Nonlinear PDE models with stochastic fractional perturbation

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Outline

1 Introduction

2 Step 1: identify the (future) regularity of $u$

3 Step 2: solve the equation in the regular case $\alpha_{d,H} > 0$

4 Step 3: the rough case $\alpha_{d,H} \leq 0$
We consider the following general nonlinear SPDE model:

\[ \mathcal{L} u = u^2 + \dot{B}, \quad t \in [0, T], \quad x \in \mathbb{R}^d, \]

where:

- \( \mathcal{L} \) can be either
  - \((h)\) the heat operator: \( \mathcal{L}^{(h)} u = \partial_t u - \Delta u \quad (u_0 = \phi) \)
  - \((w)\) the wave operator: \( \mathcal{L}^{(w)} u = \partial_t^2 u - \Delta u \quad (u_0 = \phi_1, (\partial_t u)_0 = \phi_2) \)
  - \((s)\) the Schrödinger operator: \( \mathcal{L}^{(s)} u = i\partial_t u - \Delta u \quad (u_0 = \phi) \)

- \( \dot{B} \) is a space-time fractional noise.
**Space-time fractional noise**

**Definition.** We call a space-time fractional Brownian motion of Hurst index $H = (H_0, H_1, \ldots, H_d) \in (0, 1)^{d+1}$ any centered Gaussian process $B : \Omega \times ([0, T] \times \mathbb{R}^d) \to \mathbb{R}$ with covariance given by

$$
\mathbb{E} [B_s(x)B_t(y)] = R_{H_0}(s, t) \prod_{i=1}^{d} R_{H_i}(x_i, y_i),
$$

where $R_H(a, b) := \frac{1}{2} (|a|^{2H} + |b|^{2H} - |a-b|^{2H})$.

**Definition.** We call a **space-time fractional noise** of Hurst index $H \in (0, 1)^{d+1}$ the derivative (in the sense of the distributions)

$$
\dot{B} := \partial_t \partial_{x_1} \cdots \partial_{x_d} B.
$$

**Remark.** $H_0 = H_1 = \ldots = H_d = \frac{1}{2} \implies$ classical space-time white noise.
Possible motivations

\[ \mathcal{L}u = u^2 + \dot{B}, \quad t \in [0, T], \ x \in \mathbb{R}^d. \]  

(1)

Eq. (1) is the most basic **stochastic perturbation of the classical PDE**

\[ \mathcal{L}u = u^2 \]

\[ \rightarrow \text{“Nonlinear PDE with power nonlinearity”} \]

Eq. (1) is the most basic **nonlinear extension of the stochastic PDE**

\[ \mathcal{L}u = \dot{B} \]

whose solution is explicitly given by

\[ u = G \ast \dot{B} \]

where \( G \) is the Green kernel associated with \( \mathcal{L} \).
Why a fractional noise?

\[ \mathcal{L} u = u^2 + \dot{B}, \quad t \in [0, T], \quad x \in \mathbb{R}^d. \]

- More general than a white noise + no need for “martingale” property
- Study the transition

\[ H_i \approx 1 \]
Regular (classical) PDE

\[ \overset{\text{Continuous transition}}{\longrightarrow} \]

\[ H_i = \frac{1}{2} \]
White noise
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Step 1: identify the (future) regularity of \( u \)

The starting point of our analysis is the mild form of the equation:

\[
\begin{align*}
  u &= G_t \ast \phi + G \ast u^2 + \, \bullet ,
\end{align*}
\]

where \( \phi \) is a regular initial condition, \( G \) is the Green kernel associated with \( \mathcal{L} \), and the symbol \( \, \bullet \) refers to "linear solution"

\[
\, \bullet := G \ast \dot{B}.
\]

\[\Rightarrow\] The regularity of \( u \) is expected to be the same as the one of \( \, \bullet \).

\[\Rightarrow\] Step 1: Identify the regularity of \( \, \bullet \).

Due to the roughness of the noise \( \dot{B} \), the exact definition and regularity of

\[
\, \bullet := G \ast \dot{B}
\]

are not exactly standard issues...
Step 1: identify the (future) regularity of $u$

The starting point of our analysis is the mild form of the equation:

$$u = G_t \ast \phi + G \ast u^2 + \circ,$$

where $\phi$ is a regular initial condition, $G$ is the Green kernel associated with $\mathcal{L}$, and the symbol $\circ$ refers to "linear solution"

$$\circ := G \ast \dot{B}.$$

$\implies$ The regularity of $u$ is expected to be the same as the one of $\circ$.

$\implies$ **Step 1:** Identify the regularity of $\circ$.

