Multiple Choice Questions 3 : Solutions

1. Answer (iv). Using the chain rule

$$\frac{\partial}{\partial x}e^{h(x,y,z)} = e^{h(x,y,z)}\frac{\partial h}{\partial x}(x,y,z) = e^{xy^2z^3}\frac{\partial}{\partial x}(xy^2z^3) = e^{xy^2z^3}y^2z^3.$$

By doing the same for y and z, we finally get

$$(\nabla f)^T = e^{xy^2z^3}\left(y^2z^3, 2xyz^3, 3xy^2z^2\right).$$

2. Answer (iv). The length of the graph of f between a and b is given by

$$\int_a^b \sqrt{1+|f'(t)|^2}\mathrm{d}t.$$

Since $f'(t) = (e^{t/2}, e^{-t/2})$,

$$|f'(t)|^2 = \frac{e^{2t}}{2}(1+e^{-4t}),$$

and thus

$$1+|f'(t)|^2 = 1+\frac{e^{2t}}{2}(1+e^{-4t}) = \frac{e^{2t}}{2}(1+2e^{-2t}+e^{-4t}) = \frac{e^{2t}}{2}(1+e^{-2t})^2.$$

Therefore, the length is given by

$$\int_1^2 \frac{e^t(1+e^{-2t})}{\sqrt{2}}\mathrm{d}t = \frac{1}{\sqrt{2}}(e^2-e+e^{-1}-e^{-2}).$$

3. Answer (i).

$$D_x f = \begin{cases} x(2\ln|x|+1) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Since we can prolonge $x \longmapsto x(2\ln|x|+1)$ by continuity at 0, and that $x(2\ln|x|+1)$ is independent of y, we get $D_{yx}f \equiv 0$. Doing as well for $D_{xy}f$ we get that $D_{xy}f \equiv 0$, and thus $(ii), (iii)$ and (iv) are false and thus (iv) is correct. In fact,

$$D_{xx}f = \frac{\partial}{\partial x}x(2\ln|x|+1) = 2\ln|x|+3$$

that is not continuous at 0.

4. Answer (ii). Since $\|\cdot\|_\infty$ and $\|\cdot\|_2$ are equivalent, using $\|\boldsymbol{h}\|_\infty$ despite of $\|\boldsymbol{h}\|_2$ won't change the result. So

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{f(\boldsymbol{h}) - f(\boldsymbol{0}) - \langle \boldsymbol{h}, \boldsymbol{t}\rangle}{\|\boldsymbol{h}\|_2} = 0 \iff \lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{f(\boldsymbol{h}) - f(\boldsymbol{0}) - \langle \boldsymbol{h}, \boldsymbol{t}\rangle}{\|\boldsymbol{h}\|_\infty} = 0.$$

So it's either (i) or (ii). By the way, if $f \in \mathcal{C}^1(\mathbb{R}^2)$ then $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is differentiable. But the converse is false, i.e. f can be differentiable even if $f \notin \mathcal{C}^1(\mathbb{R}^2)$. For example

$$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}.$$

We have that

$$\frac{\partial f}{\partial x}(\boldsymbol{0}) = \lim_{x\to 0} \frac{x^2 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0,$$
$$\frac{\partial f}{\partial y}(\boldsymbol{0}) = 0,$$

and

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sqrt{x^2+y^2}} = 0,$$

since

$$\left| \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sqrt{x^2+y^2}} \right| \leq \left| \frac{x^2 \sin\left(\frac{1}{x}\right)}{\sqrt{x^2}} \right| = |x| \left| \sin\left(\frac{1}{x}\right) \right| \underset{(x,y)\to(0,0)}{\longrightarrow} 0.$$

But the $\frac{\partial f}{\partial x}$ is not continuous at 0. Indeed,

$$\frac{\partial f}{\partial x}(x,0) = 2x \sin\left(\frac{1}{x}\right) \cos\left(\frac{1}{x}\right).$$

By taking

$$x_n = \frac{1}{2\pi n} \quad \text{and} \quad y_n = \frac{1}{\frac{\pi}{2} + 2\pi n},$$

we get

$$\lim_{n\to\infty} \frac{\partial f}{\partial x}(x_n, 0) = -1 \neq 0 = \lim_{n\to\infty} \frac{\partial f}{\partial x}(y_n, 0).$$

Therefore $\frac{\partial f}{\partial x}$ is not continuous and thus f is not $\mathcal{C}^1$.

5. Answer (iv). Observe that $\frac{x^2-9y^2}{x^2+y^2}$ is not defined at $(0,0)$. Therefore, its domain is $\mathbb{R}^2\backslash\{(0,0)\}$. Let consider first $f(x,0)$ for $x \neq 0$. We have that $f(x,0) = 1$ for all $x \neq 0$, and thus

$$f\left(\frac{1}{n}, 0\right) = 1.$$

On the other hand, $f(0,y) = -9$ for all $y \neq 0$, and thus

$$\lim_{n\to\infty} f\left(0, \frac{1}{n}\right) = -9.$$

Therefore, $x_n = (\frac{1}{n}, 0)$ and $y_n = (0, \frac{1}{n})$ define to sequences that converge to 0 but such that

$$\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n).$$

Therefore, the limit at 0 doesn't exist.

6. Answer (iv).

$$\lim_{y\to 0^+} f(1,y) = \lim_{y\to 0} \frac{1}{y}$$

doesn't exist. Therefore f is not continuous at $(1,0)$, and in particular neither differentiable.

7. Answer (iii). Since $f'(1/2) = g'(1/4) = 0$, neither f nor g are regular. But

$$h'(t) = \left(-8\left(t - \frac{1}{2}\right), -2\left(t - \frac{1}{2}\right)\right) \neq 0$$

for all t, therefore h is regular.

8. Answer (i). We have that f is continuous over $\mathbb{R}^2 \backslash \{(x,x) \mid x \in \mathbb{R}\}$. Let show that f is also continuous over $\{(x,x) \mid x \in \mathbb{R}\}$. We have that

$$\lim_{y\to x} \frac{xe^x - ye^y}{x-y} = \lim_{y\to x} \frac{xe^x - ye^x + ye^x - ye^y}{x-y} \underset{(*)}{=} e^x + \lim_{y\to x} y\frac{e^y - e^x}{y-x} = e^x + xe^x = (x+1)e^x.$$

We used the fact that

$$\lim_{y\to x} \frac{e^y - e^x}{y-x} = \frac{d}{dy}e^y\bigg|_{y=x}.$$