Ergodicity & CLT for SPDE
Stochastic Analysis, Random Fields & Applications
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Joint work with
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The Basic Problem

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x))\eta(t, x) \text{ for } t > 0 \text{ and } x \in \mathbb{R}^d \]
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- \( \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \) for \( t > 0 \) and \( x \in \mathbb{R}^d \)
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- \( \sigma : \mathbb{R} \to \mathbb{R} \) is non random and Lipschitz continuous;

Here is a simulation for the parabolic Anderson model driven by space-time white noise \( \sigma = \sigma(u) = u \), and \( t = 1 \) [thanks to Kunwoo Kim].
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  \text{Cov}[\eta(t, x), \eta(s, y)] = \delta_0(t - s)f(x - y) \quad \text{for } s, t \geq 0, x, y \in \mathbb{R}^d.
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The ergodic behavior of \( x \mapsto u(t, x) \).
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- I.e., \( \text{Cov} \left[ \int \phi \, d\eta, \int \psi \, d\eta \right] = \int_0^\infty (\phi(s), \psi(s) \ast f)_{L^2(\mathbb{R}^d)} \, ds. \)
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**Theorem (Dalang, 1999)**

The solution to SPDE exists and is “unique” and is continuous in \( L^k(\mathbb{P}) \) for all \( k \geq 1 \) provided that

\[
\int_{\mathbb{R}^d} \frac{\hat{f}(dz)}{1 + \|z\|^2} < \infty. \quad (D)
\]

This is NAS when \( \sigma \propto 1 \).
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Example

\[ f = \delta_0 \] and \[ \hat{f} = \delta_0 \] is in \( L^1(\mathbb{R}^d) \) [Wiener algebra] and \( f \in L^2(\mathbb{R}^d) \) \[ \Leftrightarrow \hat{f} \in L^2(\mathbb{R}^d) \] and \( d \leq 3 \) [Cauchy–Schwarz ineq.]

\[ f(x) \propto \| x \|^{2\beta} \] where \( \beta \in (0, d \wedge 2) \) [Riesz kernels]
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2. \( f, \hat{f} \in L^1(\mathbb{R}^d) \) [Wiener algebra]
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It is not difficult to see that $x \mapsto u(t, x)$ is stationary for all $t > 0$.
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\[\hat{f}\{0\} = 0 \iff f([-N,N]^d) = o(N^d) \text{ as } N \to \infty\]

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(a) \( u(t) \) is ergodic for all \( t > 0 \) if \( \hat{f}\{0\} = 0 \) [iff when \( \sigma \propto 1 \): Maruyama (1949); Dym-McKean (1976)]
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\[ \hat{f}\{0\} = 0 \quad \text{iff} \quad \frac{1}{|B(0, N)|} \int_{B(0, N)} f(x) \, dx \to 0 \quad \text{as} \quad N \to \infty \]
Let \( \{X(a)\}_{a \in \mathbb{R}^d} \) be a stationary random field. It is \textit{weakly mixing} [mixing in the sense of ergodic theory] if for all \( a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathbb{R}^d \) and \( A_1, \ldots, A_m, B_1, \ldots, B_m \in \mathbb{R} \),

\[
P \{ X(a_i) < A_i , \; X(R + b_j) < B_j , \; \forall i \leq n, j \leq m \} \rightarrow P \{ X(a_i) < A_i , \; \forall i \leq n \} P \{ X(b_j) < B_j , \; \forall j \leq m \} \text{ as } \|R\| \rightarrow \infty
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Theorem (Chen-K-Nualart-Pu, 2019+)

\( u(t) \) is mixing for all \( t > 0 \) if for some, hence all, \( \lambda > 0 \),

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Let \( \{X(a)\}_{a \in \mathbb{R}^d} \) be a stationary random field. It is weakly mixing [mixing in the sense of ergodic theory] if for all \( a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathbb{R}^d \) and \( A_1, \ldots, A_m, B_1, \ldots, B_m \in \mathbb{R}, \)

\[ P \left\{ X(a_i) < A_i, X(R + b_j) < B_j, \forall i \leq n, j \leq m \right\} \]

\[ \rightarrow P \left\{ X(a_i) < A_i, \forall i \leq n \right\} P \left\{ X(b_j) \leq B_j, \forall j \leq m \right\} \text{ as } \|R\| \rightarrow \infty \]

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\( u(t) \) is mixing for all \( t > 0 \) if for some, hence all, \( \lambda > 0, \)

\[ \lim_{\|x\| \to \infty} \int_{\mathbb{R}^d} \frac{e^{ix \cdot z} \hat{f}(dz)}{\lambda + \|z\|^2} = 0. \]
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**Example**

