Multiple Choice Questions 2 : Solutions

1. Answer (*iv*). The line and the circle intersect at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and thus, we have to find the tangent vector on the point for each curve. For the line, the slope is given by $v_1 = \left(1, -\frac{2}{\sqrt{3}}\right)$. For the circle, we derivate the fonction

$$\begin{aligned} f : [0,1] &\longrightarrow \mathbb{R} \\ x &\longmapsto \sqrt{1-x^2} \end{aligned}$$

and compute $f'(1/2) = -\frac{1}{\sqrt{3}}$. Therefore $v_2 = \left(-1, \frac{1}{\sqrt{3}}\right)$ and thus

$$\cos(\theta) = \frac{\langle v_1, v_2 \rangle}{\|v_1\|\|v_2\|} = \frac{5}{\sqrt{28}}.$$

Notice that to get v_2 we could also have parametrized the circle as $\varphi(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in [0, 2\pi[$ and compute $\dot{\gamma}(\frac{\pi}{3})$ (since the circle and the line intersect when $\theta = \frac{\pi}{3}$).

2. Answer (*iii*). The "if" has to be understood as an "if" of a definition, i.e. as an "if and only if".

(*i*) Counter-example : any periodic function (for example $x \longmapsto \cos(x)$)

(*ii*) Counter-example :

$$f(x) = \begin{cases} 1 & x \in \{\frac{1}{n} \mid n \in \mathbb{N}^*\} \cup \{0\} \\ 0 & \text{otherwise.} \end{cases}$$

The function f is obviously not continuous at 0 and we have that

$$\lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0) = 1.$$

(*iii*) The property is equivalent to : *if* $(x_n)_{n\in\mathbb{N}}$ *converge to* x_0*, then* $f(x_n) \to f(x_0)$. We can prove the result using the definition of continuity of f. Indeed, let $\varepsilon > 0$. since f is continuous at x_0, there is $\delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. Since $x_n \to x_0$, there is $N \in \mathbb{N}$ s.t. $|x_n - x_0| < \delta$ whenever $n \geq N$. Therefore, if $n \geq N$, we have that $|x_n - x_0| < \delta$, and thus $|f(x_n) - f(x_0)| < \varepsilon$. We conclude that $f(x_n) \to f(x_0)$.

(*iv*) Counter-example : Let

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

We will now prove that if $(x_n)_{n\in\mathbb{N}}$ is a sequence that converge to 0 and such that $\big(f(x_n)\big)_{n\in\mathbb{N}}$ converge, then there is $N \in \mathbb{N}$ s.t. $x_n = 0$ for all $n \geq N$ (and thus, the property of the statement will by verified, but f is not continuous).

Suppose by contradiction that $(x_n)_{n\in\mathbb{N}}$ converge to 0 such that $\big(f(x_n)\big)_{n\in\mathbb{N}}$ converge, but that for all $N\in\mathbb{N}$, there is $n\geq N$ s.t. $x_n\neq 0$. Let $M>0$. Since $(x_n)_{n\in\mathbb{N}}$ converge, there is $N_M\in\mathbb{N}$ s.t. $|x_n|<\frac{1}{M}$ for all $n\geq N_M$. However, by our assumption, there is $n_{N_M}\geq N_M$ s.t. $x_{n_{N_M}}\neq 0$, and thus $|f(x_{n_{N_M}})|=\frac{1}{|x_{n_{N_M}}|}>M$. We have proved that

$$\forall M>0, \exists n_M\in\mathbb{N}: |f(x_{n_M})|>M,$$

i.e. that

$$\lim_{M\to\infty} x_{n_M}=0 \quad\text{and}\quad \lim_{n\to\infty} f(x_{n_M})=+\infty,$$

what contradict the fact that $\big(f(x_n)\big)_{n\in\mathbb{N}}$ converge. The claim follow.

