Incompressible viscous fluids in the plane and SPDEs on graphs, in the presence of fast advection and non-smooth noise

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joint work with Guangyu Xi

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The equation

We consider here an incompressible flow in $\mathbb{R}^2$, with stream function $H(x)$, $x \in \mathbb{R}^2$. Some particles are moving together with the flow.

If we denote by $u(t,x)$ the density of the particles at time $t \geq 0$ and position $x \in \mathbb{R}^2$, then the function $u(t,x)$ satisfies the Liouville equation

$$\begin{cases} \partial_t u(t,x) = \left\langle \bar{\nabla} H(x), \nabla u(t,x) \right\rangle, & t > 0, \quad x \in \mathbb{R}^2, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^2. \end{cases} \tag{1}$$

Suppose now that the flow has a small viscosity and the particles take part in a slow reaction, with a deterministic and a stochastic component, as described by the equation

$$\left\{\begin{array}{rl} \partial_t \hat{u}_\epsilon(t,x) = & \dfrac{\epsilon}{2}\,\Delta \hat{u}_\epsilon(t,x) + \left\langle \bar{\nabla} H(x), \nabla \hat{u}_\epsilon(t,x)\right\rangle \\ & \\ & +\epsilon\, b(\hat{u}_\epsilon(t,x)) + \sqrt{\epsilon}\, g(\hat{u}_\epsilon(t,x))\partial_t \mathcal{W}(t,x), \\ & \\ \hat{u}_\epsilon(0,x) = & \varphi(x), \quad x \in \mathbb{R}^2. \end{array}\right. \tag{2}$$

Here, $0 < \epsilon << 1$ is a small parameter, included in equation (2) in such a way that all perturbation terms have strength of the same order, as $\epsilon \downarrow 0$.

Our goal

We are interested in

studying the limiting behavior of $\hat{u}_\epsilon(t,x)$, as $\epsilon \downarrow 0$, on a time interval of order $\frac{1}{\epsilon}$.

Actually, on that time scale there is a non-trivial limit, which coincides with

the solution of a suitable SPDE defined on a graph.

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- There exists a constant $c > 0$ such that

$$H(x) \geq c\,|x|^2, \;\; |\nabla H(x)| \geq c\,|x|, \;\; \Delta H(x) \geq c,$$

when $|x|$ is large enough.

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For convenience, we assume

$$\min_{x \in \mathbb{R}^2} H(x) = 0.$$

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Notice that the spatially homogeneous Wiener processes can be represented as

$$\mathcal{W}(t,x) = \sum_{j=1}^{\infty} \widehat{u_j m}(x)\beta_j(t),$$

where $\{u_j\}$ is an orthonormal basis of $L^2_{(s)}(\mathbb{R}^2,\mu)$ and $\{\beta_j\}$ is a sequence of independent Brownian motions.

Finally,

$b, g : \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz continuous non-linearities.

[1] It is the dual of the closure of $\mathcal{S}(\mathbb{R}^2)$ w.r.t. the scalar product $\langle \hat{\mu}, \varphi \star \psi_{(s)} \rangle$.

Finally,

$$b, g : \mathbb{R}^2 \to \mathbb{R} \text{ are Lipschitz continuous non-linearities.}$$

For every $\rho \in L^1(\mathbb{R}^2)$, $u \in L^2(\mathbb{R}^2, \rho\, dx)$ and v in the reproducing kernel[1] RK of $\mathcal{W}$, we shall denote

$$B(u)(x) = b(u(x)), \quad [G(u)v](x) = g(u(x))v(x), \quad x \in \mathbb{R}^2.$$

[1] It is the dual of the closure of $\mathcal{S}(\mathbb{R}^2)$ w.r.t. the scalar product $\langle \hat{\mu}, \varphi \star \psi_{(s)} \rangle$.

Under the above conditions and suitable other conditions on ρ,

for any $T > 0$ and $p \geq 1$, equation (2) admits a unique mild solution $\hat{u}_\epsilon \in L^p(\mathbf{\Omega}; C([0,T]; L^2(\mathbb{R}^2, \rho\, dx))$, for every fixed $\epsilon > 0$.

