Three examples at the boundary of mean field theory

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Virtual Seminar on Stochastic Analysis, Random Fields and Applications (continuation of Ascona)

Plan of the lecture

- 0: classical mean field
- 1: towards local interaction
- 2: *Example 1: individual based SIR*
- 3: *Example 2: cluster of cells*
- 4: environmental noise
- 5: scaling limit to independent noise
- 6: *Example 3: from cloud droplets to raindrops*

Classical mean field (partial exposition; e.g. Sznitman lect. notes)

$$dX_t^i = \frac{1}{N}\sum_{j=1}^{N} K\left(X_t^i - X_t^j\right) dt + \sigma dW_t^i \qquad i = 1, ..., N$$
$$X_0^i \text{ i.i.d. } \mu_0$$

$$\text{empirical measure: } \mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}}$$

K bounded Lipschitz: the family of laws of $\mu_\cdot^N$ is tight on $C\left([0, T]; M\left(\mathbb{R}^d\right)\right)$. Unique limit point μ_t, solving

$$\langle \mu_t, \phi \rangle = \langle \mu_0, \phi \rangle + \int_0^t \langle \mu_s, (K * \mu_s) \cdot \nabla\phi \rangle \, ds + \frac{\sigma^2}{2} \int_0^t \langle \mu_s, \Delta\phi \rangle \, ds.$$

Identity for the empirical measure

Key element for the sequel is the identity satisfied by the empirical measure:

$$d\left\langle \mu_t^N, \phi \right\rangle = \frac{1}{N}\sum_{i=1}^{N} d\phi\left(X_t^i\right) = \frac{1}{N}\sum_{i=1}^{N} \nabla\phi\left(X_t^i\right) dX_t^i + \frac{\sigma^2}{2}\Delta\phi\left(X_t^i\right) dt$$

$$\begin{aligned}\left\langle \mu_t^N, \phi \right\rangle &= \left\langle \mu_0^N, \phi \right\rangle + \int_0^t \left\langle \mu_s^N, \left(K * \mu_s^N\right) \cdot \nabla\phi \right\rangle ds \\ &\quad + \frac{\sigma^2}{2}\int_0^t \left\langle \mu_s^N, \Delta\phi \right\rangle ds + M_t^N\end{aligned}$$

where $\mathbb{E}\left[\left|M_t^N\right|^2\right] \sim \frac{1}{N}$ and

$$\left\langle \mu_s^N, \left(K * \mu_s^N\right) \cdot \nabla\phi \right\rangle \to \left\langle \mu_s, \left(K * \mu_s\right) \cdot \nabla\phi \right\rangle .$$

Local interactions

$$X_0^i \text{ i.i.d. } \mu_0 \Rightarrow X_t^{i,N} \text{ almost } \in \text{ compact set}$$

hence typical distance of nearest neighbor is

$$\left| X_t^{i,N} - X_t^{j,N} \right| \sim N^{-1/d}.$$

Can we consider only local interactions, namely

$$dX_t^i = \frac{1}{N} \sum_{j=1}^{N} N^{\gamma} K \left(N^{\beta} \left(X_t^i - X_t^j \right) \right) dt + \sigma dW_t^i$$

where K is compact support (for suitable β, γ)? Problem: tightness of $\mu_{\cdot}^N$ and

$$\lim_{N \to \infty} \left\langle \mu_s^N, \left(N^{\gamma} \int K \left(N^{\beta} (x - y) \right) \mu_s^N (dy) \right) \cdot \nabla \phi \right\rangle = ?$$

Local interactions

It is a very difficult but central problem. Different levels:

- local interaction in the motion of particles (as above)
- local interaction in mechanisms of change of species, proliferation and death.

Let us start from the second ones, perhaps easier.
Strictly speaking this is the realm of hydrodynamic limits instead of mean field ones but I will see them from the perspective of mean field theory.