Due to the roughness of the noise $\dot{B}$, the exact definition and regularity of $\circ := G \ast \dot{B}$ are not exactly standard issues...
Step 1: identify the (future) regularity of $u$

The starting point of our analysis is the mild form of the equation:

$$u = G_t \ast \phi + G \ast u^2 + \dot{B},$$

where $\phi$ is a regular initial condition, $G$ is the Green kernel associated with $L$, and the symbol $\dot{B}$ refers to “linear solution”:

$$\dot{B} := G \ast \dot{B}.$$

$\Longrightarrow$ The regularity of $u$ is expected to be the same as the one of $\dot{B}$.

$\Longrightarrow$ **Step 1:** Identify the regularity of $\dot{B}$.

Due to the roughness of the noise $\dot{B}$, the exact definition and regularity of

$$\dot{B} := G \ast \dot{B}$$

are not exactly standard issues...
Study of $G * \dot{B} \rightarrow$ based on an approximation procedure:
Start from a $C^\infty$ approximation $B^n$ of $B$, and study the convergence of

$$n := G * \dot{B}^n, \quad \text{where } \dot{B}^n := \partial_t \partial_{x_1} \cdots \partial_{x_d} B^n.$$

The approximation $B^n$ can for instance be given by:

(i) A mollifying sequence, that is, for $\rho$ test-function with $\int_{R^{d+1}} \rho = 1$,

$$B^n := \rho_n * B \ , \quad \text{where } \rho_n(s, x) = 2^{(d+1)n} \rho(2^n s, 2^n x).$$

(ii) A discrete approximation along a grid: if $(t, x) \in [i/n, (i+1)/n] \times [j/n, (j+1)/n]$,

$$\dot{B}_{t,x}^n := B_{i+1/n, j+1/n} - B_{i/n, j+1/n} - B_{i+1/n, j/n} + B_{i/n, j/n}.$$
Step 1: identify the (future) regularity of $u$

Study of $\hat{\circ} := G \ast \dot{B} \to$ based on an approximation procedure:
Start from a $C^\infty$ approximation $B^n$ of $B$, and study the convergence of

$\hat{\circ}^n := G \ast \dot{B}^n$, where $\dot{B}^n := \partial_t \partial_{x_1} \cdots \partial_{x_d} B^n$.

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(i) A **mollifying sequence**, that is, for $\rho$ test-function with $\int_{\mathbb{R}^{d+1}} \rho = 1$,

$$B^n := \rho_n \ast B,$$

where $\rho_n(s,x) = 2^{(d+1)n} \rho(2^n s, 2^n x)$.

(ii) A **discrete approximation** along a grid: if $(t,x) \in \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$,

$$\dot{B}^n_{t,x} := B_{\frac{i+1}{n}, \frac{j+1}{n}} - B_{\frac{i}{n}, \frac{j+1}{n}} - B_{\frac{i+1}{n}, \frac{j}{n}} + B_{\frac{i}{n}, \frac{j}{n}}.$$
Step 1: identify the (future) regularity of $u$

Study of $\circ := G \ast \dot{B} \rightarrow$ based on an approximation procedure:
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(ii) A discrete approximation along a grid: if $(t, x) \in \left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$,

$$\dot{B}^n_{t, x} := B_{\frac{i+1}{n}, \frac{i+1}{n}} - B_{\frac{i}{n}, \frac{i+1}{n}} - B_{\frac{i+1}{n}, \frac{j}{n}} + B_{\frac{i}{n}, \frac{j}{n}}.$$
**Step 1: identify the (future) regularity of $u$**

**Proposition.** Let $H = (H_0, H_1, \ldots, H_d) \in (0, 1)^{d+1}$. Then, for all $T > 0$ and $\alpha < \alpha_{d,H}$, the sequence $(\mathbf{B}^n = G \ast \dot{B}^n)_{n \geq 1}$ converges (almost surely) to some limit in the space

$$C([0, T]; \mathcal{W}^\alpha(\mathbb{R}^d)),$$

with $\alpha_{d,H} \in \mathbb{R}$ defined as

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where $H_+ := \sum_{i=1}^{d} H_i$.

The limit is the same for the two approximations (i)-(ii) of $B^n$. 

