Linear-Quadratic Stochastic Differential Games on Directed Chain Networks

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at UC Santa Barbara

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Coupled Directed Chain and Mean Field SDEs

Consider the finite system of particles $(X^{(u)}_{t,i}\,,\ t \ge 0\,,\ i = 1, \ldots, N)$ defined by the system of stochastic differential equations

$$dX^{(u)}_{t,i} \,=\, b\big(t, X^{(u)}_{t,i}, \widehat{F}^{(u)}_{t,i}\big)\,dt + dW_{t,i}\,; \quad t \ge 0\,, \quad i = 1, \ldots, N-1$$

where

$$\widehat{F}^{(u)}_{t,i}(\cdot) \,:=\, u \cdot \delta_{X^{(u)}_{t,i+1}}(\cdot) + (1-u)\cdot \frac{1}{N}\sum_{j=1}^{N} \delta_{X^{(u)}_{t,j}}(\cdot)\,, \quad i = 1, \ldots, N$$

with the **periodic condition** $X_{t,N+1} = X_{t,1}$.
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Ref: *Directed Chain Stochastic Differential Equations* with N. Detering and T. Ichiba, SPTA 2020

Infinite Dimensional System ($N \to \infty$)

Assume $X_{0,i}^{(u)}$ are i.i.d. and independent of the W's. **In the limit $N \to \infty$**, in distribution, one given particle (say the first one) can be described by the following equation for $(X^{(u)}, \widetilde{X}^{(u)})$:

$$dX_t^{(u)} = b(t, X_t^{(u)}, F_t^{(u)})\, dt + dB_t\, ; \quad t \geq 0\, ,$$

driven by a Brownian motion $(B_t, t \geq 0)$, where $F_\cdot^{(u)}$ is the weighted probability measure

$$F_t^{(u)}(\cdot) := u \cdot \delta_{\widetilde{X}_t^{(u)}}(\cdot) + (1-u) \cdot \mathcal{L}_{X_t^{(u)}}(\cdot)$$

with

$$\text{Law}(X_t^{(u)},\, t \geq 0) \equiv \text{Law}(\widetilde{X}_t^{(u)},\, t \geq 0)$$
$$\sigma(\widetilde{X}_t^{(u)},\, t \geq 0) \perp\!\!\!\perp \sigma(B_t\, ,\, t \geq 0)\, .$$

The Linear Case: Infinite Dimensional OU

Choosing $b(t,x,\mu) := -\int_{\mathbb{R}}(x-y)\mu(dy)$, the problem becomes:

$$dX_t^{(u)} = u\,(\widetilde{X}_t^{(u)} - X_t^{(u)})\,dt + (1-u)(\mathbb{E}[X_t^{(u)}] - X_t^{(u)})\,dt + dB_t$$

For a fixed initial value $X_0^{(u)} = 0$, we have $\mathbb{E}[X_t^{(u)}] = \mathbb{E}[\widetilde{X}_t^{(u)}] = 0$

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Explicitly solvable Gaussian pair $(X_t^{(u)}, \widetilde{X}_t^{(u)})$:

$$\begin{aligned}
X_t^{(u)} &= u\int_0^t e^{-(t-s)}\widetilde{X}_s^{(u)}\,ds + \int_0^t e^{-(t-s)}\,dB_s \\
\widetilde{X}_t^{(u)} &= \sum_{k=1}^{\infty}\int_0^t \mathfrak{p}_{1,k}(t-s;u)\,dW_{s,k} \\
\mathfrak{p}_{1,k}(t-s;u) &:= \frac{u^{k-1}(t-s)^{k-1}}{(k-1)!}e^{-(t-s)}
\end{aligned}$$

where $(W^k, k\geq 1)$ are independent Brownian motions, independent of the Brownian motion B

The Linear Case: Summary

The previous formulas lead to explicit computation of variances and covariances.

