Stochastic α-Navier-Stokes model

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joint work with Luigi Manca,

collaboration with Letournel, Laurent-Nègre, Mercier, Massot, Richard, Vié, Zamansky.

CNRS & Fédération de Mathématiques de CentraleSupélec & Université Paris-Saclay

Virtual Seminar on Stochastic Analysis, Random Fields and Applications

from my office

SIMALIN project funded by French National Research Agency

Why the α-Navier-Stokes model ?

What about the physicists ?

The stochastic α-Navier-Stokes model

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Fluid dynamics

Chosen model should satisfy many properties

- phenomenological equation
- existence and uniqueness of solution
- "easy" to simulate numerically
- physical properties conserved
- many statistics of fluid respected (vorticity structures, alignment, two point statistics, flatness, etc.)
- possibility to add particles (tracer or inertial)
- transport of scalar quantities

The famous equation

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u + f$$

satisfies many properties, but...

- very difficult to simulate all the scales

  $\rightarrow$ microscopic structures and local regularities impact strongly the whole dynamic (from a mathematical point of view)
- uniqueness of global weak solution is open (also persistence or blow-up of strong solution)

Related works and models

The necessity to simplify the model has lead to many other works and models.

Fact: A precise description of the fine scales (in a turbulent flow) may be irrelevant for numerical simulation in engineering applications, thus a regularization of the Navier-Stokes equation could be meaningful (mathematically, physically and numerically).

Ideas:

- temporal averaging
- Lagrangian averaging
- Leray regularization
- adding viscosity to other model
- adding hyper-viscosity to Navier-Stokes equation
- deconvolution model of turbulence
- adding supplementary terms
- and the nowadays famous "regularization by noise"

The α-Navier-Stokes equation

Sometimes there are different names for the same model, but not strictly equivalence of all the models...

- (LANS-α) Lagrangian averaging $\rightarrow$ Marsden-Shkoller (2001) Coutand-Peirce-Shkoller (2002) Caraballo-Real-Taniguchi (2006) Deugoue-Sango (2009)
- (viscous Camassa-Holm) adding viscosity to other model $\rightarrow$ Foais-Holm-Titi (2002)

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- Leray regularization

$\rightarrow$ Guermond-Oden-Prudhomme (2003) (perturbation - frame invariant)

$\rightarrow$ Rebholz (2008) (higher orders)

$\rightarrow$ Li-Liu-Xie (2018) with ergodicity for fractional dissipation and noise.

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- temporal averaging $\rightarrow$ Chen-Foias-Holm-Olson-Titi-Wynne (1999)
- adding hyper-viscosity to Navier-Stokes equation
- deconvolution model of turbulence $\rightarrow$ Layton-Lewandowski (2008)
- adding supplementary terms

$\rightarrow$ $|u|^{2\alpha} r$ like Brickman-Forchheimer $\rightarrow$ Bessaih-Trabelsi-Zorgati (2016) and Bessaih-Millet (2018) for stochastic

$\rightarrow$ $u \times e_3$ as rotating term $\rightarrow$ Chemin-Desjardin-Gallagher-Grenier (2006)

$\rightarrow$ $\sim |u|^{r-1} r$ like Kalantarov-Zelik (2012) and Rockner-Zhang-Zhang (2009-2010) for the tamed stochastic NS.

- Navier-Stokes-Voigt, Clark-α, etc.

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Numerical simulations of the Navier-Stokes equation

(DNS) We know how to simulate the Navier-Stokes equation on a very fine grid in a very large domain with huge amplitude in the scales of phenomenon.