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4. Step 3: the rough case $\alpha_{d,H} \leq 0$
The regular case

Assume that $\alpha_{d,H} > 0$, so $\mathcal{W} \in \mathcal{C}([0, T]; \mathcal{W}^\alpha)$ for every $0 < \alpha < \alpha_{d,H}$.

$$u = G_t * \phi + G * u^2 + \mathbb{q}, \quad t \in [0, T], \ x \in \mathbb{R}^d.$$  

We can interpret $u^2$ as a standard product of functions

**Theorem.** Assume that $\alpha_{d,H} > 0$.

Then, almost surely, Equation (2) admits a unique solution in $\mathcal{C}([0, T]; \mathcal{W}^\alpha)$, for $0 < \alpha < \alpha_{d,H}$ and $T > 0$ small enough, provided $d$ satisfies

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The regular case

Assume that $\alpha_{d,H} > 0$, so $\mathbb{P} \in \mathcal{C}([0, T]; \mathcal{W}^\alpha)$ for every $0 < \alpha < \alpha_{d,H}$.

$$u = G_t \ast \phi + G \ast u^2 + \mathbb{P}, \quad t \in [0, T], \ x \in \mathbb{R}^d.$$ (2)

$\implies$ We can interpret $u^2$ as a standard product of functions.

**Theorem.** Assume that $\alpha_{d,H} > 0$.

Then, almost surely, Equation (2) admits a unique solution in $\mathcal{C}([0, T]; \mathcal{W}^\alpha)$, for $0 < \alpha < \alpha_{d,H}$ and $T > 0$ small enough, provided $d$ satisfies

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Analysis of the rough case

If $\alpha_{d,H} \leq 0$, then must be treated as a distribution of negative order:

$$
\mathbb{W} \in C([0, T]; \mathcal{W}^\alpha), \quad \text{for } \alpha < \alpha_{d,H} \leq 0.
$$

**Problem:** how to interpret the non-linearity $u^2$ in the equation

$$
u = G_t \ast \phi + G \ast u^2 + \mathbb{W}, \quad t \in [0, T], \ x \in \mathbb{R}^d.
$$

**Da Prato-Debussche trick:** consider the equation satisfied by the process $v := u - \mathbb{W}$, namely

$$
v = G_t \ast \phi + G \ast v^2 + 2G \ast (v \cdot \mathbb{W}) + G \ast (\mathbb{W})^2.
$$

**Strategy:**

1. Use renormalization and stochastic arguments to interpret $(\mathbb{W})^2$
2. Use the properties of $G$ to control $G \ast (v \cdot \mathbb{W})$ and $G \ast (\mathbb{W})^2$ as functions
3. Solve Equation (3) in a suitable space of functions.
Analysis of the rough case

If $\alpha_{d,H} \leq 0$, then $\mathcal{N}$ must be treated as a distribution of negative order:

$$\mathcal{N} \in C([0, T]; \mathcal{W}^\alpha), \quad \text{for } \alpha < \alpha_{d,H} \leq 0.$$  

**Problem:** how to interpret the non-linearity $u^2$ in the equation

$$u = G_t \ast \phi + G \ast u^2 + \mathcal{N}, \quad t \in [0, T], \ x \in \mathbb{R}^d.$$  

**Da Prato-Debussche trick:** consider the equation satisfied by the process $v := u - \mathcal{N}$, namely

$$v = G_t \ast \phi + G \ast v^2 + 2G \ast (v \cdot \mathcal{N}) + G \ast (\mathcal{N}^2).$$  \hspace{1cm} (3)  

**Strategy:**

(1) Use renormalization and stochastic arguments to interpret $(\mathcal{N}^2)$

(2) Use the properties of $G$ to control $G \ast (v \cdot \mathcal{N})$ and $G \ast (\mathcal{N}^2)$ as functions

(3) Solve Equation (3) in a suitable space of functions.
Analysis of the rough case

If $\alpha_{d,H} \leq 0$, then $\mathbb{H}$ must be treated as a distribution of negative order:

$$\mathbb{H} \in \mathcal{C}([0, T]; \mathcal{W}^\alpha), \quad \text{for } \alpha < \alpha_{d,H} \leq 0.$$

**Problem:** how to interpret the non-linearity $u^2$ in the equation

$$u = G_t \ast \phi + G \ast u^2 + \mathbb{H}, \quad t \in [0, T], \ x \in \mathbb{R}^d.$$

**Da Prato-Debussche trick:** consider the equation satisfied by the process $v := u - \mathbb{H}$, namely

$$v = G_t \ast \phi + G \ast v^2 + 2G \ast (v \cdot \mathbb{H}) + G \ast (\mathbb{H})^2. \quad (3)$$

**Strategy:**

1. Use renormalization and stochastic arguments to interpret $(\mathbb{H})^2$
2. Use the properties of $G$ to control $G \ast (v \cdot \mathbb{H})$ and $G \ast (\mathbb{H})^2$ as functions
3. Solve Equation (3) in a suitable space of functions.
Interpretation of $\left(\bullet\right)^2$

**Wick renormalization:** consider the approximation $\hat{G}^n := G \ast \hat{B}^n$ and set

$$\hat{n}^n(t, x) := \left(\hat{G}^n(t, x)\right)^2 - \sigma^n(t, x),$$

with $\sigma^n(t, x) := \mathbb{E}\left[\left(\hat{G}^n(t, x)\right)^2\right]$.