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This means that there exists a unique adapted process $\hat{u}_\epsilon \in L^p(\boldsymbol{\Omega}; C([0,T]; L^2(\mathbb{R}^2, \rho\, dx))$, such that

$$\hat{u}_\epsilon(t) = \hat{S}_\epsilon(t)\varphi + \epsilon \int_0^t \hat{S}_\epsilon(t-s) B(\hat{u}_\epsilon(s))\, ds + \sqrt{\epsilon} \int_0^t \hat{S}_\epsilon(t-s)\, G(\hat{u}_\epsilon(s))\, d\mathcal{W}(s),$$

where $\hat{S}_\epsilon(t)$ is the semigroup associated with the operator

$$\hat{\mathcal{L}}_\epsilon \varphi(x) = \frac{\epsilon}{2} \Delta\varphi(x) + \left\langle \bar{\nabla} H(x), \nabla\varphi(x) \right\rangle, \qquad x \in \mathbb{R}^2.$$

The problem

It is possible to check that, if $\hat{u}_\epsilon(t,x)$ is the solution of equation (2) and $u(t,x)$ is the solution of equation (1), then for every $T>0$ and $\eta>0$

$$\lim_{\epsilon\to 0}\mathbb{P}\left(\sup_{t\in[0,T]}|\hat{u}_\epsilon(t,x)-u(t,x)|>\eta\right)=0,$$

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But on large time intervals, growing together with ϵ^{-1}, the difference $\hat{u}_\epsilon(t,x)-u(t,x)$ can have order 1, as $\epsilon\downarrow 0$.

To describe the long-time behavior of the particle density, we define

$$u_\epsilon(t,x) := \hat{u}_\epsilon(t/\epsilon, x), \qquad t \geq 0, \quad x \in \mathbb{R}^2.$$

With this change of time, the new function $u_\epsilon(t,x)$ solves the equation

$$\left\{\begin{array}{rl} \partial_t u_\epsilon(t,x) = & \dfrac{1}{2}\,\Delta u_\epsilon(t,x) + \dfrac{1}{\epsilon}\,\langle \bar{\nabla} H(x), \nabla u_\epsilon(t,x)\rangle \\[10pt] & + b(u_\epsilon(t,x)) + g(u_\epsilon(t,x))\partial_t \mathcal{W}(t,x), \\[10pt] u_\epsilon(0,x) = & \varphi(x), \quad x \in \mathbb{R}^2, \end{array}\right. \tag{3}$$

for some spatially homogeneous Wiener process $\mathcal{W}(t,x)$.

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In particular, we will see that, in order to describe the limit of $u_\epsilon(t,x)$, one should consider SPDEs on a non standard setting, where the space variable changes on the graph Γ obtained by identifying all points in each connected component of the level sets of the Hamiltonian H.

The linear deterministic problem

For every $\epsilon > 0$, we consider the Cauchy problem

$$\begin{cases} \partial_t v_\epsilon(t,x) = \mathcal{L}_\epsilon v_\epsilon(t,x), & t > 0, \quad x \in \mathbb{R}^2, \\ v_\epsilon(0,x) = \varphi(x), & x \in \mathbb{R}^2, \end{cases}$$

where $\mathcal{L}_\epsilon$ is the second order uniformly elliptic differential operator defined by

$$\mathcal{L}_\epsilon v(x) = \frac{1}{2}\Delta v(x) + \frac{1}{\epsilon}\left\langle \bar{\nabla} H(x), \nabla v(x) \right\rangle, \quad x \in \mathbb{R}^2.$$

As well known, the solution $v_\epsilon(t, x)$ can be represented in terms of the Markov transition semigroup $S_\epsilon(t)$ associated with $\mathcal{L}_\epsilon$.