Example 1: individual based SIR

S = Susceptible
I = Infected
R = Recovered.
Classical SIR model:

$$\begin{aligned}
\frac{dS}{dt} &= -\frac{IS}{\tau_{\inf}} \\
\frac{dI}{dt} &= \frac{IS}{\tau_{\inf}} - \frac{I}{\tau_{rec}} \\
\frac{dR}{dt} &= \frac{I}{\tau_{rec}}.
\end{aligned}$$

Several variants with spatial structure. Next view: particle systems (individual based model).

Example 1: individual based SIR

Individuals have a location $X^1, ..., X^N$ which doesn't change (their home). They can meet and infect each other. State $V_t^1, ..., V_t^N$

$$V_t^i \in \{S, I, R\}.$$

$$\text{Configuration: } \eta_t \in \{S, I, R\}^N.$$

Markov, finite state space, continuous time, infinitesimal generator

$$\mathcal{L}F(\eta) = \mathcal{L}_{S\to I}F(\eta) + \mathcal{L}_{I\to R}F(\eta)$$

$$\mathcal{L}_{S\to I}F(\eta) = \sum_{i:V^i=S}\sum_{j:V^j=I}\frac{1}{N}\lambda_N\left(X^i, X^j\right)\left[F\left(\eta^{V^i\to I}\right) - F(\eta)\right]$$

Example 1: individual based SIR

$$\text{emp. meas.: } \mu_t^{V,N} = \frac{1}{N}\sum_{i=1}^{N}\delta\left(V_t^i = V\right)\delta_{X^i}, \qquad V = S, I, R$$

$$d\left\langle \mu_t^{V,N}, \phi \right\rangle = \mathcal{L}\left(\frac{1}{N}\sum_{i=1, V_t^i = V}\phi\left(X^i\right)\right) dt + dM_t^N$$

$$\begin{aligned}\left\langle \mu_t^{S,N}, \phi \right\rangle &= \left\langle \mu_0^{S,N}, \phi \right\rangle \\ &\quad - \int_0^t \int\int \phi(x)\,\lambda_N(x,y)\,\mu_s^{I,N}(dy)\,\mu_s^{S,N}(dx)\,ds + M_t^N\end{aligned}$$

(only zero order terms) (equation for $\left\langle \mu_t^{I,N}, \phi \right\rangle$ is similar, for $\left\langle \mu_t^{R,N}, \phi \right\rangle$ is easier).

Example 1: individual based SIR, mean field regime

$$\lambda_N(x,y) = \frac{K(x,y)}{\tau_{\text{inf}}}$$

$$\begin{aligned}\left\langle \mu_t^{S,N}, \phi \right\rangle &= \left\langle \mu_0^{S,N}, \phi \right\rangle \\ &\quad - \int_0^t \int \int \phi(x) \frac{K(x,y)}{\tau_{\text{inf}}} \mu_s^{I,N}(dy)\, \mu_s^{S,N}(dx)\, ds + M_t^N\end{aligned}$$

The family of laws of $\left(\mu_\cdot^{S,N}, \mu_\cdot^{I,N}, \mu_\cdot^{R,N}\right)$ is tight on $D\left([0,T]; M\left(\mathbb{R}^d\right)\right)^3$. Unique limit point $\left(\mu_\cdot^S, \mu_\cdot^I, \mu_\cdot^R\right)$, solving

$$\begin{aligned}\left\langle \mu_t^{S}, \phi \right\rangle &= \left\langle \mu_0^{S}, \phi \right\rangle + \int_0^t \int \int \phi(x) \frac{K(x,y)}{\tau_{\text{inf}}} \mu_s^{I}(dy)\, \mu_s^{S}(dx)\, ds \\ & \quad etc.\end{aligned}$$

Example 1: individual based SIR, mean field regime

Differential form, for densities

$$\mu_t^S(dx) = f_S(t,x)\,dx$$
$$etc.$$

:

$$\frac{\partial f_S(t,x)}{\partial t} = -f_S(t,x)\int \frac{K(x,y)}{\tau_{\text{inf}}} f_I(t,y)\,dy$$
$$\frac{\partial f_I(t,x)}{\partial t} = f_S(t,x)\int \frac{K(x,y)}{\tau_{\text{inf}}} f_I(t,y)\,dy - \frac{1}{\tau_{\text{rec}}} f_I(t,x)$$
$$\frac{\partial f_R(t,x)}{\partial t} = \frac{1}{\tau_{\text{rec}}} f_I(t,x)$$

Details in forthcoming paper with Francesco Grotto, Andrea Papini and Cristiano Ricci. But our main motivation is the following local case.