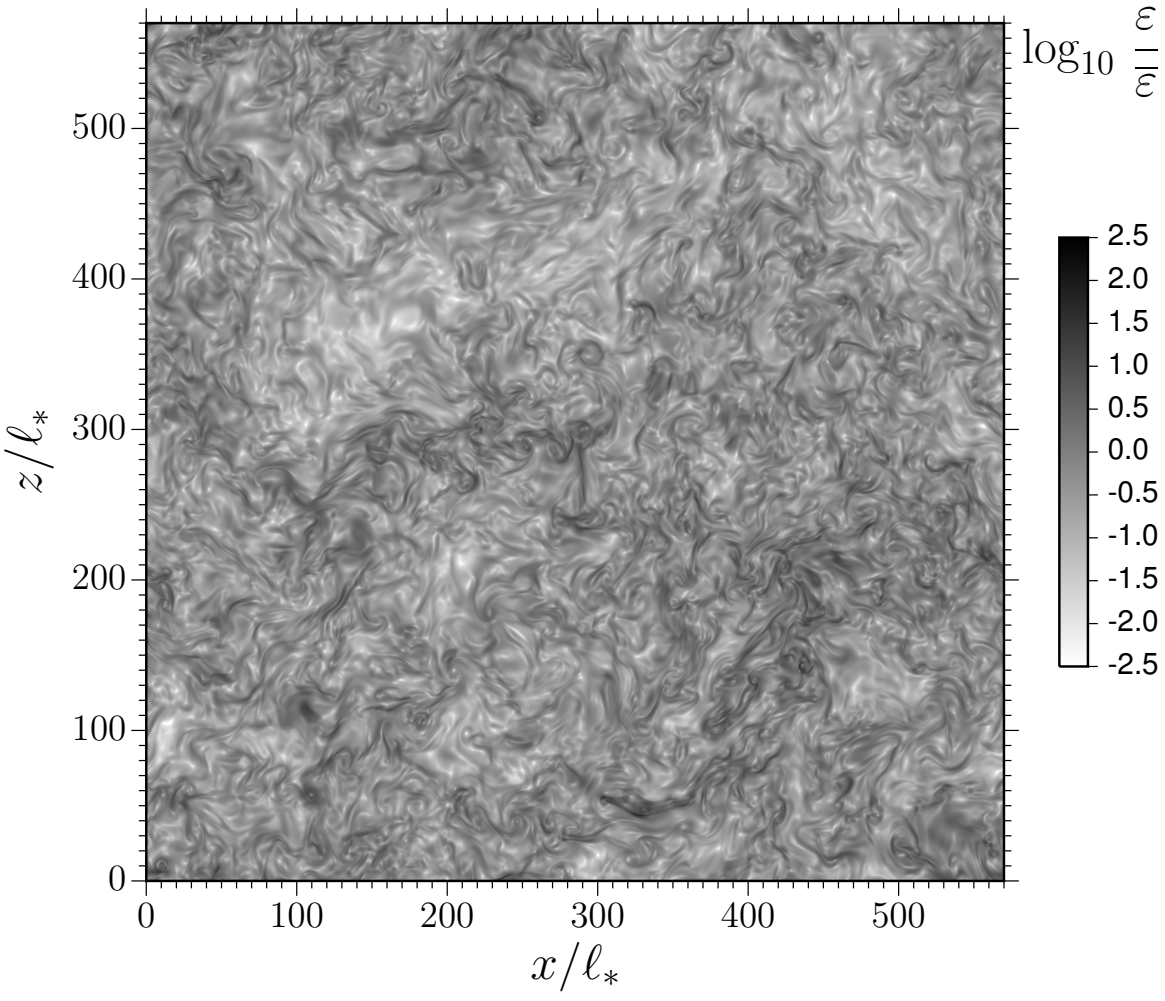


Figure: Dissipation field - credits: Rémi Zamansky

Numerical simulations of the Navier-Stokes equation

Coupling with the transport of a scalar value