Different behaviors for different values of u in the linear Gaussian case:

u	Interaction Type	Asymptotic Variance	Asymptotic Dependence (Propagation of chaos)
$u = 0$	Pure mean-field	Stabilized	Independent
$u \in (0, 1)$	Mixed interaction	Stabilized	Dependent
$u = 1$	Pure directed chain	**Explosive**	Dependent

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Next, we introduce a **game feature** with the following question in mind:

- **Is this dynamics an equilibrium?**

N-Player Directed Chain Game Model

With Yichen Feng and Tomoyuki Ichiba, submitted 2020

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$$dX_t^i = \alpha_t^i dt + \sigma dW_t^i, \quad i = 1, \cdots, N,$$

The system starts at time $t = 0$ from $i.i.d.$ random variables $X_0^i = \xi_i$ independent of the Brownian motions and such that $\mathbb{E}(\xi_i) = 0$.

Player i chooses its own strategy α^i in order to minimize its objective function of the form:

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left\{\int_0^T \left(\frac{1}{2}(\alpha_t^i)^2 + \frac{\epsilon}{2}(X_t^{i+1} - X_t^i)^2\right) dt + \frac{c}{2}(X_T^{i+1} - X_T^i)^2\right\}$$

for constants $\epsilon > 0$ and $c \geq 0$, and a BC for J^N:

Periodic BC: $X^{N+1} = X^1$

Free BC: autonomous stochastic control for X^N

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We are looking at an LQ differential game on a directed network since X^i interacts only with X^{i+1} through the cost functions.

Construction of Open-Loop Nash Equilibria

The **Hamiltonian** for player i is given by:

$$H^i(x^1,\cdots,x^N,y^{i,1},\cdots,y^{i,N},\alpha^1,\cdots,\alpha^N)=\sum_{k=1}^{N}\alpha^k y^{i,k}+\frac{1}{2}(\alpha^i)^2+\frac{\epsilon}{2}(x^{i+1}-x^i)^2$$

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The **adjoint processes** $Y_t^i = (Y_t^{i,j}; j = 1, \cdots, N)$ and $Z_t^i = (Z_t^{i,j,k}; j, k = 1, \cdots, N)$ for $i = 1, \cdots, N$ solve the BSDE:

$$\begin{aligned} dY_t^{i,j} &= -\partial_{x^j} H^i(X_t, Y_t^i, \alpha_t)dt + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k \\ &= -\epsilon(X_t^{i+1} - X_t^i)(\delta_{i+1,j} - \delta_{i,j})dt + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k, \\ Y_T^{i,j} &= c(X_T^{i+1} - X_T^i)(\delta_{i+1,j} - \delta_{i,j}) \end{aligned}$$

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For $j\neq i$ or $i+1$, $dY_t^{i,j}=\sum_{k=1}^N Z_t^{i,j,k}dW_t^k$ and $Y_T^{i,j}=0$ implies $Z_t^{i,j,k}=0$.

Construction of Open-Loop Nash Equilibria (continued)

By **Pontryagin stochastic maximum principle**, we get an open-loop Nash equilibrium by minimizing the Hamiltonian H^i with respect to α^i:

$$\partial_{\alpha^i} H^i = y^{i,i} + \alpha^i = 0 \quad \textit{leads to the choice:} \quad \hat{\alpha}^i = -y^{i,i}.$$

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$$Y_t^{i,i} = \sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j},$$

for some deterministic scalar functions ϕ_t's satisfying the terminal conditions: $\phi_T^{N,0} = c, \quad \phi_T^{N,1} = -c, \quad \phi_T^{N,k} = 0$ for $k \geq 2$.

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Note that one needs $\phi_t^{N,i,j}$ in the (non-stationary) free BC case

Construction of Open-Loop Nash Equilibria (continued)

The **forward equations** become

$$dX_t^i = -\sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j} dt + \sigma dW_t^i$$

Differentiating the ansatz $Y_t^{i,i} = \sum_{j=0}^{N-1} \phi_t^{N,j} X_t^{i+j}$, we get:

$$\begin{aligned} dY_t^{i,i} &= \sum_{j=0}^{N-1} [X_t^{i+j} \dot{\phi}_t^{N,j} dt + \phi_t^{N,j} dX_t^{i+j}] \\ &= \sum_{j=0}^{N-1} X_t^{i+j} \dot{\phi}_t^{N,j} dt - \sum_{j=0}^{N-1} \phi_t^{N,j} \sum_{k=0}^{N-1} \phi_t^{N,k} X_t^{i+j+k} dt + \sum_{j=0}^{N-1} \sigma \phi_t^{N,j} dW_t^{i+j} \end{aligned}$$