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Namely,

$$v_\epsilon(t,x) = S_\epsilon(t)\varphi(x) = \mathbb{E}_x\,\varphi(X_\epsilon(t)), \qquad x \in \mathbb{R}^2,$$

where $X_\epsilon(t)$ is the solution of the SDE

$$dX_\epsilon(t) = \frac{1}{\epsilon}\,\bar{\nabla} H(X_\epsilon(t))\,dt + dw(t), \qquad X_\epsilon(0) = x \in \mathbb{R}^2,$$

for some 2-dimensional Brownian motion $w(t)$, defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$.

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Clearly, the first fundamental goal is studying the limiting behavior of the semigroup $S_\epsilon(t)$, as $\epsilon \downarrow 0$.

Some notations

For every $z \geq 0$, we denote by $C(z)$ the z-level set

$$C(z) = \{x \in \mathbb{R}^2 : H(x) = z\} = \bigcup_{k=1}^{N(z)} C_k(z).$$

If $X(t)$ is the solution of the Hamiltonian system

$$\dot{X}(t) = \bar{\nabla} H(X(t)),$$

for every $x \in \mathbb{R}^2$ we have

$$X(0) = x \Longrightarrow X(t) \in C_{k(x)}(H(x)), \qquad t \geq 0,$$

where $C_{k(x)}(x)$ is the connected component of the level set $C(H(x))$, containing x.

Now, for every $z \geq 0$ and $k = 1, \ldots, N(z)$, we define

$$T_k(z) = \oint_{C_k(z)} \frac{1}{|\nabla H(x)|}\, dl_{z,k},$$

where $dl_{z,k}$ is the length element on $C_k(z)$.

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Moreover, the probability measure

$$d\mu_{z,k} := \frac{1}{T_k(z)} \frac{1}{|\nabla H(x)|}\, dl_{z,k}$$

is invariant for the Hamiltonian system on the level set $C_k(z)$.

The identification map Π and the graph Γ

If we identify all points in $\mathbb{R}^2$ belonging to the same connected component of a given level set $C(z)$ of the Hamiltonian H,

we obtain a graph Γ, given by several intervals $I_1, \dots I_n$ and vertices $O_1, \dots, O_m$.

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The vertices will be of two different types,

external and internal vertices.

External vertices correspond to local extrema of H, while internal vertices correspond to saddle points of H.

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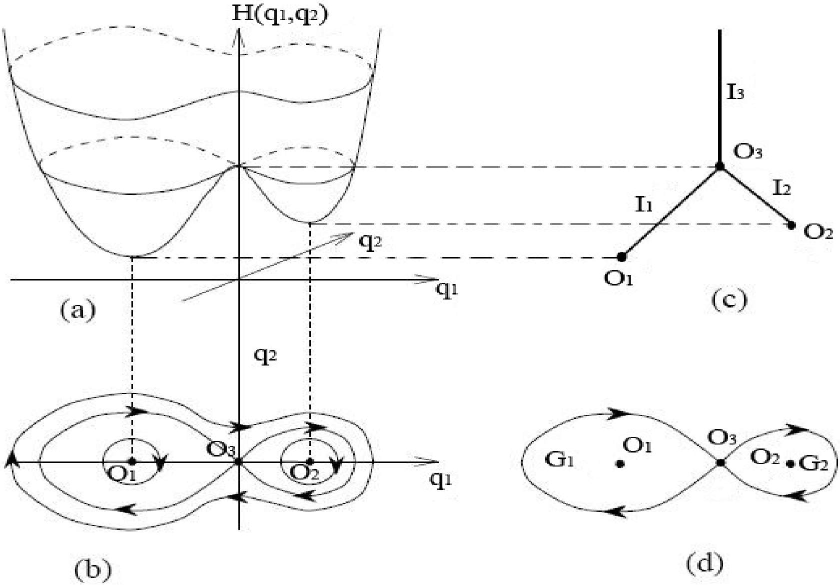
We shall denote by $\Pi : \mathbb{R}^2 \to \Gamma$ the identification map.


