Ergodicity & CLT for SPDE

Stochastic Analysis, Random Fields & Applications

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Acknowledgements

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NSF

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- $\partial_t u(t\,,x) = \frac{1}{2}\Delta u(t\,,x) + \sigma(u(t\,,x))\eta(t\,,x)$ for $t > 0$ and $x \in \mathbb{R}^d$

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- Here is a simulation for the parabolic Anderson model driven by space-time white noise $\boxed{d = 1,\ f = \delta_0,\ \sigma(u) = u\ ,\ \text{and } t = 1}$ [thanks to Kunwoo Kim]:

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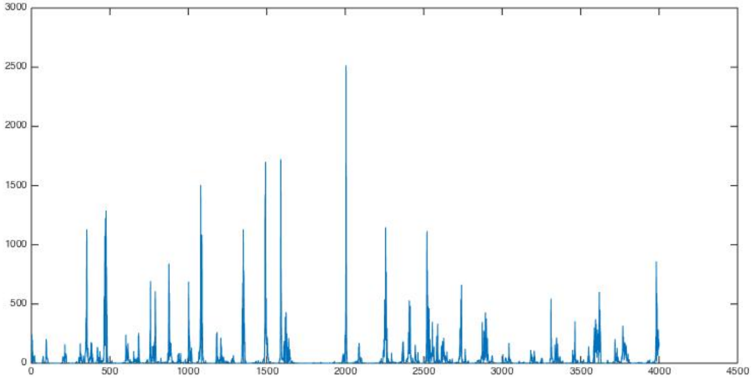

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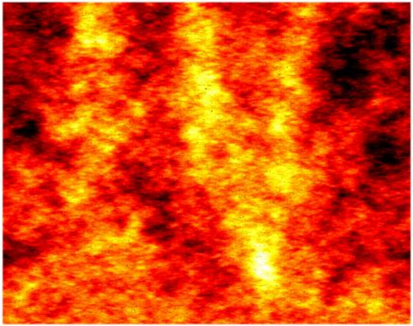

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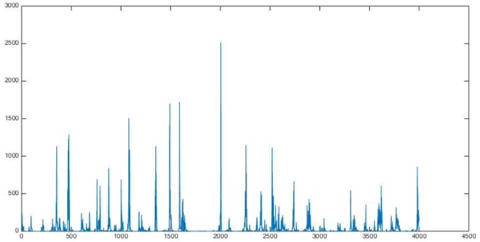

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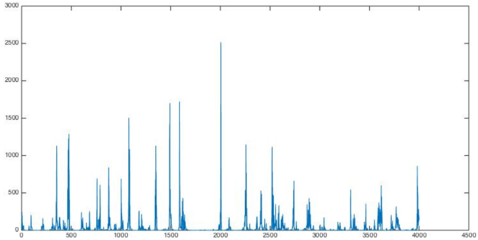

- The ergodic behavior of $x \mapsto u(t\,, x)$.

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Theorem (Dalang, 1999)

The solution to SPDE exists and is "unique" and is continuous in $L^k(\mathrm{P})$ for all $k \geqslant 1$ provided that

$$\int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}z)}{1 + \|z\|^2} < \infty. \qquad \text{(D)}$$

This is NAS when $\sigma \propto 1$.

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4. $f(x) \propto \|x\|^{-\beta}$ where $\beta \in (0\,, d \wedge 2)$ [Riesz kernels]

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3. If $f(\mathrm{d}x) \ll \mathrm{d}x$ then

$$\hat{f}\{0\} = 0 \quad \text{iff} \quad \frac{1}{|B(0\,,N)|}\int_{B(0,N)} f(x)\,\mathrm{d}x \to 0 \quad \text{as } N \to \infty$$

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Example

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$$N^{d/2}\left(\frac{1}{N^d}\int_{[0,N]^d} g(u(t\,,x))\,\mathrm{d}x - \mathrm{E}[g(u(t\,,0))]\right) \Rightarrow \mathrm{N}(0\,,\tau_g^2) \qquad \textit{as } N \to \infty.$$

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1. (D): $\int_{\mathbb{R}^d} \frac{\hat{f}(\mathrm{d}z)}{1+\|z\|^2} < \infty$ implies spatial ergodicity
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The effect of initial data

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$x \mapsto U(t,x) = \dfrac{u(t,x)}{p_t(x)}$ *is stationary for all* $t>0$, *where* $p_t(x) = \dfrac{\mathrm{e}^{-x^2/(2t)}}{\sqrt{2\pi t}}$.

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$U(t)$ *is weakly mixing* $\forall t > 0$, *and* $S_{N,t} := N^{-1}\int_0^N [U(t,x)-1]\,\mathrm{d}x$ *satisfies*

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$$\mathrm{Var}(F) = \int_0^\infty \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}z\,\|\mathrm{E}\left[D_{s,z}F \mid \mathcal{F}_s\right]\|_2^2 \leqslant \mathrm{E}\left(\|DF\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})}\right)$$

- A Clark-Ocone Formula (Chen-K-Nualart-Pu, 19+): $\forall F \in \mathbb{D}^{1,2}$,

$$F = \mathrm{E}F + \int_{(0,\infty)\times\mathbb{R}^d} \mathrm{E}\left[D_{s,z}F \mid \mathcal{F}_s\right]\eta(\mathrm{d}s\,\mathrm{d}z) = \mathrm{E}F + \int v\,\mathrm{d}\eta$$

- $\therefore$ If $\text{Var}\langle DF, v\rangle_{\mathcal{H}} \ll 1$, then $F - \mathrm{E}F \approx$ normal, in TV
- E.g., if $f = \delta_0$ and $d = 1$ (space-time white noise), then Poincaré ineq.

$$\text{Var}(F) = \int_0^\infty \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}z\, \|\mathrm{E}\left[D_{s,z}F \mid \mathcal{F}_s\right]\|_2^2 \leqslant \mathrm{E}\left(\|DF\|^2_{L^2(\mathbb{R}_+\times\mathbb{R})}\right)$$

- More generally: $\forall F \in \mathbb{D}^{1,2}$,

$$\text{Var}(F) \leqslant \mathrm{E}\left(\|DF\|^2_{\mathcal{H}}\right).$$

Thank you!


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