Construction of Open-Loop Nash Equilibria (continued)

Comparing with the **backward equations**, the martingale terms give:

$$Z_t^{i,i,0} = 0;\ Z_t^{i,i,k} = \sigma \phi_t^{N,N+k-i} \text{ for } 1 \le k < i;\ Z_t^{i,i,k} = \sigma \phi_t^{N,k-i} \text{ for } i \le k \le N.$$

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From the drift terms:

$$\begin{aligned}
\dot\phi_t^{N,0} &= \phi_t^{N,0}\cdot\phi_t^{N,0} + \sum_{i=1}^{N-1}\phi_t^{N,i}\phi_t^{N,N-i} - \epsilon,\ \ \phi_T^{N,0} = c,\\
\dot\phi_t^{N,1} &= \phi_t^{N,0}\cdot\phi_t^{N,1} + \phi_t^{N,1}\cdot\phi_t^{N,0} + \sum_{i=2}^{N-1}\phi_t^{N,i}\phi_t^{N,N+1-i} + \epsilon,\ \ \phi_T^{N,1} = -c,\\
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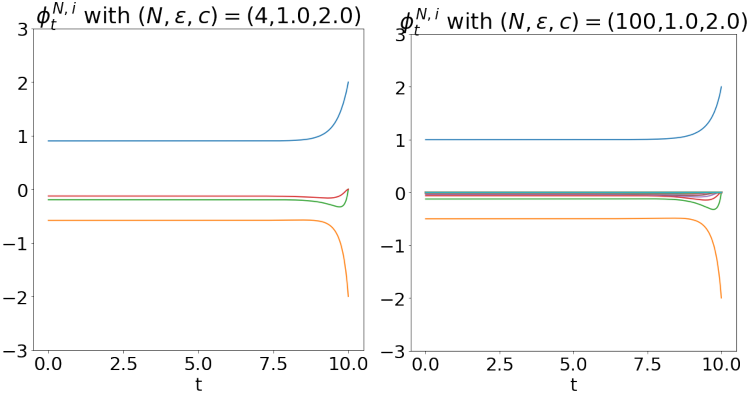
\end{aligned}$$

Matrix Riccati equation:

$$\dot\Phi^N(t) = \Phi^N(t)\Phi^N(t) - \mathbf{E}\,, \quad \Phi^N(T) := \mathbf{C}$$

Convergence as $N \to \infty$: numerical results



Blue line: $\phi_t^{N,0} \to 1$. Orange line $\phi_t^{N,1} \to -\frac{1}{2}$, Other lines: $\phi_t^{N,k} \to 0$ for $k \geq 2$. Left: $N = 4$. Right: $N = 100$.

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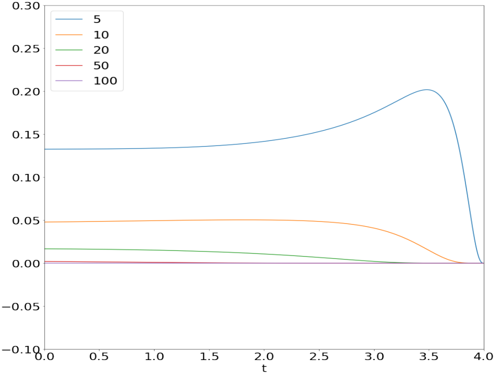

$\sum_{k=1}^{N-1} \phi_t^{N,k} \phi_t^{N,N-k}$ for different values of N

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$$\begin{aligned} \dot{\phi}_t^0 &= \phi_t^0 \cdot \phi_t^0 - \epsilon, \quad \phi_T^0 = c, \\ \dot{\phi}_t^1 &= 2\phi_t^0 \cdot \phi_t^1 + \epsilon, \quad \phi_T^1 = -c, \\ k \geq 2: \quad \dot{\phi}_t^k &= \phi_t^0 \cdot \phi_t^k + \phi_t^1 \cdot \phi_t^{k-1} + \cdots + \phi_t^{k-1} \cdot \phi_t^1 + \phi_t^k \cdot \phi_t^0, \quad \phi_T^k = 0. \end{aligned}$$

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\end{aligned}
$$

Define $S_t(z) = \sum_{k=0}^{\infty} z^k \cdot \phi_t^{(k)}$ where $0 \leq z < 1$ and $\phi_t^{(k)} = \phi_t^k$ to avoid confusion.