\( u(t) \) is mixing for all \( t > 0 \) if \( \hat{f}(dx) \ll dx \) [(D) + Riemann–Lebesgue]
A Hierarchy of conditions

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x))\eta(t, x) \]  

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\[ (D): \int_{\mathbb{R}^d} \frac{\hat{f}(dz)}{1 + \|z\|^2} < \infty \text{ implies spatial ergodicity} \]
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1. (D): \[ \int_{\mathbb{R}^d} \frac{\hat{f}(dz)}{1 + ||z||^2} < \infty \] implies spatial ergodicity

2. (D) + \[ \lim_{||z|| \to \infty} \int_{\mathbb{R}^d} \frac{e^{iz \cdot x} \hat{f}(dz)}{1 + ||z||^2} = 0 \] implies mixing

All are NAS when \( \sigma \propto 1 \); \( \exists \) FCLTs; and the CLT holds in total variation!

\[ \text{Theorem (Chen-K-Nualart-Pu 2020+)} \]

If \( 0 < f(R^d) < \infty \) then for all \( g \in \text{Lip} \) with \( g(0) = 0 \) and \( \text{Lip}(g) = 1 \),

\[ N^{d/2} \left( \frac{1}{N^d} \int_0^N dg(u(t,x)) dx - E[g(u(t,0))] \right) \Rightarrow N(0, \tau^2 g) \text{ as } N \to \infty. \]
A Hierarchy of conditions

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad \mid \quad u(0, x) \equiv 1 \]

\[ \text{Cov}[\eta(t, x), \eta(s, y)] = \delta_0(t - s)f(x - y) \quad \mid \quad \int_{\mathbb{R}^d} \hat{f}(dx)/(1 + \|x\|^2) < \infty \cdots (D) \]

1. (D): \( \int_{\mathbb{R}^d} \frac{\hat{f}(dz)}{1 + \|z\|^2} < \infty \) implies spatial ergodicity

2. (D) + \( \lim_{\|z\| \to \infty} \int_{\mathbb{R}^d} \frac{e^{iz \cdot x} \hat{f}(dz)}{1 + \|z\|^2} = 0 \) implies mixing

**Theorem (Chen-K-Nualart-Pu 2020+)**

If \( 0 < f(\mathbb{R}^d) < \infty + (D) \), then for all \( g \in \text{Lip} \) with \( g(0) = 0 \) and \( \text{Lip}(g) = 1 \),

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3. All are NAS when \( \sigma \propto 1; \exists \text{FCLTs}; \) and the CLT holds in total variation!
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*If* \( 0 < f(\mathbb{R}^d) < \infty + (D) \), *then for all* \( g \in \text{Lip} \) *with* \( g(0) = 0 \) *and* \( \text{Lip}(g) = 1 \),

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**Theorem (Chen-K-Nualart-Pu 2020+)**

If \( 0 < f(\mathbb{R}^d) < \infty \) + (D), then for all \( g \in \text{Lip} \) with \( g(0) = 0 \) and \( \text{Lip}(g) = 1 \),

\[ N^{d/2} \left( \frac{1}{N^d} \int_{[0,N]^d} g(u(t, x)) \, dx - \mathbb{E}[g(u(t, 0))] \right) \Rightarrow \mathcal{N}(0, \tau_g^2) \quad \text{as} \ N \to \infty. \]

3. All are NAS when \( \sigma \propto 1 \); \( \exists \) FCLTs; and the CLT holds in total variation!

4. The assumption \( f(\mathbb{R}^d) < \infty \) can be dropped sometimes, but the \( N^{d/2} \)-rate of CLT can then change

5. \( g \in \text{Lip} \) can be extended in various directions
The effect of initial data

\[ \partial_t u(t, x) = \frac{1}{2} \partial_x^2 u(t, x) + u(t, x) \eta(t, x) \quad \mid \quad u(0) = \delta_0, \ d = 1 \]

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Suppose instead that \( u(0) = \delta_0 \), and let us specialize to \( f = \delta_0 \) and \( d = 1 \) [we understand \( d > 1 \) as well, to a very good extent]
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**Proposition (Amir-Corwin-Quastel 2011)**

\[ x \mapsto U(t, x) = \frac{u(t, x)}{p_t(x)} \text{ is stationary for all } t > 0, \text{ where } p_t(x) = \frac{e^{-x^2/(2t)}}{\sqrt{2\pi t}}. \]
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**Theorem (Chen-K-Nualart-Pu 2020+)**