H(q1,q2)
q2
q1
(a)
I3
O3
I2
I1
O2
O1
(c)
q2
O3
O1
O2
q1
(b)
G1
O1
O3
O2
G2
(d)

A limiting result

Freidlin and Wentcell in 2002 have studied

the limiting behavior, as $\epsilon \downarrow 0$, of the (non Markov) process $\Pi_\epsilon(t) := \Pi(X_\epsilon(t))$, $t \geq 0$, in the space $C([0, T]; \Gamma)$, for any fixed $T > 0$ and $x \in \mathbb{R}^2$.

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the process Π_ϵ, which describes the slow component of the motion X_ϵ, converges, in the sense of weak convergence of distributions in $C([0, T]; \Gamma)$, to a diffusion process $\bar{Y}$.

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Namely, they have proven that for any bounded and continuous functional $F : C([0,T];\Gamma) \to \mathbb{R}$ and $x \in \mathbb{R}^2$

$$\lim_{\epsilon \to 0} \mathbb{E}_x F(\Pi_\epsilon(\cdot)) = \bar{\mathbb{E}}_{\Pi(x)} F(\bar{Y}(\cdot)).$$

The process $\bar{Y}$ has been described in terms of its generator $\bar{L}$, which is given by suitable differential operators $\bar{\mathcal{L}}_k$ within each edge I_k of the graph and by certain gluing conditions at the interior vertices O_i of the graph.

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Nevertheless,

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In what follows,

we shall denote by $\bar{S}(t)$ the semigroup associated with $\bar{Y}(t)$.

Back to the linear problem

Since the solution of the problem

$$\begin{cases} \dfrac{\partial v_\epsilon}{\partial t}(t,x,y) = \dfrac{1}{2}\,\Delta v_\epsilon(t,x) + \dfrac{1}{\epsilon}\langle \bar{\nabla} H(x), \nabla v_\epsilon(t,x)\rangle, \\ \\ v_\epsilon(0,x) = \varphi(y), \end{cases}$$

is given by

$$v_\epsilon(t,x) = S_\epsilon(t)\varphi(x) = \mathbb{E}_x \varphi(X_\epsilon(t)),$$

in order to study the asymptotics of v_ϵ one would like to use the limit

$$\lim_{\epsilon\to 0} \mathbb{E}_x F(\Pi(X_\epsilon(\cdot))) = \bar{\mathbb{E}}_{\Pi(x)} F(\bar{Y}(\cdot)).$$

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But things are more complicated...

Functions defined on Γ and $\mathbb{R}^2$

We fix here a continuous mapping $\gamma : \Gamma \to (0, +\infty)$ such that

$$\sum_{k=1}^{n} \int_{I_k} \gamma(z,k)\, T_k(z)\, dz < \infty,$$

where, we recall,

$$T_k(z) = \oint_{C_k(z)} \frac{1}{|\nabla H(x)|}\, dl_{z,k}.$$

If we define

$$\gamma^\vee(x) = \gamma(\Pi(x)), \qquad x \in \mathbb{R}^2,$$

we have that

$$\gamma^\vee \in L^1(\mathbb{R}^2).$$

In what follows, we shall define

$$H_\gamma = L^2(\mathbb{R}^2, \gamma^\vee(x)\, dx),$$

and

$$\bar{H}_\gamma = L^2(\Gamma, \nu_\gamma),$$

where the measure ν_γ is defined as

$$\nu_\gamma(A) := \sum_{k=1}^{n} \int_{I_k} \mathbb{I}_A(z,k)\, \gamma(z,k)\; T_k(z)\, dz, \qquad A \subseteq \mathcal{B}(\Gamma).$$

How to go from functiond defined on Γ to functions defined on $\mathbb{R}^2$

- For every $u : \mathbb{R}^2 \to \mathbb{R}$ and $(z, k) \in \Gamma$ we define

$$u^\wedge(z,k) = \oint_{C_k(z)} u(x)\, d\mu_{z,k},$$

where

$$d\mu_{z,k} := \frac{1}{T_k(z)} \frac{1}{|\nabla H(x)|}\, dl_{z,k}.$$

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$$d\mu_{z,k} := \frac{1}{T_k(z)} \frac{1}{|\nabla H(x)|}\, dl_{z,k}.$$

