Nonlinear PDE models with stochastic fractional perturbation

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Outline

1. Introduction
2. Step 1: identify the (future) regularity of u
3. Step 2: solve the equation in the regular case $\alpha_{d,H} > 0$
4. Step 3: the rough case $\alpha_{d,H} \leq 0$

The general model

We consider the following general nonlinear SPDE model:

$$\mathcal{L}u = u^2 + \dot{B}, \quad t \in [0,T], x \in \mathbb{R}^d,$$

where:

- $\mathcal{L}$ can be either

  (h) the heat operator: $\mathcal{L}^{(\mathbf{h})} u = \partial_t u - \Delta u \quad (u_0 = \phi)$

  (w) the wave operator: $\mathcal{L}^{(\mathbf{w})} u = \partial_t^2 u - \Delta u \quad (u_0 = \phi_1, (\partial_t u)_0 = \phi_2)$

  (s) the Schrödinger operator: $\mathcal{L}^{(\mathbf{s})} u = \imath \partial_t u - \Delta u \quad (u_0 = \phi)$

- $\dot{B}$ is a **space-time fractional noise**

Space-time fractional noise

Definition. We call a space-time fractional Brownian motion of Hurst index $H = (H_0, H_1, \dots, H_d) \in (0,1)^{d+1}$ any centered Gaussian process $B : \Omega \times ([0,T] \times \mathbb{R}^d) \to \mathbb{R}$ with covariance given by

$$\mathbb{E}\big[B_s(x)B_t(y)\big] = R_{H_0}(s,t) \prod_{i=1}^{d} R_{H_i}(x_i, y_i) \ ,$$

$$\text{where} \quad R_H(a,b) := \frac{1}{2}\left(|a|^{2H} + |b|^{2H} - |a-b|^{2H}\right) \ .$$

Definition. We call a **space-time fractional noise** of Hurst index $H \in (0,1)^{d+1}$ the derivative (in the sense of the distributions)

$$\dot{B} := \partial_t \partial_{x_1} \cdots \partial_{x_d} B.$$

Remark. $H_0 = H_1 = \ldots = H_d = \frac{1}{2} \implies$ classical space-time white noise.

Possible motivations

$$\mathcal{L}u = u^2 + \dot{B}, \quad t \in [0, T], x \in \mathbb{R}^d. \tag{1}$$

Eq. (1) is the most basic **stochastic perturbation of the classical PDE**

$$\mathcal{L}u = u^2$$

$\longrightarrow$ "Nonlinear PDE with power nonlinearity"

Eq. (1) is the most basic **nonlinear extension of the stochastic PDE**

$$\mathcal{L}u = \dot{B}$$

whose solution is explicitly given by

$$u = G * \dot{B}$$

where G is the Green kernel associated with $\mathcal{L}$.

Why a fractional noise ?

$$\mathcal{L}u = u^2 + \dot{B}, \quad t \in [0, T], x \in \mathbb{R}^d.$$

- More general than a white noise + no need for "martingale" property

- Study the transition

$$\begin{matrix} H_i \approx 1 \\ \text{Regular (classical) PDE} \end{matrix} \quad \xrightarrow{\text{Continuous transition}} \quad \begin{matrix} H_i = \frac{1}{2} \\ \text{White noise} \end{matrix}$$

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Step 1: identify the (future) regularity of u

The starting point of our analysis is the mild form of the equation:

$u = G_t * \phi + G * u^2 +$  ,

where ϕ is a regular initial condition, G is the Green kernel associated with $\mathcal{L}$, and the symbol  refers to "linear solution"

 $:= G * \dot{B}\,.$

$\Longrightarrow$ The regularity of u is expected to be the same as the one of
$\Longrightarrow$ **Step 1:** Identify the regularity of .