\( U(t) \) is weakly mixing \( \forall t > 0 \), and \( S_{N,t} := N^{-1} \int_0^N [U(t, x) - 1] \, dx \) satisfies

\[ \sqrt{\frac{N}{\log N}} S_{N,t} \xrightarrow{TV} N(0, 2t) \quad \text{as } N \to \infty. \]
The effect of initial data

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad \mid \quad u(0) = \delta_0, \quad d \geq 1 \]

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**Theorem (K-Nualart-Pu 2020+)**

\[ \exists \text{ CLT in TV. And:} \]

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Cov[\( \eta(t, x), \eta(s, y) \)] = \( \delta_0(t-s)f(x-y) \) \quad | \quad \int_{\mathbb{R}^d} \hat{f}(dx)/(1 + \|x\|^2) < \infty \cdots (D)

Theorem (K-Nualart-Pu 2020+)

\[ \exists \text{ CLT in TV. And:} \]

1. If \( d = 1 \) and \( f(\mathbb{R}) < \infty \), then \( \text{Var}(S_{N,t}) \asymp N^{-1} \log N \). Moreover:

\[ \int_{\mathbb{R}^d} \hat{f}(dx)/(1 + \|x\|^2) < \infty \cdots (D) \]
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   (b) If \( f \) is a Rajchman measure, then \( \text{Var}(S_{N,t}) \sim t f(\mathbb{R}) N^{-1} \log N \).
The effect of initial data

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad \mid u(0) = \delta_0, \; d \geq 1 \]

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2. If \( d \geq 2 \), then \( \mathcal{R}(f) := \lim_{N \to \infty} N\text{Var}(S_{N,t}) \) exists and is in \((0, \infty] \).
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\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad | \quad u(0) = \delta_0, \ d \geq 1 \]

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**Theorem (K-Nualart-Pu 2020+)**

∃ CLT in TV. And:

1. If \( d = 1 \) and \( f(\mathbb{R}) < \infty \), then \( \text{Var}(S_{N,t}) \asymp N^{-1} \log N \). Moreover:
   - (a) If \( f = c\delta_0 \) for some \( c > 0 \) then \( \text{Var}(S_{N,t}) \sim 2tf(\mathbb{R})N^{-1} \log N \)
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2. If \( d \geq 2 \), then \( R(f) := \lim_{N \to \infty} N\text{Var}(S_{N,t}) \) exists and is in \((0, \infty]\).

3. If \( d \geq 2 \), then \( R(f) < \infty \iff \int_{\mathbb{R}^d} \frac{f(dx)}{\|x\|^{d-1}} < \infty \iff \int_{\mathbb{R}^d} \frac{\hat{f}(dx)}{\|x\|} < \infty \);
The effect of initial data

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad \text{with} \quad u(0) = \delta_0, \quad d \geq 1 \]

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1. **If** \( d = 1 \) and \( f(\mathbb{R}) < \infty \), \( \text{then} \ Var(S_{N,t}) \asymp N^{-1} \log N \). Moreover:
   
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3. **If** \( d \geq 2 \), \( \mathcal{R}(f) < \infty \iff \int_{\mathbb{R}^d} \frac{f(dx)}{\|x\|^{d-1}} < \infty \iff \int_{\mathbb{R}^d} \frac{\hat{f}(dx)}{\|x\|} < \infty \);

4. **If** \( d \geq 2 \) and \( f(dx) = \|x\|^{-\beta} \) for \( \beta \in (0, d \land 2) \), \( \text{then:} \ \exists \ 3 \text{ different phases:} \)
The effect of initial data

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad \mid \quad u(0) = \delta_0, \quad d \geq 1 \]

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**Theorem (K-Nualart-Pu 2020+)**

If \( d \geq 2 \) and \( f(dx) = \|x\|^{-\beta} dx \) for \( \beta \in (0, d \wedge 2) \) then \( \exists \) CLT in TV, and:
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1. If \( \beta \in (0, 1) \cup (1, d \wedge 2) \), then \( \text{Var}(S_{N,t}) \sim CN^{-\beta^\wedge(2-\beta)} \).
The effect of initial data

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If \( d \geq 2 \) and \( f(dx) = \|x\|^{-\beta} dx \) for \( \beta \in (0, d \wedge 2) \) then \( \exists \) CLT in TV, and:

1. If \( \beta \in (0, 1) \cup (1, d \wedge 2) \), then \( \text{Var}(S_{N,t}) \sim CN^{-\beta \wedge (2-\beta)} \);
2. If \( \beta = 1 < d \wedge 2 \), then \( \text{Var}(S_{N,t}) \sim C'N^{-1} \log N \).
We know of various such results, unfortunately all different at the technical level. But there are high-level proofs that one implements differently in different settings:

\( F \approx E_F \) if \( \text{Var}(F) \approx 0 \)

We will soon see how we can approximate \( \text{Var}(F) \) using Malliavin calculus (Poincaré ineq.)