Example 1: individual based SIR, local interaction

$$\lambda_N(x,y) = \frac{1}{\tau_{\text{inf}}} N\lambda\left(N^{1/d}(x-y)\right)$$

$$\left\langle \mu_t^{S,N}, \phi \right\rangle = \left\langle \mu_0^{S,N}, \phi \right\rangle$$

$$-\int_0^t \int \int \phi(x) \frac{1}{\tau_{\text{inf}}} N\lambda\left(N^{1/d}(x-y)\right) \mu_s^{I,N}(dy)\, \mu_s^{S,N}(dx)\, ds + M_t^N$$

Intuition: if the limit measures have densities,

$$\left\langle \mu_t^{S}, \phi \right\rangle = \left\langle \mu_0^{S}, \phi \right\rangle + \int_0^t \int \phi(x) \frac{1}{\tau_{\text{inf}}} \mu_s^I(x)\, \mu_s^S(x)\, dxds$$

$$\frac{\partial f_S(t,x)}{\partial t} = -\frac{1}{\tau_{\text{inf}}} f_S(t,x)\, f_I(t,x)$$

This is our result (F.-Grotto-Papini-Ricci) under suitable assumptions.

Other local zero-order interactions

The previous example is paradigmatic of local zero-order interactions. Other examples (among many others):

- rate of proliferation (e.g. cells) depending on the state of nearest particles (-> FKKP equations; see for instance F. - Leimbach - Olivera '19)
- change of species as above (for general models, see several works of Karl Oelschläger)
- Smoluchowski coagulation equation (Großkinsky-Klingenberg-Oelschläger '04, Hammond-Rezakhanlou '07 and many others)
- neuron activity (Delarue-Inglis-Rubenthaler-Tanré '15, F.-Priola-Zanco '19)

Scheme of a general approach

$$d\mu_t^{S,N} = -\left(\lambda_N * \mu_t^{I,N}\right)\mu_t^{S,N} dt + dM_t^N$$

where λ_N are classical mollifiers. The intuition is that

$$\begin{array}{rcl} \mu_t^{S,N}(dx) & \rightharpoonup & f_S(t,x)\,dx \\ \mu_t^{I,N}(dx) & \rightharpoonup & f_I(t,x)\,dx \end{array}$$

$$\partial_t f_S(t,x) = -f_I(t,x)\, f_S(t,x)\,.$$

Assume we prove tightness of $\left(\mu_\cdot^{S,N}, \mu_\cdot^{I,N}\right)$ and existence of densities for the limit measures. How to prove that

$$\left\langle \left(\lambda_N * \mu_t^{I,N}\right)\mu_t^{S,N}, \phi \right\rangle \to \langle f_I(t)\, f_S(t), \phi\rangle \qquad ?$$

Auxiliary mollifiers

Introduce auxiliary mollifiers θ_δ (smooth, symmetric, compact support) converging to delta Dirac as $\delta \to 0$. rewrite

$$\left(\lambda_N * \mu_t^{I,N}\right) \mu_t^{S,N} = \left(\lambda_N * \theta_\delta * \mu_t^{I,N}\right) \mu_t^{S,N} + R_t^{N,\delta}$$

so that the original equation

$$d\mu_t^{S,N} = -\left(\lambda_N * \mu_t^{I,N}\right) \mu_t^{S,N} dt + dM_t^N$$

becomes

$$d\mu_t^{S,N} = -\left(\lambda_N * \theta_\delta * \mu_t^{I,N}\right) \mu_t^{S,N} dt + R_t^{N,\delta} dt + dM_t^N.$$

Now we can take the limit as $N \to \infty$.