Due to the roughness of the noise $\dot{B}$, the exact definition and regularity of

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Step 1: identify the (future) regularity of u

Study of  $:= G * \dot{B}$ $\rightarrow$ **based on an approximation procedure:**
Start from a $\mathcal{C}^\infty$ approximation B^n of B, and study the convergence of

 $$^{n} := G * \dot{B}^n, \quad \text{where } \dot{B}^n := \partial_t \partial_{x_1} \cdots \partial_{x_d} B^n.$$

The approximation B^n can for instance be given by:

(i) A **mollifying sequence**, that is, for ρ test-function with $\int_{\mathbb{R}^{d+1}} \rho = 1$,

$$B^n := \rho_n * B\,, \quad \text{where } \rho_n(s,x) = 2^{(d+1)n} \rho(2^n s, 2^n x)\,.$$

(ii) A **discrete approximation** along a grid: if $(t,x) \in [\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}]$,

$$\dot{B}^n_{t,x} := B_{\frac{i+1}{n},\frac{j+1}{n}} - B_{\frac{i}{n},\frac{j+1}{n}} - B_{\frac{i+1}{n},\frac{j}{n}} + B_{\frac{i}{n},\frac{j}{n}}\,.$$

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Step 1: identify the (future) regularity of u

Proposition. Let $H = (H_0, H_1, \ldots, H_d) \in (0,1)^{d+1}$.
Then, for all $T > 0$ and $\alpha < \alpha_{d,H}$, the sequence ( $^n = G * \dot{B}^n)_{n \geq 1}$ converges (almost surely) to some limit  in the space

$$\mathcal{C}([0,T]; \mathcal{W}^{\alpha}(\mathbb{R}^d)),$$

with $\alpha_{d,H} \in \mathbb{R}$ defined as

$\mathcal{L}$	α_H
$\mathcal{L}^{(\mathbf{h})} = \partial_t - \Delta$	$\alpha^{(\mathbf{h})}_{d,H} := 2H_0 + H_+ - d$
$\mathcal{L}^{(\mathbf{w})} = \partial_t^2 - \Delta$	$\alpha^{(\mathbf{w})}_{d,H} := H_0 + H_+ - (d - \frac{1}{2})$
$\mathcal{L}^{(\mathbf{s})} = \imath\partial_t - \Delta$	$\alpha^{(\mathbf{s})}_{d,H} := 2H_0 + H_+ - (d+1)$

where $H_+ := \sum_{i=1}^{d} H_i$.

The limit  is the same for the two approximations **(i)**-**(ii)** of B^n.

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The regular case

Assume that $\alpha_{d,H} > 0$, so ⫯ $\in \mathcal{C}([0,T];\mathcal{W}^{\alpha})$ for every $0 < \alpha < \alpha_{d,H}$.

$$u = G_t * \phi + G * u^2 + ⫯, \quad t \in [0,T],\ x \in \mathbb{R}^d. \qquad (2)$$

$\Longrightarrow$ We can interpret u^2 as a standard product of functions

Theorem. Assume that $\alpha_{d,H} > 0$.

Then, almost surely, Equation (2) admits a unique solution in $\mathcal{C}([0,T];\mathcal{W}^{\alpha})$, for $0 < \alpha < \alpha_{d,H}$ and $T > 0$ small enough, provided d satisfies

$\mathcal{L}$	d	Key ingredient
$\mathcal{L}^{(\mathbf{h})} = \partial_t - \Delta$	$d \geq 1$	Properties of the heat semigroup
$\mathcal{L}^{(\mathbf{w})} = \partial_t^2 - \Delta$	$1 \leq d \leq 4$	(Wave) Strichartz inequalities
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Analysis of the rough case

If $\alpha_{d,H} \leq 0$, then Ψ must be treated as a distribution of negative order:

$$\Psi \in \mathcal{C}([0,T];\mathcal{W}^{\alpha}), \quad \text{for } \alpha < \alpha_{d,H} \leq 0.$$

Problem: how to interpret the non-linearity u^2 in the equation

$$u = G_t * \phi + G * u^2 + \Psi, \quad t \in [0,T], x \in \mathbb{R}^d.$$

Da Prato-Debussche trick: consider the equation satisfied by the process $v := u - \Psi$, namely

$$\boxed{v = G_t * \phi + G * v^2 + 2\, G * (v \cdot \Psi) + G * (\Psi)^2}. \qquad (3)$$

Strategy:

(1) Use renormalization and stochastic arguments to interpret $(\Psi)^2$

(2) Use the properties of G to control $G * (v \cdot \Psi)$ and $G * (\Psi)^2$ as functions

(3) Solve Equation (3) in a suitable space of functions.