3. Answer *(ii)*. For the first assertion, recall that $\|f\|_\infty=\max\limits_{y\in[0,1]}|f(y)|$. The function F_1 is a linear operator, so to prove the continuity over $\mathcal{C}([0,1])$, it's enough to prove the continuity at 0 (why ?). We have that

$$|F_1(f)|\leq\int_0^1|f(x)|\mathrm{d}x\leq\int_0^1\|f\|_\infty\mathrm{d}x=\|f\|_\infty,$$

and thus, if $\varepsilon>0$ and $\delta=\varepsilon$, we have that

$$\|f\|_\infty<\delta \implies |F_1(f)|<\varepsilon,$$

what prove the claim. For the second one, let $f\equiv 1$ on $[0,1]$. For all n, set

$$g_n(x)=\begin{cases}1 & x\in\left[0,1-\frac{1}{n}\right]\\ n(1-x) & x\in\left[1-\frac{1}{n},1\right]\end{cases},$$

so that $g_n(1)=0$ for all $n\in\mathbb{N}^*$. Therefore, for all $n\in\mathbb{N}^*$,

$$|F_2(f)-F_2(g_n)|=1.$$

However,

$$\|f-g_n\|_2^2=\int_{1-\frac{1}{n}}^1\Big(1-n(1-x)\Big)\mathrm{d}x\underset{x=1-u}{=}\int_0^{\frac{1}{n}}(1-nu)^2\mathrm{d}u=\frac{1}{3n}\xrightarrow[n\to\infty]{}0.$$

To conclude, we found a sequence $(g_n)_{n\in\mathbb{N}^*}$ that converge to f but such that $\big(F_2(g_n)\big)_{n\in\mathbb{N}}$ doesn't converge to $F_2(f)$. Therefore F_2 is not continuous.

4. Remark that $f_1(1)=f_2(1)$ and thus, both cuves intersects at $t=1$. We have

$$f_1'(t)=(1,2,2t)$$
$$f_2'(t)=(0,-1,3t^2),$$

there value at $t=1$ gives the tangent vectors of both curves at the intersection point : $v_1=(1,2,2)$ and $v_2=(0,-1,3)$. We can conclude that

$$\cos(\theta)=\frac{\langle v_1,v_2\rangle}{\|v_1\|\|v_2\|}=\frac{4}{3\sqrt{10}}.$$

5. Answer (iii). Recall that a set is complete is all Cauchy sequence of E converge in E (i.e. the limit is also in E).

For the first assertion, let

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1, \end{cases}$$

and for all $n \in \mathbb{N}^*$, let define

$$g_n(x) = x^n, \quad x \in [0,1].$$

For all $m, n \in \mathbb{N}$ s.t. $1 \leq n \leq m$, we have

$$\|g_m - g_n\|_4^4 = \int_0^1 (x^n - x^m)^4 \mathrm{d}x \leq \int_0^1 x^{4n} \mathrm{d}x = \frac{1}{4n} \underset{n\to\infty}{\longrightarrow} 0,$$

and thus $(g_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence. Moreover $(g_n)_{n\in\mathbb{N}^*}$

$$\|g_n - f\|_4^4 = \int_0^1 x^{4n} \mathrm{d}x = \frac{1}{4n} \underset{n\to\infty}{\longrightarrow} 0,$$

and thus $(g_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence of E that converge to the function f which is not in E (since it's not continuous). Therefore E is not complete.

For the second one, let define a Cauchy sequence of E by $x_n = (u_n, v_n)$, i.e. $u_n^2 + v_n^2 \leq 1$. In particular, $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are both in $\mathbb{R}$ that is complete. Therefore, they both converge in $\mathbb{R}$. Let denote u and v there respective limits. Then we have that

$$1 \geq \lim_{n\to\infty} (u_n^2 + v_n^2) = u^2 + v^2,$$

and thus

$$x_n = (u_n, v_n) \underset{n\to\infty}{\longrightarrow} (u, v) \in E.$$

We have proved that all Cauchy sequence of E converge in E. Therefore E is complete.

6. Answer (iv). For the first assertion, the derivative of the curve is given by

$$f'(t) = \Big(2\sin(t)\cos(t), -2\sin(t)\cos(t)\Big).$$

Since $f'(0) = f'(\pi/2) = f'(\pi) = 0$, the curve is not regular. For the second one, the curve is not continuous.

7. Answer (i). Let $(y_n)_{n\in\mathbb{N}}$ a sequence that converge to x s.t. $y_n \neq x$ for all n. Since f is bounded, the sequence $\big(f(y_n)\big)_{n\in\mathbb{N}}$ is bounded, and thus, there is a subsequence that converge. Let $\big(f(y_{n_k})\big)_{k\in\mathbb{N}}$ such a sequence, and let set $x_k = y_{n_k}$. Let denote y the limit of $\big(f(x_n)\big)_{n\in\mathbb{N}}$. Then we have that

$$x_n \longrightarrow x \quad \text{and} \quad f(x_n) \longrightarrow y,$$

what prove the claim.