Lemma in the spirit of defect measures

From

$$d\mu_t^{S,N} = -\left(\lambda_N * \theta_\delta * \mu_t^{I,N}\right)\mu_t^{S,N} dt + R_t^{N,\delta} dt + dM_t^N.$$

under suitable assumptions, which include tightness of $\left(\mu_\cdot^{S,N}, \mu_\cdot^{I,N}\right)$ and existence of L^2 densities for the limit measures, we deduce:

Lemma

The limit exists

$$\lim_{\delta\to 0}\lim_{N\to\infty}\left\langle R_t^{N,\delta}, \phi\right\rangle$$

and

$$\left\langle \mu_t^S, \phi\right\rangle = \left\langle \mu_0^S, \phi\right\rangle + \int_0^t \int \phi(x)\,\mu_s^I(dx)\,\mu_s^S(dx)\,ds + \lim_{\delta\to 0}\lim_{N\to\infty}\left\langle R_t^{N,\delta}, \phi\right\rangle.$$

Analysis of the remainder

How to prove that $R_t^{N,\delta}$, defined as

$$R_t^{N,\delta} = \left(\lambda_N * \mu_t^{I,N}\right) \mu_t^{S,N} - \left(\lambda_N * \theta_\delta * \mu_t^{I,N}\right) \mu_t^{S,N}$$

goes to zero in distribution?
Two interpretations:

- local interaction with infected individuals $\lambda_N * \mu_t^{I,N}$ is similar to interaction with a macroscopic average of them: $\lambda_N * \theta_\delta * \mu_t^{I,N}$ (local equilibrium)
- the family of random fields (*empirical densities*)

$$\rho_t^{I,N}(x) := \left(\lambda_N * \mu_t^{I,N}\right)(x)$$

has suitable continuity properties in x, uniform in N.

Other interpretations

Recall

$$R_t^{N,\delta} = \left(\lambda_N * \mu_t^{I,N}\right)\mu_t^{S,N} - \left(\lambda_N * \theta_\delta * \mu_t^{I,N}\right)\mu_t^{S,N}$$

- *Commutator estimates*: the problem is similar to the smallness of

$$\theta_\delta * (b \cdot u_N) - b \cdot (\theta_\delta * u_N)$$

- *Closure problem*: similar also to the smallness of

$$\langle u_N v_N \rangle - \langle u_N \rangle \cdot \langle v_N \rangle$$

(this analogy is important in view of granular corrections)

Oelschläger solution

Without boring with other details, let us say that in F.-Grotto-Papini-Ricci we have solved the problem by following ideas of Karl Oelschläger:

- write an identity for the empirical density $\rho_t^{I,N}(x) := \left(\lambda_N * \mu_t^{I,N}\right)(x)$
- use PDE ideas to prove its regularity (in particular its hölderianity uniform in N).

The price is rescaling as

$$N^{\beta}\lambda\left(N^{\beta/d}(x-y)\right) \qquad \text{for some } \beta < \beta_0 \leq 1$$

instead of $N\lambda\left(N^{1/d}(x-y)\right)$. It means interaction with infinitely many individuals, still infinitesimal with respect to the total (intermediate between mean field and nearest neighbor).

Relevant literature for Examples 1 and 2

- Oelschläger, K., A law of large numbers for moderately interacting diffusion processes *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*(1985) 279–322.
- Oelschläger, K., Large systems of interacting particles and the porous medium equation. *Journal of differential equations*(1990) 294-346.
- Varadhan, SRS, Scaling limits for interacting diffusions *Communications in mathematical physics* (1991), 313–353.
- Uchiyama, K. , Pressure in classical statistical mechanics and interacting brownian particles in multi-dimensions *Annales Henri Poincaré.* (2000), 98-113.

Example 2: Brownian particles with local interaction

Even more difficult is the case of local interaction in the dynamics. In order to appreciate the difficulty it is necessary to start from particles at *unitary distance* in $\mathbb{R}^d$:

$$dY_t^i = -\sum_{j=1}^{N} \nabla V\left(Y_t^i - Y_t^j\right) dt + \sigma dB_t^i$$

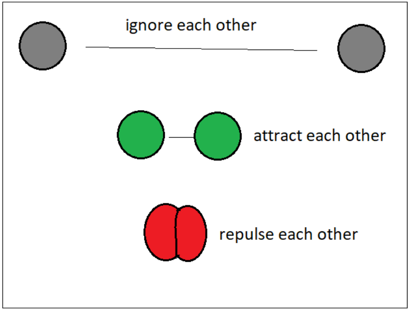
with compact support potential $V : \mathbb{R}^d \to \mathbb{R}$. Assume Y_0^i i.i.d. in or near $\left[0, N^{1/d}\right]^d$.