Analysis of the rough case

If $\alpha_{d,H} \leq 0$, then ⚲ must be treated as a distribution of negative order:

$$\text{⚲} \in \mathcal{C}([0,T];\mathcal{W}^{\alpha}), \quad \text{for } \alpha < \alpha_{d,H} \leq 0.$$

Problem: how to interpret the non-linearity u^2 in the equation

$$u = G_t * \phi + G * u^2 + \text{⚲}, \quad t \in [0,T],\ x \in \mathbb{R}^d.$$

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Interpretation of $(\mathbf{I})^2$

Wick renormalization: consider the approximation $\mathbf{I}^n := G * \dot{B}^n$ and set

$$\mathbf{V}^n(t,x) := (\mathbf{I}^n(t,x))^2 - \sigma^n(t,x) \,, \quad \text{with } \sigma^n(t,x) := \mathbb{E}\big[(\mathbf{I}^n(t,x))^2\big] \,.$$

Proposition. Recall that $\mathbf{I} \in \mathcal{C}([0,T];\mathcal{W}^{\alpha})$ for every $\alpha < \alpha_{d,H} \leq 0$.

Then, for all $d \geq 1$, $T > 0$ and $\alpha < \alpha_{d,H} \leq 0$, the sequence $(\mathbf{V}^n)_{n \geq 1}$ converges (almost surely) to some limit $\mathbf{V}$ in the space $\mathcal{C}([0,T];\mathcal{W}^{2\alpha})$, provided $\alpha_{d,H}$ satisfy

$\mathcal{L}$	$\alpha_{d,H}$
$\partial_t - \Delta$	$\alpha^{(\mathbf{h})}_{d,H} > -\frac{1}{2}$
$\partial_t^2 - \Delta$	$\alpha^{(\mathbf{w})}_{d,H} > -\frac{1}{4}$
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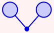
Wellposedness results in the rough case

Recall that  $\in \mathcal{C}([0,T];\mathcal{W}^{\alpha})$, for every $\alpha < \alpha_{d,H} \leq 0$.

Theorem. Almost surely, and for $T > 0$ small enough, the equation

$$v = G_t * \phi + G * v^2 + 2\, G * (v \cdot$$  $$) + G *$$ 

admits a unique solution in a suitable space of functions, provided d and $\alpha_{d,H}$ satisfy

$\mathcal{L}$	d	$\alpha_{d,H}$	Key ingredient
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$\imath\partial_t - \Delta$	$1 \leq d \leq 3$	$\alpha^{(\mathbf{s})}_{d,H} \approx 0_-$	Local regularizing properties

Remark. • $d = 2,\ H_0 = H_1 = H_2 = \frac{1}{2} \quad \Longrightarrow \quad \alpha^{(\mathbf{w})}_{d,H} = 0.$

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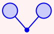
$v = G_t * \phi + G * v^2 + 2\, G * (v \cdot$ $) + G *$ 

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Thank you !

A. Deya: A non-linear wave equation with fractional perturbation. *Ann. Probab.* **47 (2019), no. 3, 1775-1810.**

A. Deya: On a non-linear 2D fractional wave equation. To appear in *Ann. Inst. H. Poincaré Probab. Statist* **56 (2020), no.1, 477-501.**

A. Deya, N. Schaeffer and L. Thomann: A non-linear Schrödinger equation with fractional perturbation. *Submitted*.

A possible interpretation of renormalization

Corollary. In the setting of the "rough" theorem, let $(u^n)_{n\geq 1}$ be the sequence of solutions to the renormalized equation

$$\begin{cases} \mathcal{L}u^n = (u^n)^2 - \sigma^n + \dot{B}^n \,, \quad t \in [0,T], x \in \mathbb{R}^d \,, \\ u^n(0,.) = \phi \,. \end{cases}$$

Then, almost surely, there exists a time $T_0 > 0$ such that $u^n \to u$ in the space $\mathcal{C}([0,T_0];\mathcal{W}^\alpha)$, for $\alpha < \alpha_{H,d} \leq 0$.