- For every $f : \Gamma \to \mathbb{R}$ and $x \in \mathbb{R}^2$ we define

$$f^\vee(x) = f(\Pi(x)).$$

The semigroup $S_\epsilon(t)$ in the weighted space H_γ

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In particular, $S_\epsilon(t)$ can be extended to a contraction semigroup on $L^\kappa(\mathbb{R}^2)$, for every $\kappa \geq 1$ and $\epsilon > 0$.

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We assume that

the semigroup $S_\epsilon(t)$ is well defined on H_γ, for every $\epsilon > 0$. Moreover, for every fixed $T > 0$, there exists $c_T > 0$ such that

$$\|S_\epsilon(t)\|_{\mathcal{L}(H_\gamma)} \leq c_T, \qquad t \in [0, T], \qquad \epsilon > 0.$$

In fact, we can prove that

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Another fundamental assumption is that

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$$\frac{dT_k(z)}{dz} \neq 0, \quad (z, k) \in \Gamma.$$

In particular, we rule out the case $H(x) = |x|^2$.

Convergence of the semigroups

In C.-Freidlin [19], it has been proven that

$$\lim_{\epsilon\to 0}\ \sup_{t\in[\tau,T]} \left|\mathbb{E}_x\, u(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)}\, u^\wedge(\bar{Y}(t))\right| = 0, \tag{4}$$

for any $u \in C_b(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, and for any $0 \leq \tau \leq T$.

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for any $u \in C_b(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$, and for any $0 \leq \tau \leq T$.

This means that

$$\lim_{\epsilon\to 0}\ \sup_{t\in[\tau,T]} |S_\epsilon(t)u(x) - \bar{S}(t)u^\wedge(\Pi(x))| = 0.$$

Moreover, the limit is true in H_γ and $\bar{H}_\gamma$.

How to prove limit (4)

Limit (4) is not a straightforward consequence of the limit

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$$\lim_{\epsilon\to 0} \mathbb{E}_x F(\Pi(X_\epsilon(\cdot))) = \bar{\mathbb{E}}_{\Pi(x)} F(\bar{Y}(\cdot)).$$

Actually, (4) is a consequence of the following two limits

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|\mathbb{E}_x\, u(X^\epsilon(t)) - \mathbb{E}_x\, u^\wedge(\Pi(X_\epsilon(t)))\right| = 0, \tag{5}$$

and

$$\lim_{\epsilon\to 0} \sup_{t\in[\tau,T]} \left|\mathbb{E}_x u^\wedge(\Pi(X_\epsilon(t))) - \bar{\mathbb{E}}_{\Pi(x)} u^\wedge(\bar{Y}(t))\right| = 0, \tag{6}$$

that have to be valid for any $0 < \tau < T$ and $x \in \mathbb{R}^2$ and for any $u \in C_b(\mathbb{R}^2)$.

The SPDE on the graph Γ

We introduce now the following SPDE on the graph Γ

$$\begin{cases} \partial_t \bar{u}(t,z,k) = \bar{L}\,\bar{u}(t,z,k) + b(\bar{u}(t,z,k)) + g(\bar{u}(t,z,k))\,\partial_t \bar{\mathcal{W}}(t,z,k), \\ \bar{u}(0,z,k) = \varphi^\wedge(z,k), \quad (z,k) \in \Gamma, \end{cases} \tag{7}$$

where $\bar{L}$ is the generator of the limiting Markov process $\bar{Y}(t)$.

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The random forcing $\bar{\mathcal{W}}$ is defined by

$$\bar{\mathcal{W}}(t,z,k) = \sum_{j=1}^{\infty} (\widehat{u_j m})^{\wedge}(z,k)\,\beta_j(t), \qquad t\geq 0 \quad (z,k)\in\Gamma.$$

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Here we are only assuming that μ has a density $m \in L^p(\mathbb{R}^2)$, for some $p > 1$, and things are more complicated.

