{1}{2}t^2-6t} \Rightarrow A(t) = -7e^{-\frac{1}{2}t^2-6t} + c, c \in \mathbb{R}$ //

Then we get our answer : $u_1(t) = -7 + ce^{\frac{1}{2}t^2+6t}, c \in \mathbb{R}$.