\( Nourdin-Peccati (2011) \) have proved that if \( F = \int v \, d\eta \in D^1,2 \) has variance one, then
\[
\text{dTV}(F, N(0,1)) \leq 2 \sqrt{\text{Var} \langle DF, v \rangle} H,
\]
where \( H \) denotes the Hilbert space associated to the cov of noise \( \eta \)

\( E\langle DF, v \rangle = \text{Var} F = 1 \) [Gaussian integration by parts]
We know of various such results, unfortunately all different at the technical level. But there are high-level proofs that one implements differently in different settings:

**Ergodicity/Weak mixing**

1. For ergodicity we need that $F \approx E_F$ if $\text{Var}(F) \approx 0$.
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**CLT in TV**

(Malliavin–Stein method)

1. Nourdin-Peccati (2011) have proved that if $F = \int v \, d\eta \in D_1^1$, has variance one, then $d_{\text{TV}}(F, N(0, 1)) \leq 2 \sqrt{\text{Var} \langle DF, v \rangle}_{\mathcal{H}}$, where $\mathcal{H}$ denotes the Hilbert space associated to the cov of noise.
2. $\langle DF, v \rangle_{\mathcal{H}} = \text{Var} F = 1$ [Gaussian integration by parts].

Common point: $\text{Var} \cdots \ll 1$.
We know of various such results, unfortunately all different at the technical level. But there are high-level proofs that one implements differently in different settings:

1. Ergodicity/Weak mixing

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2. CLT in TV

   (Malliavin–Stein method)

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   where $H$ denotes the Hilbert space associated to the cov of noise $\eta$

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Ergodicity/Weak mixing

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**CLT in TV (Malliavin–Stein method)**

(a) Nourdin-Peccati (2011) have proved that if $F = \int v \, d\eta \in \mathbb{D}^{1,2}$ has variance one, then

$$d_{TV}(F, N(0, 1)) \leq 2 \sqrt{\text{Var}\langle DF, v\rangle_{\mathcal{H}}},$$

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(b) $E\langle DF, v \rangle_{\mathcal{H}} = \text{Var}F = 1$ [Gaussian integration by parts]
Ergodicity & CLT in TV

1. We know of various such results, unfortunately all different at the technical level. But there are high-level proofs that one implements differently in different settings:

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   (b) We will soon see how we can approximate $\text{Var}(F)$ using Malliavin calculus (Poincaré ineq.)

3. CLT in TV (Malliavin–Stein method)
   (a) Nourdin-Peccati (2011) have proved that if $F = \int v \, d\eta \in D^{1,2}$ has variance one, then

   $$d_{TV}(F, N(0,1)) \leq 2 \sqrt{\text{Var} \langle DF, v \rangle_{\mathcal{H}}},$$

   where $\mathcal{H}$ denotes the Hilbert space associated to the cov of noise $\eta$
   (b) $E \langle DF, v \rangle_{\mathcal{H}} = \text{Var}F = 1$ [Gaussian integration by parts]

4. Common point: $\text{Var}(\cdots) \ll 1$
Malliavin Calculus & the Poincaré Inequality

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad | \quad u(0, x) \equiv 1 \]

Cov[\eta(t, x), \eta(s, y)] = \delta_0(t - s) f(x - y) \quad | \quad \int_{\mathbb{R}^d} \hat{f}(dx)/(1 + \|x\|^2) < \infty \cdots (D)

- **A Clark-Ocone Formula** (Chen-K-Nualart-Pu, 19+): \( \forall F \in \mathbb{D}^{1,2}, \)

\[ F = E[F] + \int_{(0, \infty) \times \mathbb{R}^d} E[D_{s,z}F | \mathcal{F}_s] \eta(ds \, dz) = E[F] + \int v \, d\eta \]
Malliavin Calculus & the Poincaré Inequality

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad | \quad u(0, x) \equiv 1 \]
\[ \text{Cov}[\eta(t, x), \eta(s, y)] = \delta_0(t-s)f(x-y) \quad | \quad \int_{\mathbb{R}^d} \hat{f}(dx)/(1 + \|x\|^2) < \infty \cdots (D) \]

- A Clark-Ocone Formula (Chen-K-Nualart-Pu, 19+): \( \forall F \in D^{1,2}, \)