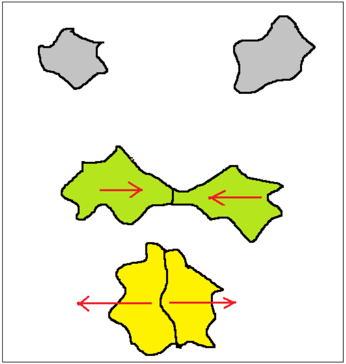
Example 2: cluster of cells

Our interest (F.-Leocata-Ricci '19) started looking at biological cell-dynamics

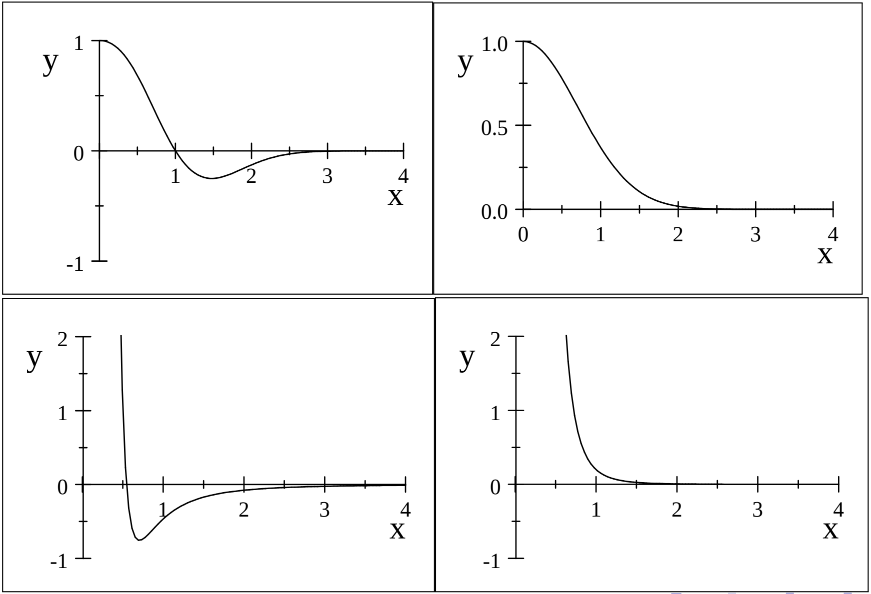
repulsion: volume constraint

attraction: cell adhesion.





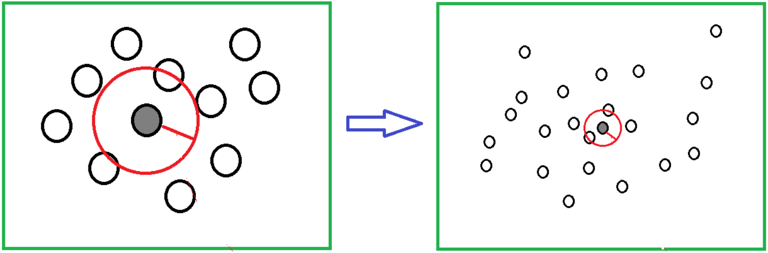
Examples of interaction potentials



Rescaling local interaction

$$X_t^i := \frac{1}{N^{1/d}} Y^i_{t \cdot N^{2/d}} \qquad W_t^i := \frac{1}{N^{1/d}} B^i_{t \cdot N^{2/d}}$$

$$dX_t^i = -N^{1/d} \sum_{j=1}^{N} \nabla V \left(N^{1/d} \left(X_t^i - X_t^j \right) \right) dt + \sigma dW_t^i$$