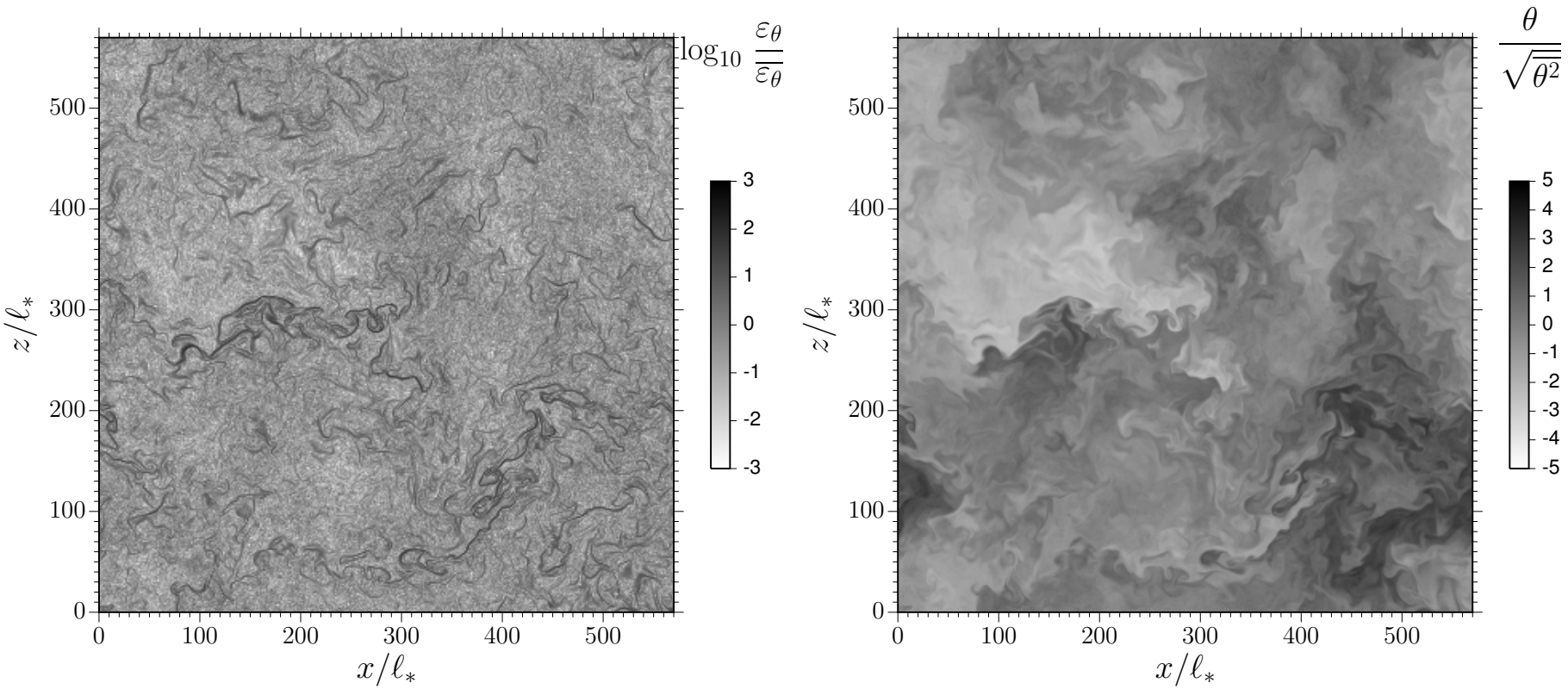


Figure: Scalar dissipation $\varepsilon_\theta := \|\nabla\theta\|$ and renormalized temperature - credits: Rémi Zamansky