Then

$$
\dot{S}_t(z) = (S_t(z))^2 - \epsilon(1-z), \quad S_T(z) = c(1-z).
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Taking $z = 0$ gives $\phi_t^{(0)}$ explicitly, and $z = 1$ implies $\sum_{k=1}^{\infty} \phi_t^{(k)} = -\phi_t^{(0)}$.

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The model **under equilibrium** can be written as an **∞-dim OU**:

$$\begin{aligned}
dX_t^i &= -\sum_{j=0}^{\infty} \phi_t^j X_t^{i+j} dt + \sigma dW_t^i \\
&= \phi_t^0 \Big[\sum_{j=1}^{\infty} \left(\frac{-\phi_t^j}{\phi_t^0} \right) X_t^{i+j} - X_t^i \Big] dt + \sigma dW_t^i,
\end{aligned}$$

which is not exactly the model presented at the beginning.

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To simplify the presentation we assume $\epsilon = 1$, and we take the limit $T \to \infty$ so that the ϕ_t^k's become constant numbers. They satisfy the recurrence relation: $\phi^0 = 1$, $\phi^1 = -\frac{1}{2}$, and $\sum_{k=0}^{n} \phi^k \phi^{n-k} = 0$ for $n \geq 2$, which is related to the recurrence relation of **Catalan numbers**.

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Let $p_1 = -\phi^1 = \frac{1}{2}$, $p_k = -\phi^k = \frac{(2k-3)!}{(k-2)!k!} \frac{1}{2^{2k-2}}$ for $k \geq 2$. We consider a continuous-time Markov chain $M(\cdot)$ in the state space $\mathbb{N}$ with generator

matrix $\mathbf{Q} = \begin{pmatrix} -1 & p_1 & p_2 & p_3 & \cdots \\ 0 & -1 & p_1 & p_2 & \ddots \\ 0 & 0 & -1 & p_1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$.

Equilibrium Dynamics

The infinite particle system $(X_t^1, X_t^2, \cdots)$ is represented as the solution of the stochastic evolution equation:

$$d\mathbf{X}_t = \mathbf{Q}\,\mathbf{X}_t dt + d\mathbf{W}_t,$$

with its solution $\mathbf{X}_t = e^{t\mathbf{Q}}\mathbf{x_0} + \int_0^t e^{(t-s)\mathbf{Q}} d\mathbf{W}_s$, assuming $\mathbf{X}_0 = 0$ w.l.o.g.

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Taking advantage of the **Jordan block structure** of $\mathbf{Q}^2$:

$$\mathbf{Q}^2 = \begin{pmatrix} 1 & -1 & 0 & \cdots & \\ 0 & 1 & -1 & \ddots & \\ & \ddots & \ddots & \ddots & \end{pmatrix},$$

one can show that

$$X_t^j = \sum_{k=j}^{\infty} \int_0^t \frac{(t-s)^{2(k-j)}}{(k-j)!} \rho_{k-j}(-(t-s)^2)\, e^{-(t-s)} dW_s^k,$$

where the functions ρ_k's are given by

Variance Stabilization

$$\rho_k(-\nu^2) = \frac{e^\nu}{2^k \nu^k}\sqrt{\frac{2\nu}{\pi}}\, K_{k-\frac{1}{2}}(\nu)\,; \quad k \geq 1\,, \quad \rho_0 \equiv 1\,,$$

where $K_n(x)$ is the modified Bessel function of the second kind, i.e.,

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$$\mathrm{Var}(X_t^1) = \sum_{k=1}^{\infty} \int_0^t \frac{2}{\pi} \frac{\nu^{2k+1}}{(k!)^2\, 4^k} \left(K_{k-\frac{1}{2}}(\nu)\right)^2 d\nu + \frac{1 - e^{-2t}}{2}$$

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but we also show that **asymptotic dependence persists.**

Mixed Games

As before:

$$dX_t^i = \alpha_t^i dt + \sigma dW_t^i; \quad i = 1, 2, \cdots .$$

By choosing α_t^i, player i tries to minimize:

$$J^i(\alpha^1, \cdots) = \mathbb{E}\Big\{ \int_0^T \Big[\frac{1}{2}(\alpha_t^i)^2 + u\frac{\epsilon}{2}(X_t^{i+1} - X_t^i)^2 + (1-u)\frac{\epsilon}{2}(m_t - X_t^i)^2\Big] dt$$
$$+u\frac{c}{2}(X_T^{i+1} - X_T^i)^2 + (1-u)\frac{c}{2}(m_T - X_T^i)^2\Big\},$$

for some constants $\epsilon > 0$, $c \geq 0$ and $u \in [0, 1]$.