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- prove that for every q large enough and $0 < \tau < t < T$;

$$\mathbb{E}\sup_{s\in[0,t]}|\int_0^s S_\epsilon(s-r)G(u_\epsilon(r))d\mathcal{W}(s) - \int_0^s \bar{S}(s-r)G(\bar{u}(r)^\vee)d\bar{\mathcal{W}}(r)|^q_{H_\gamma}$$

$$\leq C_{q,T}\int_\tau^t \mathbb{E}\sup_{r\in[\tau,s]}|u_\epsilon(r)-\bar{u}(r)^\vee|^q_{H_\gamma}\,ds + H_{\epsilon,2}(T),$$

for some $H_{\epsilon,2}(T)$ such that

$$\lim_{\epsilon\to 0} H_{\epsilon,2}(T) = 0.$$

A more refined analysis

Suppose $G_\epsilon(t,x,y)$ is the kernel corresponding to $S_\epsilon(t)$, i.e.

$$S_\epsilon(t)u(x) = \int_{\mathbb{R}^2} G_\epsilon(t,x,y)u(y)dy, \qquad x \in \mathbb{R}^2.$$

The limit result mentioned before implies that for any fixed (t,x), kernels $G_\epsilon(t,x,\cdot)$ converge weakly to some $\bar{G}(t,x,\cdot)$, which satisfies

$$\bar{S}(t)^\vee u(x) = \int_{\mathbb{R}^2} \bar{G}(t,x,y)u(y)dy.$$

We needed to show that for any $(t, x, y) \in (0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$

$$\sup_{\epsilon>0} G_\epsilon(t, x, y) \leq \frac{C}{t} \exp\left(-\frac{(\sqrt{H(y)+1} - \sqrt{H(x)+1})^2}{4Ct}\right).$$

In particular, given any compact $K \subset \mathbb{R}^2$, there exist c and R depending on K such that for any $t \in (0, \infty)$ and $y \in \mathbb{R}^2$.

$$\sup_{x \in K} G_\epsilon(t, x, y) \leq \begin{cases} \frac{c}{t} & |y| \leq R \\ \frac{c}{t} \exp\left(-\frac{|y|^2}{Ct}\right) & |y| > R. \end{cases}$$

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Due to the weak convergence of $G_\epsilon(t,x,y)$ to $\bar{G}(t,x,y)$, the same bounds are valid for $\bar{G}(t,x,y)$.

In particular, if we denote by M the *multiplication operator* defined by $M(\psi)\xi = \psi\xi$, we can show that

$$\|S_\epsilon(t)M(\psi)\|^2_{L_{(HS)}(RK,H_\gamma)} \leq C_T\|m\|_{L^p}\, t^{-(p-1)/p}|\psi|^2_{H_\gamma},$$

for each $0 \leq t \leq T$ and $\psi \in H_\gamma$.

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Moreover,

$$\|\bar{S}(t)M(\psi)\|^2_{L_{(HS)}(R\bar{K},\bar{H}_\gamma)} \leq C_T\|m\|_{L^p}\, t^{-(p-1)/p}|\psi|^2_{\bar{H}_\gamma},$$

for all $0 \leq t \leq T$ and $\psi \in \bar{H}_\gamma$.

A more refined limit

We can prove that for any $\psi \in H_\gamma$, for any fixed $0 < \tau < T$

$$\lim_{\epsilon \to 0} \sup_{t \in [\tau, T]} \sum_{j=1}^{\infty} \left|\left(S_\epsilon(t) - \bar{S}(t)^\vee\right)(\psi e_j)\right|_{H_\gamma}^2 = 0,$$

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where $\{e_j\}$ is a complete orthonormal system for the reproducing kernel.

This limit allows to treat the stochastic convolution and conclude that the following limit for the solutions of the SPDE's holds.