Numerical simulations of the Navier-Stokes equation

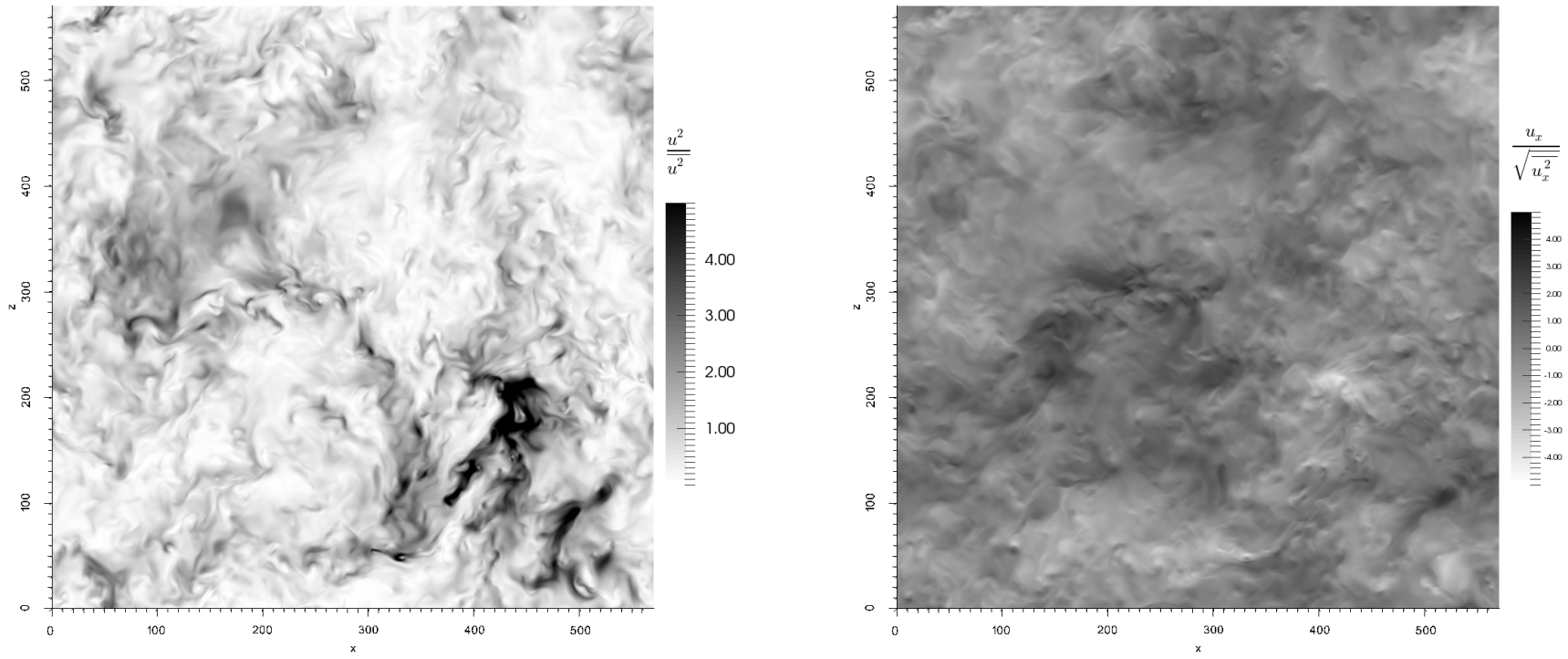
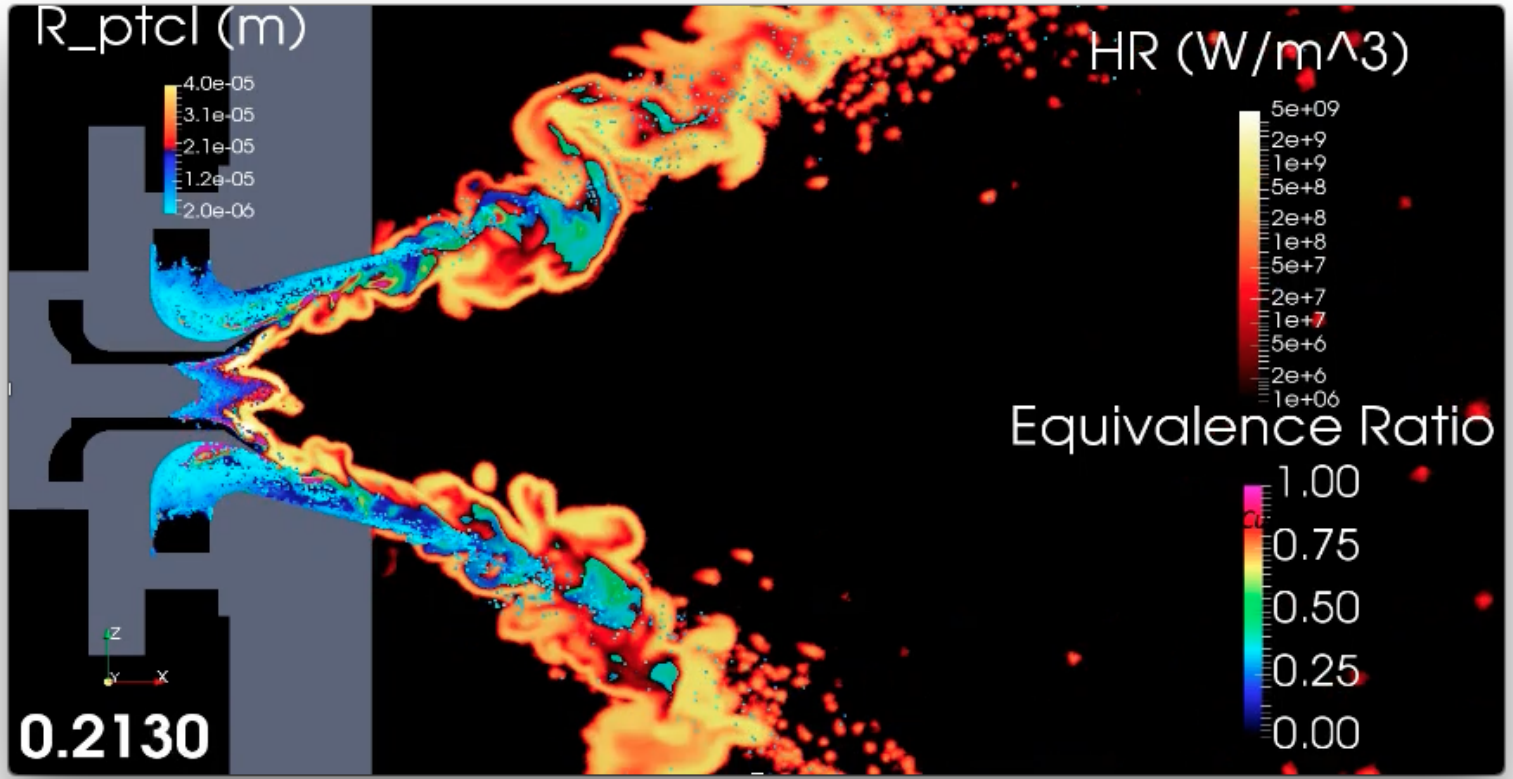