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One needs to be careful and start with the N-player game or simply set $m_t = \mathbb{E}\{X_t^i\}$.

Open-Loop Nash Equilibria

The generalized Hamiltonian for individual i is given by:

$$H^i(x^1, x^2, \cdots, y^{i,1}, y^{i,2}, \cdots, \alpha^1, \alpha^2, \cdots) = \sum_{k=1}^{\infty} \alpha^k y^{i,k} + \frac{1}{2}(\alpha^i)^2 + u\frac{\epsilon}{2}(x^{i+1} - x^i)^2 + (1-u)\frac{\epsilon}{2}(m_t - x^i)^2$$

Then, one writes the generalized BSDEs for the adjoint processes and the ansatz:

$$Y_t^{i,i} = u\sum_{j=i}^{\infty} \phi_t^{j-i} X_t^j - (1-u)\psi_t(m_t - X_t^i)$$

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The computational machinery presented in the case $u = 0$ can be extended to the case $u > 0$ by using a **killed Catalan Markov chain**. That leads to computation of variances and covariances.

Asymptotic Behavior

$$\begin{aligned}\mathrm{Var}(X_t^i) &= \sum_{k=1}^{\infty}\int_0^t \frac{2u^{2k}}{\pi(k!)^2 4^k}\nu^{2k+1}(K_{k-\frac{1}{2}}(\nu))^2 d\nu + \frac{1-e^{-2t}}{2} \\ &\longrightarrow \frac{1}{2}(1-\frac{u^2}{2})^{-\frac{1}{2}} \quad \text{as} \quad t\to\infty\end{aligned}$$

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To summarize:

u	Interaction Type	Asymptotic Variance	Asymptotic Dependence
$u = 0$	Pure mean-field	Stabilized	Independent
$u \in (0,1)$	Mixed interaction	Stabilized	Dependent
$u = 1$	Pure directed chain	**Stabilized**	Dependent

Extension: Directed Tree Networks

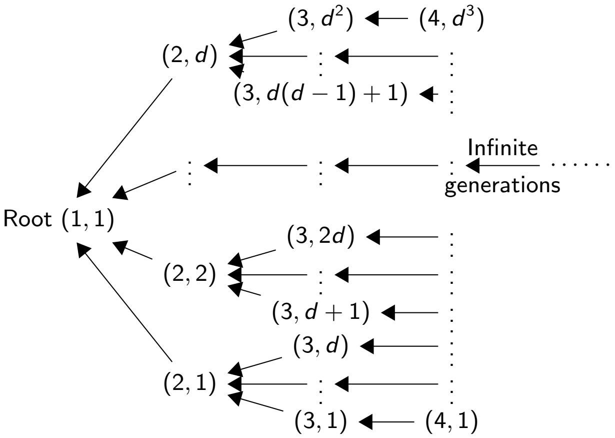

Figure: Directed Tree Network with d direct descendants

Extension: Directed Tree Networks

At the n^{th} generation:

$$dX_t^{n,k} = \alpha_t^{n,k} dt + \sigma dW_t^{n,k}, \quad 0 \leq t \leq T$$

Objective:

$$J^{n,k} = \mathbb{E}\left\{ \int_0^T \left[\frac{1}{2}(\alpha_t^{n,k})^2 + \frac{\epsilon}{2}\left(\overline{X}_t^{n+1,k} - X_t^{n,k}\right)^2 \right] dt + \frac{c}{2}\left(\overline{X}_T^{n+1,k} - X_T^{n,k}\right)^2 \right\},$$

where $\overline{X}_\cdot^{n,k} := \frac{1}{d}\sum_{i=(k-1)d+1}^{kd} X_\cdot^{n,i}$ for some constants $\epsilon > 0$ and $c \geq 0$ and for $n, k \geq 1$.