Brownian particles with local interaction

Let us see in more detail the rescaling of the potential:

$$dX_t^i = -\frac{1}{N}\sum_{j=1}^{N} N^{1/d} N \left(\nabla V\right)\left(N^{1/d}\left(X_t^i - X_t^j\right)\right) dt + \sigma dW_t^i$$

$$\begin{aligned} V_N(x) &:= NV\left(N^{1/d}x\right) \qquad \text{(classical mollifiers)} \\ \nabla V_N(x) &= N^{1/d} N \left(\nabla V\right)\left(N^{1/d}x\right) \end{aligned}$$

$$dX_t^i = -\frac{1}{N}\sum_{j=1}^{N} \nabla V_N\left(X_t^i - X_t^j\right) dt + \sigma dW_t^i.$$

Gradients of classical mollifiers appear!

Brownian particles with local interaction

Summarizing, we study a system in a "bounded" region, with interaction

$$dX_t^i = -\frac{1}{N}\sum_{j=1}^{N} \nabla V_N\left(X_t^i - X_t^j\right) dt + \sigma dW_t^i$$

$$V_N(x) := NV\left(N^{1/d}x\right) \qquad \text{(classical mollifiers).}$$

Heuristically the mean field equation ($V_N(x) = V_{N_0}(x)$) would be

$$\langle \mu_t, \phi \rangle = \langle \mu_0, \phi \rangle + \int_0^t \langle \mu_s, \nabla (V_{N_0} * \mu_s) \cdot \nabla \phi \rangle \, ds + \frac{\sigma^2}{2} \int_0^t \langle \mu_s, \Delta \phi \rangle \, ds.$$

If $\mu_s(dx) = f_s(x)\,dx$ and

$$(V_{N_0} * f_s)(x) \underset{N_0 \to \infty}{\longrightarrow} C_V f_s(x) \qquad C_V = \int V(x)\,dx$$

we get

$$\langle f_t, \phi \rangle = \langle f_0, \phi \rangle + C_V \int_0^t \langle f_s, \nabla f_s \cdot \nabla \phi \rangle \, ds + \frac{\sigma^2}{2} \int_0^t \langle f_s, \Delta \phi \rangle \, ds$$

Brownian particles with local interaction

$$\begin{aligned}\partial_t f &= \frac{\sigma^2}{2}\Delta f + C_V \operatorname{div}(f \nabla f) \\ &= \frac{\sigma^2}{2}\Delta f + \frac{C_V}{2}\Delta f^2.\end{aligned}$$

- Rigorously proved by Oelschläger for repulsive potentials when

$$V_N(x) = N^\beta V\left(N^{\beta/d}x\right) \qquad \text{for some } \beta < \beta_0 \leq 1$$

  instead of $V_N(x) := NV\left(N^{1/d}x\right)$
- For $V_N(x) := NV\left(N^{1/d}x\right)$, $d = 1$, repulsive potential, Varadhan '91 obtains

$$\partial_t f = \frac{\sigma^2}{2}\Delta f + \Delta Q_V(f)$$

  but the form of $Q_V(f_t)$ is unknown.

Brownian particles with local interaction

$$\partial_t f = \frac{\sigma^2}{2} \Delta f + \Delta Q_V (f)$$

- For $V_N(x) := NV(N^{1/d}x)$, $d = 1$, repulsive potential: Varadhan '91.
- Extended by Uchyama '00 to $d \geq 1$ and some attracting-repulsive potentials, still $Q_V(f_t)$ unknown.
- Remark: for many lattice systems like the zero-range process, $Q_V(f)$ is computable.

Conjectures on the nearest neighbour case

In F.-Leocata-Ricci 2019 we have discussed numerically and heuristically the shape of $Q_V(f)$ in the limit equation

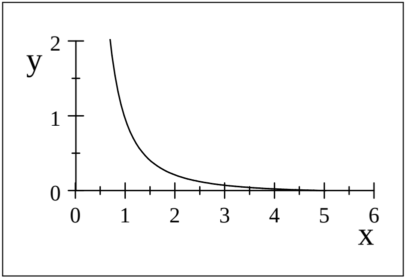
$$\partial_t f = \frac{\sigma^2}{2}\Delta f + \Delta Q_V(f)$$

- $C_V = \int V(x)\, dx \in (0,\infty)$: our investigation confirms

  $$\partial_t f = \frac{\sigma^2}{2}\Delta f + \frac{C_V}{2}\Delta f^2$$

  also in repulsive+attracting cases. Uchiyama '00 has a rigorous result of type $Q_V(f) \sim \frac{C_V}{2} f^2$ for $f \to \infty$.
- $C_V < 0$: strongly unclear.
- $C_V = +\infty$: next two slides.