The limit theorem

For any initial condition $\varphi \in H_\gamma$, $q \geq 1$ and $0 < \tau < T$ we have

$$\lim_{\epsilon \to 0} \mathbf{E} \sup_{t \in [\tau, T]} |u_\epsilon(t) - \bar{u}(t)^\vee|^q_{H_\gamma}$$

$$= \lim_{\epsilon \to 0} \mathbf{E} \sup_{t \in [\tau, T]} |u_\epsilon(t)^\wedge - \bar{u}(t)|^q_{H_\gamma} = 0,$$

where u_ϵ and $\bar{u}$ are the unique mild solutions of the SPDE on $\mathbb{R}^2$ and of the SPDE on Γ, respectively.

A weaker type of convergence

One of the key assumptions in order to prove that

$$\lim_{\epsilon \to 0} \sup_{t \in [\tau, T]} \left| \mathbb{E}_x u(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} u^\wedge(\bar{Y}(t)) \right| = 0,$$

for any $u \in H_\gamma$ and $0 < \tau < T$, is

$$\frac{dT_k(z)}{dz} \neq 0, \qquad (z, k) \in \Gamma. \tag{8}$$

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Assumption (8) allows to say that if $\alpha \in (4/7, 2/3)$ then for every $u \in C_b^2(\mathbb{R}^2)$ and for every compact set $K \in \mathbb{R}^2$

$$\lim_{\epsilon \to 0} \sup_{x \in K} \left| \mathbb{E}_x u(X_\epsilon(\epsilon^\alpha)) - (u^\wedge)^\vee(x) \right| = 0.$$

What does it happen when (8) is not verified?

We have tried to understand if it is still possible to have some limit in this case, and have proven that for any $0 \leq \tau < T$ and any compact set $K \subset \mathbb{R}^2$,

$$\lim_{\epsilon \to 0} \sup_{x \in K} \left| \int_\tau^T \left[\mathbb{E}_x \varphi(X_\epsilon(t)) - \bar{\mathbb{E}}_{\Pi(x)} \varphi^\wedge(\bar{Y}(t)) \right] \theta(t) dt \right| = 0$$

for any $\varphi \in C_b(\mathbb{R}^2)$ and $\theta \in C_b([\tau, T])$.

The previous limit allowed us to prove that when

$$b = 0, \qquad g = \text{constant}$$

for any fixed $T > 0$, $q \geq 1$ and $\theta \in C([0, T])$

$$\lim_{\epsilon \to 0} \mathbb{E} \left| \int_0^T \left[u_\epsilon(t) - \bar{u}(t)^\vee \right] \theta(t) dt \right|_{H_\gamma}^q$$

$$= \lim_{\epsilon \to 0} \mathbb{E} \left| \int_0^T \left[u_\epsilon(t)^\wedge - \bar{u}(t) \right] \theta(t) dt \right|_{\bar{H}_\gamma}^q = 0.$$

Thank you

A few words about the proof

We have

$$u_\epsilon(t) - \bar{u}(t)^\vee = \left[S_\epsilon(t)\varphi - \bar{S}(t)^\vee\varphi\right]$$

$$+ \left[\int_0^t S_\epsilon(t-s)B(u_\epsilon(s))\,ds - \int_0^t \bar{S}(t-s)^\vee B(\bar{u}(s)^\vee)\,ds\right]$$

$$+ \left[\Theta_\epsilon(t) - \bar{\Theta}^\vee(t)\right] =: \sum_{i=1}^{3} I_{\epsilon,i}(t),$$

where we have defined

$$\Theta_\epsilon(t) := \int_0^t S_\epsilon(t-s)\,G(u_\epsilon(s))\,d\mathcal{W}(s),$$

$$\bar{\Theta}(t) := \int_0^t \bar{S}(t-s)\,G(\bar{u}(s))\,d\bar{\mathcal{W}}(s).$$