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Hamiltonian:

$$H^{n,k} = \sum_{m=1}^{M_n} \sum_{l=1}^{d^{m-1}} \alpha^{m,l} y^{n,k;m,l} + \frac{1}{2}(\alpha^{n,k})^2 + \frac{\epsilon}{2}(\overline{x}^{n+1,k} - x^{n,k})^2,$$

where **only finitely many** $Y_t^{n,k;m,l}$'s will be non-zero for every given (n,k). Here, $M_n \in \mathbb{N}$ represents a depth of this finite dependence depending on n with $M_n > n$ for $n \ge 1$.

Extension: Directed Tree Networks

The **adjoint processes** $Y_t^{n,k} = (Y_t^{n,k;m,l}; m \in \mathbb{N}, 1 \leq l \leq d^{m-1})$ and $Z_t^{n,k} = (Z_t^{n,k;m,l;p,q}; m, p \in \mathbb{N}, 1 \leq l \leq d^{m-1}, 1 \leq q \leq d^{p-1})$ for $n \in \mathbb{N}, 1 \leq k \leq d^{n-1}$ are defined as the solutions of BSDEs

$$dY_t^{n,k;m,l} = -\epsilon(\overline{X}_t^{n+1,k} - X_t^{n,k})(\overline{\delta}_{m,\ell}^{n+1,k} - \delta_{m,\ell}^{n,k})dt + \sum_{p=1}^{\infty}\sum_{q=1}^{d^{p-1}} Z_t^{n,k;m,l;p,q} dW_t^{p,q},$$

$$Y_T^{n,k;m,l} = \partial_{x^{m,l}} g_{n,k}(X_T) = c(\overline{X}_T^{n+1,k} - X_T^{n,k})(\overline{\delta}_{m,\ell}^{n+1,k} - \delta_{m,\ell}^{n,k}),$$

where $\delta_{m,\ell}^{n,k} := 1$, if $(n,k) = (m,\ell)$; 0, otherwise, and $\overline{\delta}_{m,\ell}^{n,k} := \frac{1}{d}\sum_{i=(k-1)d+1}^{kd} \delta_{m,\ell}^{n,i}$.

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Open-loop Nash equilibrium plus **ansatz**:

$$\hat{\alpha}_t^{n,k} = -Y_t^{n,k;n,k} = -\sum_{m=n}^{\infty} \phi_t^{m-n} \sum_{j=0}^{d^{m-n}-1} X_t^{m,d^{m-n}k-j}$$

Extensions: Directed Tree Networks

Riccati equations:

$$\dot{\phi}_t^k = \sum_{j=0}^{k} \phi_t^j \phi_t^{k-j} - \epsilon \left(\delta_{0,k} - \frac{1}{d} \cdot \delta_{1,k} \right), \quad \phi_T^k = c \left(\delta_{0,k} - \frac{1}{d} \cdot \delta_{1,k} \right).$$

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WLOG, we assume $\epsilon = 1$ and $\sigma = 1$. By taking $T \to \infty$, we look at the stationary long-time behavior of the Riccati system. Then the system gives the recurrence relation: $\phi^0 = 1$, $\phi^1 = -1/(2d)$ and $\sum_{j=0}^{k} \phi^j \phi^{k-j} = 0$ for $k \geq 0$. By using a moment generating function method, we obtain the stationary solution:

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As in the case $d = 1$, we can use a **Catalan Markov chain** to derive explicit formulas, in particular for the **asymptotic variance:**

$$\lim_{t\to\infty} \text{Var}(X_t^{1,1}) = \frac{\sqrt{2}}{2} \cdot \left(1 + \left(\frac{d-1}{d} \right)^{1/2} \right)^{-1/2} \in \left(\frac{1}{2}, \frac{\sqrt{2}}{2} \right].$$

Work in Progress

- Stochastic Games on **Stochastic Directed Networks**

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Work in Progress

- Stochastic Games on **Stochastic Directed Networks**
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- And more $\cdots$

The end

THANKS FOR YOUR ATTENTION

and

STAY HEALTHY!