Multiple Choice Questions 1 : Solutions

1. (*i*) **False :** Take $E = E^\circ =]0, \frac{1}{2}[$ and $F = [0,1]$. We have that $\frac{1}{2} \in \partial E$ but

$$\frac{1}{2} \notin \partial(E \cup F) = \partial[0,1] = \{0,1\}.$$

(*ii*) **True :** If $x \in \partial(E \cup F)$, then by definition

$$\forall r > 0, \mathcal{B}(x,r) \cap (E \cup F)^c \neq \emptyset \quad \text{and} \quad \mathcal{B}(x,r) \cap (E \cup F) \neq \emptyset.$$

Therefore either for E or F (let say E), we have

$$\forall r > 0, \mathcal{B}(x,r) \cap (E \cup F)^c \neq \emptyset \quad \text{and} \quad \mathcal{B}(x,r) \cap E \neq \emptyset.$$

Since $E^c \supset (E \cup F)^c$,

$$\forall r > 0, \mathcal{B}(x,r) \cap E^c \neq \emptyset \quad \text{and} \quad \mathcal{B}(x,r) \cap E \neq \emptyset,$$

we finally get that $x \in \partial E$, and thus that $\partial(E \cup F) \subset \partial E \cup \partial F$. Conversely, let $x \in \partial E \cup F$. Therefore either $x \in \partial E$ or $x \in \partial F$, let say without loss of generality that $x \in \partial E$. We have that $\partial E \subset \overline{E}$ and $\overline{E} \cap \overline{F} = \emptyset$, therefore $x \in \overline{F}^c$. Since $\overline{F}$ is closed, $\overline{F}^c$ is open and thus,

$$\exists r_0 > 0 : \forall r \leq r_0, \mathcal{B}(x,r) \subset \mathcal{B}(x,r_0) \subset \overline{F}^c.$$

Therefore, for all $r \leq r_0$,

$$\mathcal{B}(x,r) \cap E \neq \emptyset, \quad \mathcal{B}(x,r) \cap E^c \neq \emptyset, \quad \mathcal{B}(x,r) \cap F = \emptyset \quad \text{and} \quad \mathcal{B}(x,r) \subset F^c.$$

Therefore, for all $r \leq r_0$,

$$\mathcal{B}(x,r) \cap E \neq \emptyset \quad \text{and} \quad \mathcal{B}(x,r) \cap (E \cup F)^c \neq \emptyset,$$

and thus

$$\mathcal{B}(x,r) \cap (E \cup F) \neq \emptyset \quad \text{and} \quad \mathcal{B}(x,r) \cap (E \cup F)^c \neq \emptyset,$$

what prove that $x \in \partial(E \cup F)$.

(*iii*) **False :** Let $E =]0,1[$ and $F = [1,2]$. We have that $1 \in \partial E$, but

$$\partial(E \cup F) = \partial]0,1] = \{0,2\},$$

while

$$E^\circ \cap F^\circ =]0,1[\cap]1,2[= \emptyset.$$

(*iv*) **False :** same counter-example as (*iii*).

2. $\mathcal{N}$ is not a norm since

$$\mathcal{N}(2x) = \frac{\|2x\|_2}{1+\|2x\|_2} = \frac{2\|x\|_2}{1+2\|x\|_2} < 2\cdot\frac{\|x\|_2}{1+\|x\|_2} = 2\mathcal{N}(x),$$

and thus the homogeneity is not valid. On the other hand, d is indeed a distance. the fact that $d(x,y)>0$ for all $x\neq y$ and $d(x,x)=0$ is clear. Let show the triangle inequality. First, let observe that

$$t \longmapsto \frac{t}{1+t}$$

is increasing on $[0,\infty)$. Also, we see that for $u,v\geq 0$,

$$\frac{u}{1+u}+\frac{v}{1+v}-\frac{u+v}{1+u+v} = \frac{u(1+v)}{(1+u)(1+v)}+\frac{v(1+u)}{(1+v)(1+u)}-\frac{u+v}{1+u+v}$$

$$= \frac{u+v+2uv}{1+u+v+uv}-\frac{u+v}{1+u+v}$$

$$= \frac{uv(u+v+2)}{(1+u+v)^2+(1+u+v)uv} \geq 0,$$

and thus

$$\frac{u+v}{1+u+v} \leq \frac{u}{1+u}+\frac{v}{1+v}, \tag{1}$$

for all $u,v\geq 0$. Using the fact that $\|\cdot\|_2$ is a norm, we get that

$$\|x-z\|_2 \leq \|x-y\|_2+\|y-z\|_2,$$

and thus, if $x,y,z\in\mathbb{R}$,

$$d(x,z)=\mathcal{N}(x-z)=\frac{\|x-z\|_2}{1+\|x-z\|_2} \leq \frac{\|x-y\|_2+\|y-z\|_2}{1+\|x-y\|_2+\|y-z\|_2}$$

$$\underset{(1)}{\leq} \frac{\|x-y\|_2}{1+\|x-y\|_2}+\frac{\|y-z\|_2}{1+\|y-z\|_2} = \mathcal{N}(x-y)+\mathcal{N}(y-z) = d(x,y)+d(y,z),$$

where the first inequality come from the fact that $t\longmapsto\frac{t}{1+t}$ is increasing. The claim follow.