Figure: Kinetic energy and first component of the speed u_1 - credits: Rémi Zamansky

Numerical simulations of the Navier-Stokes equation



Credits: Aymeric Vié.

Large Eddy Simulations

But the cost of such a simulation is not useful for engineering applications.

LES: Apply a "filter" to keep only the large eddies in the simulation.

- Gaussian filter $\bar{u}(x) := \int_{\mathcal{O}} \phi_{x,\sigma}(y) u(y) dy$
- Fourier cut-off = low pass filter $\bar{u}(x) := \int_{\mathcal{O}} sinc(k \cdot (x-y)) u(y) dy$
- Window averaging $\bar{u}(x) := \frac{1}{\lambda([-a,+a])} \int_{\mathcal{O}} 1_{[-a,+a]}(x-y) u(y) dy$

Closure of equations in LES model

But the equation is not closed. Start from the Navier-Stokes equation

$$\partial_t u - \nu \Delta u + \nabla \cdot (u \otimes u) + \frac{1}{\rho} \nabla p = 0,$$

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apply (linear, commutative) filter

$$\partial_t \bar{u} - \nu \Delta \bar{u} + \overline{\nabla \cdot (u \otimes u)} + \frac{1}{\rho} \nabla \bar{p} = 0,$$

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express the equation with only the variable $\bar{u}$

$$\partial_t \bar{u} - \nu \Delta \bar{u} + \overline{\left[\nabla \cdot \left((\bar{u})^{\#} \otimes (\bar{u})^{\#} \right) \right]} + \frac{1}{\rho} \nabla \bar{p} = 0,$$

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leads to the problem of inverting the filter $x \mapsto \bar{x}$ or at least computing the pre-image set $v^{\#} := \{u \in \mathbb{H} : \bar{u} = v\}$.

Closure of equations in LES model

The classical approach is to close the equation with a model for the right-hand-side term (error term = subgrid scale stress tensor)

$$\partial_t \bar{u} - \nu \Delta \bar{u} + \overline{\nabla \cdot \left(u \otimes u \right)} + \frac{1}{\rho} \nabla \bar{p} = 0$$

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$$\partial_t \bar{u} - \nu \Delta \bar{u} + \nabla \cdot \left(\bar{u} \otimes \bar{u} \right) + \frac{1}{\rho} \nabla \bar{p} = \nabla \cdot \left(\bar{u} \otimes \bar{u} \right) - \overline{\nabla \cdot \left(u \otimes u \right)}$$

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$$\partial_t \bar{u} - \nu \Delta \bar{u} + \nabla \cdot \left(\bar{u} \otimes \bar{u} \right) + \frac{1}{\rho} \nabla \bar{p} = \mathcal{F}(\bar{u}, \bar{u})$$

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$$\partial_t \bar{u} - \nu \Delta \bar{u} + \nabla \cdot \left(\bar{u} \otimes \bar{u} \right) + \frac{1}{\rho} \nabla \bar{p} = \dot{W}$$

Statistics are not correct

With a complex model for the error term, we can recover suitable dynamic, but statistics are not correct. Some progress have to be made in the choice of the filter and the closure term.