It is possible to show that

$$\mathbf{E} \sup_{s\in[0,t]} |I_{\epsilon,2}(s)|^p_{H_\gamma} + \mathbf{E} \sup_{s\in[0,t]} |I_{\epsilon,3}(s)|^p_{H_\gamma}$$

$$\leq c_p(T) \int_\tau^t \mathbf{E} \sup_{r\in[\tau,s]} |u_\epsilon(r) - \bar{u}(r)^\vee|^p_{H_\gamma}\, ds + R_{T,p}(\tau,\epsilon),$$

for some constant $R_{T,p}(\tau,\epsilon)$ such that

$$\lim_{\epsilon,\tau\to 0} R_{T,p}(\tau,\epsilon) = 0.$$

Therefore,

$$\mathbf{E}\sup_{s\in[\tau,t]}|u_\epsilon(s)-\bar{u}^\vee(s)|^p_{H_\gamma}$$

$$\leq c\left(\sup_{s\in[\tau,T]}|S_\epsilon(s)\varphi-\bar{S}(s)^\vee\varphi|^p_{H_\gamma}+R_{T,p}(\tau,\epsilon)\right)$$

$$+c_p(T)\int_\tau^t\mathbf{E}\sup_{r\in[\tau,s]}|u_\epsilon(r)-\bar{u}^\vee(r)|^p_{H_\gamma}\,ds,$$

and, due to the limiting result for $S_\epsilon(s)\varphi-\bar{S}(s)^\vee\varphi$, from the Gronwall lemma we can conclude.

The operator $\bar{\mathcal{L}}$

For each $k = 1, \ldots, n$, the differential operator $\bar{\mathcal{L}}_k$, acting on functions f defined on the edge I_k, has the form

$$\bar{\mathcal{L}}_k f(z) = \frac{1}{2\,T_k(z)} \frac{d}{dz} \left(\alpha_k \frac{df}{dz} \right)(z).$$

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The domain $D(\bar{L})$ is defined as the set of continuous functions on the graph Γ, that are twice continuously differentiable in the interior part of each edge of the graph, and satisfy suitable gluing conditions at the vertices.

For any vertex $O_i = (z_i, k_{i_1}) = (z_i, k_{i_2}) = (z_i, k_{i_3})$ there exist finite

$$\lim_{(z,k_{i_j})\to O_i} \bar{L} f(z, k_{i_j}), \qquad j = 1, 2, 3,$$

and the limits do not depend on the edge $I_{k_{i_j}} \sim O_i$. Moreover, for each interior vertex O_i the following gluing condition is satisfied

$$\sum_{j=1}^{3} \pm \alpha_{k_{i_j}}(z_i) d_{k_{i_j}} f(z_i, k_{i_j}) = 0,$$

where $d_{k_{i_j}}$ is the differentiation along $I_{k_{i_j}}$ and the sign $+$ is taken if the H-coordinate increases along $I_{k_{i_j}}$ and the sign $-$ is taken otherwise.

Spatially homogeneous Wiener process

This means that there exists a Gaussian random field $\mathcal{W}$ on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \{\boldsymbol{\mathcal{F}}_t\}_{t\geq 0}, \mathbf{P})$, such that

- the mapping $(t,x) \in [0+\infty) \times \mathbb{R}^2 \mapsto \mathcal{W}(t,x)$ is continuous w.r.t. $t \geq 0$ and measurable w.r.t. both variables, **P**-almost surely;
- for each $x \in \mathbb{R}^2$, the process $\mathcal{W}(t,x)$, $t \geq 0$, is a one-dimensional Wiener process;
- for every $t, s \geq 0$ and $x, y \in \mathbb{R}^2$

$$\mathbf{E}\,\mathcal{W}(t,x)\mathcal{W}(s,y) = (t \wedge s)\,\Lambda(x-y),$$

where Λ is the Fourier transform of the spectral measure μ, that is

$$\Lambda(x) = \int_{\mathbb{R}^2} e^{i\langle x,\lambda\rangle}\,\mu(d\lambda), \qquad x \in \mathbb{R}^2.$$