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\[ : \text{If } \text{Var}(DF, v) \ll 1, \text{ then } F - E[F] \approx \text{normal, in TV} \]
Malliavin Calculus & the Poincaré Inequality

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad \text{with} \quad u(0, x) \equiv 1 \]

\[ \text{Cov}[\eta(t, x), \eta(s, y)] = \delta_0(t - s) f(x - y) \quad \text{and} \quad \int_{\mathbb{R}^d} \hat{f}(dx)/(1 + \|x\|^2) < \infty \cdots (D) \]

- A Clark-Ocone Formula (Chen-K-Nualart-Pu, 19+): \( \forall F \in \mathbb{D}^{1,2} \),

\[ F = EF + \int_{(0,\infty) \times \mathbb{R}^d} \mathbb{E} [D_s,F \mid \mathcal{F}_s] \eta(ds \, dz) = EF + \int v \, d\eta \]

\[ \therefore \text{If } \text{Var}(DF, v)_{\mathcal{H}} \ll 1, \text{ then } F - EF \approx \text{normal, in TV} \]

- E.g., if \( f = \delta_0 \) and \( d = 1 \) (space-time white noise), then Poincaré ineq.

\[ \text{Var}(F) = \int_0^\infty ds \int_{-\infty}^\infty dz \| \mathbb{E} [D_s,F \mid \mathcal{F}_s] \|^2 \]
Malliavin Calculus & the Poincaré Inequality

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad \mid \quad u(0, x) \equiv 1 \]

\[ \text{Cov}[\eta(t, x), \eta(s, y)] = \delta_0(t - s) f(x - y) \quad \mid \quad \int_{\mathbb{R}^d} \hat{f}(dx)/(1 + \|x\|^2) < \infty \quad \cdots (D) \]

- A Clark-Ocone Formula (Chen-K-Nualart-Pu, 19+): \( \forall F \in \mathbb{D}^{1,2}, \)
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- \( \therefore \) If \( \text{Var} \langle DF, v \rangle_{\mathcal{H}} \ll 1, \) then \( F - EF \approx \text{normal, in TV} \)
- E.g., if \( f = \delta_0 \) and \( d = 1 \) (space-time white noise), then \( \text{Poincaré ineq.} \)

\[ \text{Var}(F) = \int_0^\infty ds \int_{-\infty}^\infty dz \| \mathbb{E} [D_{s,z}F \mid \mathcal{F}_s] \|^2 \]
A Clark-Ocone Formula (Chen-K-Nualart-Pu, 19+): \( \forall F \in \mathcal{D}^{1,2}, \)

\[
F = EF + \int_{(0,\infty) \times \mathbb{R}^d} E[D_{s,z}F \mid \mathcal{F}_s] \eta(ds \, dz) = EF + \int v \, d\eta
\]

\[
\therefore \text{If } \text{Var}(DF, v) \ll 1, \text{ then } F - EF \approx \text{normal, in TV}
\]

E.g., if \( f = \delta_0 \) and \( d = 1 \) (space-time white noise), then Poincaré ineq.

\[
\text{Var}(F) = \int_0^\infty ds \int_{-\infty}^\infty dz \left\| E[D_{s,z}F \mid \mathcal{F}_s] \right\|^2 \leq E \left( \|DF\|^2_{L^2(\mathbb{R}_+ \times \mathbb{R})} \right)
\]
Malliavin Calculus & the Poincaré Inequality

\[ \partial_t u(t, x) = \frac{1}{2} \Delta u(t, x) + \sigma(u(t, x)) \eta(t, x) \quad | \quad u(0, x) \equiv 1 \]

\[ \text{Cov}[\eta(t, x), \eta(s, y)] = \delta_0(t - s) f(x - y) \quad | \quad \int_{\mathbb{R}^d} \hat{f}(dx)/(1 + \|x\|^2) < \infty \cdots (D) \]

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\[ \therefore \text{If } \text{Var} \langle DF, v \rangle_{\mathcal{H}} \ll 1 \text{, then } F - EF \approx \text{normal, in TV} \]

- E.g., if \( f = \delta_0 \) and \( d = 1 \) (space-time white noise), then Poincaré ineq.

\[ \text{Var}(F) = \int_0^\infty ds \int_{-\infty}^\infty dz \, \| \mathbb{E} [D_{s, z} F \mid \mathcal{F}_s] \|_2^2 \leq \mathbb{E} \left( \|DF\|_{L^2(\mathbb{R}_+ \times \mathbb{R})}^2 \right) \]

- More generally: \( \forall F \in D^{1,2} \),

\[ \text{Var}(F) \leq \mathbb{E} \left( \|DF\|_{H}^2 \right) . \]
Thank you!