Repulsive not integrable: our conjecture

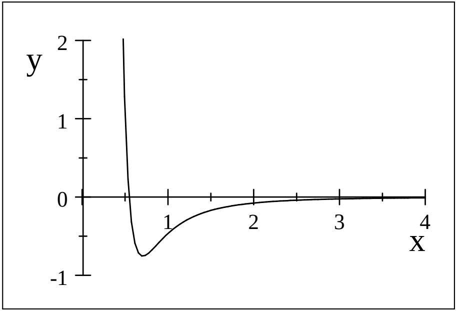


$$V(x) = \frac{1}{\alpha |x|^{\alpha}} \qquad \text{with a cut-off at } R_1$$

for $\alpha > 1$. Conjecture in $d = 1$:

$$Q_V(f) \approx \begin{cases} 0, & f < 1/R_1 \\ \frac{\alpha}{2(\alpha-1)} f^{1+\alpha} - \frac{1}{2(\alpha-1)R_1^{\alpha-1}} f^2, & f \geq 1/R_1 \end{cases}$$

Attractive-repulsive, not integrable: our conjecture



$$V(x) = \frac{R_0^\alpha}{\alpha |x|^\alpha} - \frac{R_0^\beta}{\beta |x|^\beta} \qquad \text{with a cut-off at } R_1 \text{ (min in } R_0\text{)}$$

with $\alpha > \beta$. Conjecture in $d = 1$: large f: $Q_V(f) \approx$

$$\frac{\alpha}{2(\alpha-1)} f^{1+\alpha} - \frac{\beta}{2(\beta-1)} f^{1+\beta} - \left(\frac{1}{2(\alpha-1) R_1^{\alpha-1}} - \frac{1}{2(\beta-1) R_1^{\beta-1}} \right) f^2.$$

Final subject: Environmental noise

First: classical mean field. Clearly if the system is

$$dX_t^i = b_t\left(X_t^i\right)dt + \frac{1}{N}\sum_{j=1}^{N} K\left(X_t^i - X_t^j\right)dt + \sigma dW_t^i$$

the limit equation is

$$\begin{aligned} \langle \mu_t, \phi \rangle &= \langle \mu_0, \phi \rangle + \int_0^t \langle \mu_s, (K * \mu_s) \cdot \nabla \phi \rangle \, ds \\ &\quad + \int_0^t \langle \mu_s, b_s \cdot \nabla \phi \rangle \, ds + \frac{\sigma^2}{2} \int_0^t \langle \mu_s, \Delta \phi \rangle \, ds. \end{aligned}$$

Final subject: Environmental noise

Therefore it is not surprising that in the case when $b_t(x)$ is a space-dependent white-noise-in-time

$$b_t(x)\,dt = \sum_k \sigma_k(x) \circ d\beta_t^k$$

the limit equation is the SPDE ($\sigma = 0$ for simplicity)

$$\begin{aligned} \langle \mu_t, \phi \rangle &= \langle \mu_0, \phi \rangle + \int_0^t \langle \mu_s, (K * \mu_s) \cdot \nabla \phi \rangle \, ds \\ &\quad + \sum_k \int_0^t \langle \mu_s, \sigma_k \cdot \nabla \phi \rangle \circ d\beta_s^k. \end{aligned}$$

Literature includes: Kurtz-Xiong '99, '01, Dawson-Vaillancourt 95, Coghi-F. '16, Coghi-Gess '19, Carmona-Delarue-Lacker '16, Fouque-Hu '17, Carmona-Delarue book '18, Coghi-Maurelli '19, Choi-Salem '19, Marx '20 and many others.

