Multiple Choice Questions 4 : Solutions

1. Answer (*ii*). If the partials derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ exists and are continuous at $\mathbf{0}$, then f is differentiable at 0. Indeed, let $\boldsymbol{h} = (h_1, h_2) \in \mathbb{R}^2$. Then

$$u(\boldsymbol{h}) - u(\mathbf{0}) = u(h_1, h_2) - u(0, h_2) + u(0, h_2) - u(0, 0).$$

Using the mean value theorem for one variable functions, there are $c_1, c_2 \in \mathbb{R}$ s.t. $|c_1| \leq |h_1|$ and $|c_2| \leq |h_2|$ and

$$u(h_1, h_2) - u(0, h_2) = h_1 \frac{\partial u}{\partial x}(c_1, h_2)$$

and

$$u(0, h_2) - u(0, 0) = h_2 \frac{\partial u}{\partial y}(0, c_2).$$

Finally, we obtain

$$u(\boldsymbol{h}) - u(\mathbf{0}) = h_1 \left(\frac{\partial u}{\partial x}(c_1, h_2) - \frac{\partial u}{\partial x}(\mathbf{0}) \right) + h_2 \left(\frac{\partial u}{\partial y}(0, c_2) - \frac{\partial u}{\partial y}(\mathbf{0}) \right) + \langle \nabla u(\mathbf{0}), \boldsymbol{h} \rangle$$

and thus

$$\begin{aligned} \frac{u(\boldsymbol{h}) - u(\mathbf{0}) - \langle \nabla u(\mathbf{0}), \boldsymbol{h} \rangle}{\|h\|_2} &\leq \underbrace{\frac{h_1}{\|\boldsymbol{h}\|_2}}_{\leq 1} \left(\frac{\partial u}{\partial x}(c_1, h_2) - \frac{\partial u}{\partial x}(\mathbf{0}) \right) + \underbrace{\frac{h_2}{\|\boldsymbol{h}\|_2}}_{\leq 1} \left(\frac{\partial u}{\partial y}(0, c_2) - \frac{\partial u}{\partial y}(\mathbf{0}) \right) \\ &\leq \left(\frac{\partial u}{\partial x}(c_1, h_2) - \frac{\partial u}{\partial x}(\mathbf{0}) \right) + \left(\frac{\partial u}{\partial y}(0, c_2) - \frac{\partial u}{\partial y}(\mathbf{0}) \right) \underset{\boldsymbol{h} \to \mathbf{0}}{\longrightarrow} \mathbf{0}. \end{aligned}$$

what prove the claim.

Remark : Notice that c_1 and c_2 depending on h_1 and h_2 respectively. Therefore, when $\boldsymbol{h} \to \mathbf{0}$ we also have $(c_1, c_2) \to (0, 0)$ (because $|c_1| \leq |h_1|$ and $|c_2| \leq |h_2|$), and using the continuity of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, we get

$$\lim_{\boldsymbol{h} \to \mathbf{0}} \frac{\partial u}{\partial x}(c_1, h_2) = \frac{\partial u}{\partial x}(\mathbf{0}) \quad \text{and} \quad \lim_{\boldsymbol{h} \to \mathbf{0}} \frac{\partial u}{\partial y}(0, c_2) = \frac{\partial u}{\partial y}(\mathbf{0}).$$

2. Answer (*ii*). For the first assertion, if all directional derivatives exists, since $\frac{\partial f}{\partial x}(\mathbf{0})$ is the directional derivative along $(1, 0)$ and $\frac{\partial f}{\partial y}(\mathbf{0})$ is the directional derivative along $(0, 1)$, they both exists. Therefore, the gradient exist.