**Proposition.** Recall that $\hat{G} \in C([0, T]; \mathcal{W}_\alpha)$ for every $\alpha < \alpha_{d, H} \leq 0$. Then, for all $d \geq 1$, $T > 0$ and $\alpha < \alpha_{d, H} \leq 0$, the sequence $\left(\hat{n}^n\right)_{n \geq 1}$ converges (almost surely) to some limit $\hat{n}$ in the space $C([0, T]; \mathcal{W}^{2\alpha})$, provided $\alpha_{d, H}$ satisfy

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Wick renormalization: consider the approximation $\hat{\mathcal{G}}^n := G \ast \hat{\mathcal{B}}^n$ and set

$$\hat{\mathcal{G}}^n(t, x) := (\hat{\mathcal{G}}^n(t, x))^2 - \sigma^n(t, x),$$

with $\sigma^n(t, x) := \mathbb{E}[(\hat{\mathcal{G}}^n(t, x))^2]$.

**Proposition.** Recall that $\hat{\mathcal{G}} \in \mathcal{C}([0, T]; \mathcal{W}^\alpha)$ for every $\alpha < \alpha_{d, H} \leq 0$.

Then, for all $d \geq 1$, $T > 0$ and $\alpha < \alpha_{d, H} \leq 0$, the sequence $(\hat{\mathcal{G}}^n)_{n \geq 1}$ converges (almost surely) to some limit $\hat{\mathcal{G}}$ in the space $\mathcal{C}([0, T]; \mathcal{W}^{2\alpha})$, provided $\alpha_{d, H}$ satisfy

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Wellposedness results in the rough case

Recall that \( \mathbb{O} \in \mathcal{C}([0, T]; \mathcal{W}^\alpha) \), for every \( \alpha < \alpha_{d,H} \leq 0 \).

**Theorem.** Almost surely, and for \( T > 0 \) small enough, the equation

\[
v = G_t \ast \phi + G \ast v^2 + 2G \ast (v \cdot \mathbb{O}) + G \ast \mathbb{O}
\]

admits a unique solution in a suitable space of functions, provided \( d \) and \( \alpha_{d,H} \) satisfy

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**Remark.**
\begin{itemize}
  \item \( d = 2, H_0 = H_1 = H_2 = \frac{1}{2} \implies \alpha_{d,H}^{(w)} = 0. \)
  \item \( d = 2, H_0 = 1, H_1 = H_2 = \frac{1}{2} \implies \alpha_{d,H}^{(s)} = 0. \)
\end{itemize}
Recall that \( \varphi \in \mathcal{C}([0,T];\mathcal{W}^{\alpha}) \), for every \( \alpha < \alpha_{d,H} \leq 0 \).

**Theorem.** Almost surely, and for \( T > 0 \) small enough, the equation

\[
\nu = G_t * \phi + G * \nu^2 + 2 \ G * (\nu \cdot \bullet) + \ G * \end{align}

admits a unique solution in a suitable space of functions, provided \( d \) and \( \alpha_{d,H} \) satisfy

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**Remark.**

- \( d = 2, \ H_0 = H_1 = H_2 = \frac{1}{2} \) \( \implies \alpha_{d,H}^{(w)} = 0 \).
- \( d = 2, \ H_0 = 1, \ H_1 = H_2 = \frac{1}{2} \) \( \implies \alpha_{d,H}^{(s)} = 0 \).
Thank you!


A possible interpretation of renormalization

**Corollary.** In the setting of the “rough” theorem, let \((u^n)_{n \geq 1}\) be the sequence of solutions to the renormalized equation

\[
\begin{cases}
\mathcal{L}u^n = (u^n)^2 - \sigma^n + \dot{B}^n, & t \in [0, T], x \in \mathbb{R}^d, \\
u^n(0, .) = \phi.
\end{cases}
\]

Then, almost surely, there exists a time \(T_0 > 0\) such that \(u^n \to u\) in the space \(\mathcal{C}([0, T_0]; \mathcal{W}^{\alpha})\), for \(\alpha < \alpha_{H,d} \leq 0\).