3. (i) **False :** E is not closed. Indeed, let consider the sequence defined by $x_n = \left(\frac{1}{n},\frac{1}{n^{3/2}}\right)$. Then (x_n) is a sequence of E. Moreover we have that

$$\lim_{n\to\infty} x_n = (0,0)\notin E,$$

and thus E is not closed.

(*ii*) **False :** since E is not closed.

(*iii*) **True :** Let show that E^c is closed. Remark that

$$E^c = \{(x,y) \mid y \leq x^2 \text{ or } y \geq x\} \cup (]-\infty,0] \times \mathbb{R}) \cup ([1,+\infty[\times\mathbb{R}).$$

Since $]-\infty,0]\times\mathbb{R}$ and $[1,+\infty[\times\mathbb{R}$ are closed, we just have to show that

$$\{(x,y) \mid x \in [0,1] \text{ and } (y \leq x^2 \text{ or } y \geq x)\}$$

is closed (because a finite union of closed is closed). Let $\big((x_n,y_n)\big)_{n\in\mathbb{N}}$ a convergent sequence of E. Let denote (a,b) the limit. If the limit is $(0,0)$ or $(1,1)$ then we are done. So we suppose $(a,b)\notin\{(0,0),(1,1)\}$. Let

$$A = \{n \mid y_n \leq x_n^2\} := \{n_k\}_{k\in\mathbb{N}} \quad \text{and} \quad B = \{n \mid y_n \geq x_n\} := \{m_k\}_{k\in\mathbb{N}}.$$

We necessarily have that $|A|$ or $|B|$ is finite. Indeed, if both where non-finite, then

$$b = \lim_{k\to\infty} y_{n_k} \leq \lim_{k\to\infty} x_{n_k}^2 = a^2$$

and

$$b = \lim_{k\to\infty} y_{m_k} \geq \lim_{k\to\infty} x_{n_k} = a,$$

and thus $a \leq b \leq a^2$. Since $a\in[0,1]$ we have that $a^2 \leq a$ and thus we get $a = a^2 = b$ what implies that $(a,b) = (0,0)$ or $(a,b) = (1,1)$ which is impossible by our assumption on (a,b). Suppose without loss of generality that A is finite and B is non finite. In particular, there is $k\in\mathbb{N}$ such that $n\in B$ for all $n\geq k$. Therefore, $x_n \leq y_n$ for all $n \geq k$ and thus

$$a = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n = b.$$

In particular $(a,b)\in E$, and thus E is closed. Let show that

$$\partial E = \{(x,x) \mid x\in[0,1]\} \cup \{(x,x^2) \mid x \in [0,1]\}.$$

The fact that $\partial E \subset \{(x,x) \mid x\in[0,1]\} \cup \{(x,x^2) \mid x\in[0,1]\}$ is obvious, and thus, we just have to show the converse inclusion. Also, the fact that $(0,0)$ and $(1,1)$ are in ∂E is obvious. Let $x\in]0,1[$ and consider the ball $\mathcal{B}\left((x,x),\frac{1}{n}\right)$ with $\frac{1}{n}$ small enough to have $x-\frac{1}{n} > x^2$. In particular if $m>n$, we have that $(x, x-\frac{1}{m})\in E$ and $(x,x+\frac{1}{m})\in E^c$. Therefore

$$\forall n\in\mathbb{N}, \mathcal{B}\left((x,x),\frac{1}{n}\right)\cap E \neq \emptyset \quad \text{and} \quad \mathcal{B}\left((x,x),\frac{1}{n}\right)\cap E^c \neq \emptyset.$$

Therefore $(x,x)\in\partial E$. Let $x\in]0,1[$, we consider n big enough to have $x^2+\frac{1}{n} < x$. Take $m>n$ and show that $(x,x^2+\frac{1}{m})\in E$ and $(x,x^2-\frac{1}{m})\in E^c$ what will prove the claim (to see things better, make a draw !).

4. (*i*) **False :** Take $E=[0,1]$.

(*ii*) **False :** Take $E=[0,1]$.

(iii) **True :** Let show that E^c is open. Suppose by contradiction that it's not open. So there is an $x \in E^c$ such that for all n,

$$\mathcal{B}\left(x, \frac{1}{n}\right) \cap E \neq \emptyset.$$

So let $x_n \in \mathcal{B}(x, \frac{1}{n}) \cap E$ for all n. In particular, (x_n) is a sequence of E that converge to x, and by assumption it converge in E. Therefore $x \in E$ which is a contradiction with $x \in E^c$. Therefore E^c is open, and thus E is closed.

(iv) **False :** Take $E = \mathbb{R}^n$.

5. (i) **False :** Since E is countable, it can't contain any ball. Therefore it's not open.

(ii) **False :** Same reason as previously.

(iii) **False :** Since $\lim_{n\to\infty} \left(\frac{1}{n}, e^{-(\log(n))^{3/2}}\right) = (0,0) \notin E$, it can't be closed.

(iv) **True.**