For the second assertion, we want to find a function u such that all partial derivative exists at 0 but $x \longmapsto \frac{\mathrm{d}u}{\mathrm{d}t}(x + t\boldsymbol{v})\Big|_{t=0}$ is not continuous at $\mathbf{0}$. Consider

$$u(x, y) = \begin{cases} x + y & xy \geq 0 \\ x - y & xy < 0 \end{cases}.$$

For all $\boldsymbol{v} = (v_1, v_2) \in \mathbb{R}^2$, we have that

$$\frac{u(\boldsymbol{v}t) - u(\mathbf{0})}{t} = \begin{cases} v_1 + v_2 & v_1 v_2 \geq 0 \\ v_1 - v_2 & v_1 v_2 < 0 \end{cases},$$

and thus the limit exist. However, let $\boldsymbol{x} = (x_1, x_2)$ s.t. $x_1 x_2 \neq 0$. Then

$$\frac{u(\boldsymbol{x} + \boldsymbol{v}t) - u(\boldsymbol{x})}{t} \underset{t \to 0}{\longrightarrow} \begin{cases} v_1 + v_2 & x_1 x_2 > 0 \\ v_1 - v_2 & x_1 x_2 < 0, \end{cases}$$

and thus $x \longmapsto \left.\frac{\mathrm{d}u(\boldsymbol{x}+t\boldsymbol{v})}{\mathrm{d}t}\right|_{t=0}$ is not continuous at $\mathbf{0}$.

3. Answer (iii).

$$\frac{\mathrm{d}}{\mathrm{d}t} f(1+t, 1+t) = \frac{\mathrm{d}}{\mathrm{d}t} e^{(1+t)^3} = 3(1+t)^2 e^{(1+t)^3},$$

and we evaluate at $t = 0$.

4. Answer (iii).

$$\begin{aligned} \left.\frac{\mathrm{d}f}{\mathrm{d}t}(g(t), h(t))\right|_{t=5} &= \left.\left(\frac{\partial f}{\partial x}(g(t), h(t))g'(t) + \frac{\partial f}{\partial y}(g(t), h(t))h'(t)\right)\right|_{t=5} \\ &= \frac{\partial f}{\partial x}(3,5)g'(5) + \frac{\partial f}{\partial y}(3,5)h'(5) \\ &= 4 \cdot 4 - 7 \cdot 2 \\ &= 2. \end{aligned}$$

5. Answer (ii). We have that

$$\lim_{(x,y)\to(12,4)} xy\cos(x - 3y) = 48\cos(0) = 48.$$

6. Answer (ii).

$$\left|\frac{5xy^2}{x^2 + y^2}\right| = \underbrace{\frac{y^2}{x^2 + y^2}}_{\leq 1} 5|x| \leq 5|x| \underset{(x,y)\to(0,0)}{\longrightarrow} 0.$$

7. Answer (ii).

$$\begin{aligned} \frac{\partial f}{\partial x}(x, y) &= \frac{\partial}{\partial x}(x\cos(x+y)) + \frac{\partial}{\partial x}\sin(x+y) \\ &= \cos(x+y) - x\sin(x+y) + \cos(x+y) \\ &= 2\cos(x+y) - x\sin(x+y). \end{aligned}$$

8. Answer (i). Recall that $\arctan'(x) = \frac{1}{1+x^2}$ for all $x \in \mathbb{R}$.

$$\begin{aligned} \frac{\partial f}{\partial y}(x, y) &= \frac{1}{2}x \cdot \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} \\ &= \frac{1}{2} \cdot \frac{x^2}{x^2 + y^2}. \end{aligned}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = \frac{\partial}{\partial x}\left(\frac{1}{2}\cdot\frac{x^2}{x^2+y^2}\right)$$
$$= \frac{1}{2}\cdot\frac{2x(x^2+y^2)-x^2(2x)}{(x^2+y^2)^2}$$
$$= \frac{xy^2}{(x^2+y^2)^2}.$$

9. Answer (iv).

$$\frac{\partial z}{\partial x} = -\frac{y}{x^2}f\left(\frac{x}{y}\right) + \frac{y}{x}f'\left(\frac{x}{y}\right)\cdot\frac{1}{y}$$
$$= -\frac{1}{x^2}\left(yf\left(\frac{x}{y}\right) - xf'\left(\frac{x}{y}\right)\right).$$