Environmental noise, almost uncorrelated in space

Assume the noise is invariant by space-translations, with covariance depending on a parameter ϵ:

$$Q^{\epsilon}(x-y)=\sum_{k}\sigma_{k}^{\epsilon}(x)\otimes\sigma_{k}^{\epsilon}(y).$$

If

$$Q^{\epsilon}(x)\rightarrow I\cdot\delta_{0}(dx)$$

the intuition is that particles are subject to almost independent noises and in the limit the behaviour is similar to the case of independent additive noise.

Small correlation scaling limit

In other words we conjecture that, under suitable link $N \to \epsilon_N$ and suitable assumptions on the noise coefficients, the empirical measure of the particle system

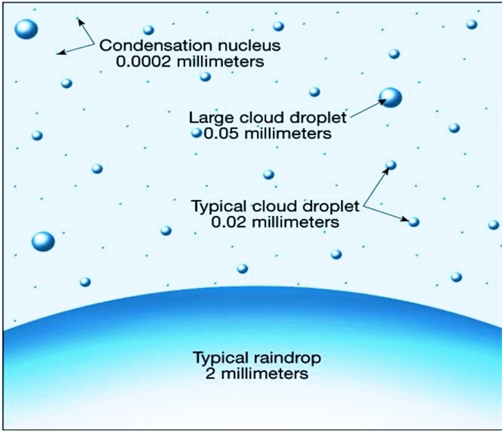
$$dX_t^i = \frac{1}{N}\sum_{j=1}^{N} K\left(X_t^i - X_t^j\right) dt + \sum_k \sigma_k^{\epsilon_N}\left(X_t^i\right) \circ d\beta_t^k$$

converges to the deterministic parabolic equation

$$\begin{aligned}\langle \mu_t, \phi \rangle &= \langle \mu_0, \phi \rangle + \int_0^t \langle \mu_s, (K * \mu_s) \cdot \nabla \phi \rangle \, ds \\ &\quad + C \int_0^t \langle \mu_s, \Delta \phi \rangle \, ds.\end{aligned}$$

Similar results on related models by Galeati '19, F.-Luo '19, F.-Luo-Galeati '20.

Example 3: rain drop formation



(from Hector Marsh)

Example 3: rain drop formation

This topic mixes several ideas illustrated above:

- particles are cloud droplets (0.02-0.5 mm)
- different species means different size
- transition of size has a rate depending on relative position (collision)
- thus interaction of zero order terms, local in space
- environmental noise: turbulent displacement instead of free fall by gravitation

Effect of turbulence on rain drop formation

The limit model seems to be a Smoluchowski equation of the form

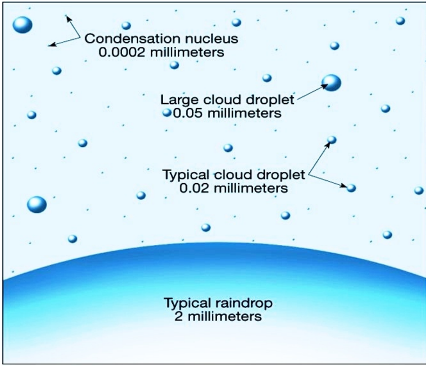
$$\partial_t f_m (t, x) = Q_+ - Q_- + \Delta f_m (t, x)$$

where

$$\begin{aligned} Q_+ &= \frac{1}{2} \int_0^m K \left(m', m - m'\right) f_{m'} (t, x) f_{m-m'} (t, x) \, dm' dt \\ Q_- &= f_m (t, x) \int_0^\infty K \left(m, m'\right) f_{m'} (t, x) \, dm' dt. \end{aligned}$$

The key aspect is that a Brownian displacement of particles should take place due to environmental noise with small correlation. Following Hammond-Rezakhanlou '07, this modifies the interaction kernel.
The exact modification of the kernel from free fall to turbulence is a major problem in cloud droplets dynamics.

Thank you!


Condensation nucleus
0.0002 millimeters
Large cloud droplet
0.05 millimeters
Typical cloud droplet
0.02 millimeters
Typical raindrop
2 millimeters