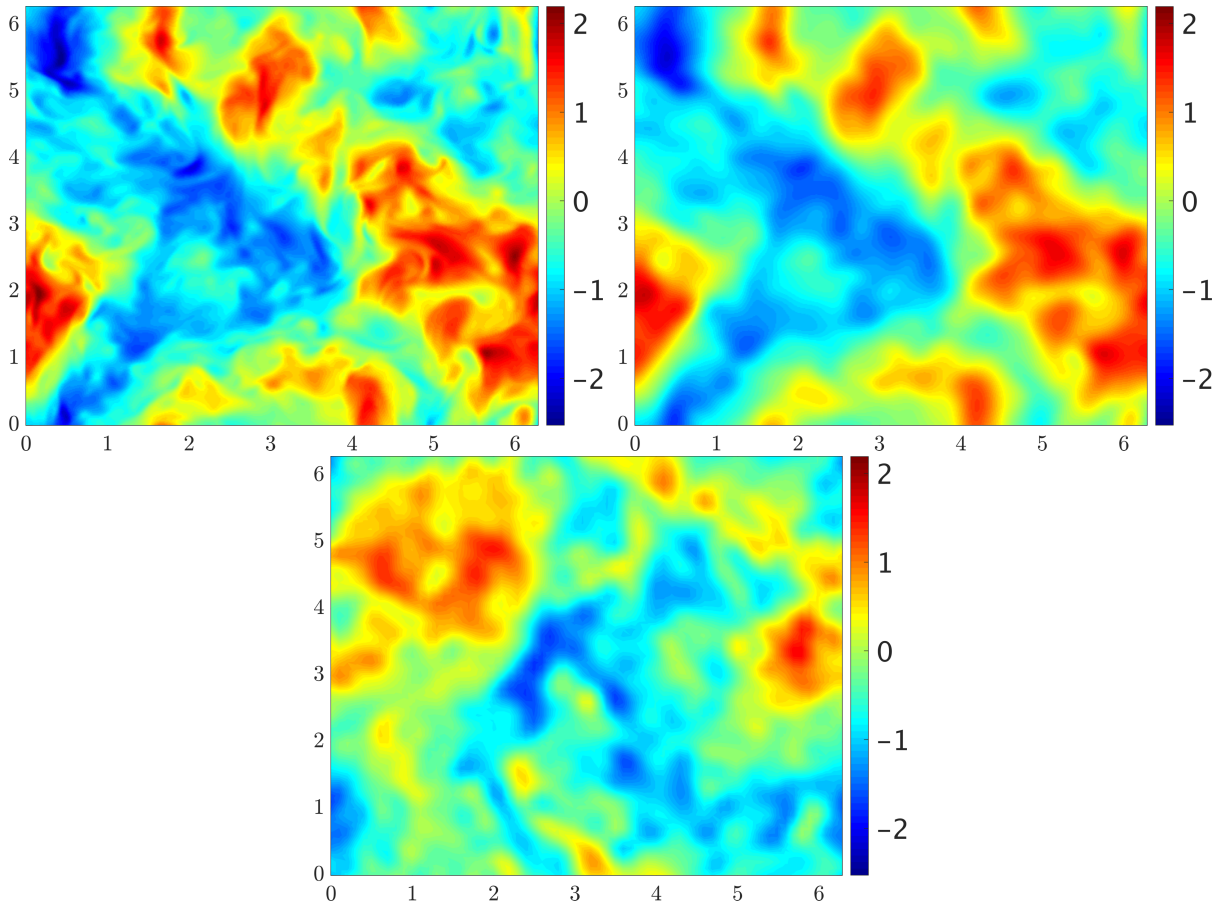

Figure: top: DNS and filtered DNS on a 1024×1024 grid. bottom: LES on a 64×64 grid. Credits: Roxane Letournel

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A two variables model

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the α-Navier-Stokes model for an incompressible fluid u is given by

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + \bar{u} \cdot \nabla u + (\nabla \bar{u})^T \cdot u = -\nabla p + f, \\ \operatorname{div} \bar{u} = 0, \\ \bar{u} = \left(I - \alpha^2 \Delta\right)^{-1} u, \end{cases}$$

where we consider a periodic fluid in a box, or a fluid with homogeneous Dirichlet condition, and the initial data is given by $u|_{t=0} = u_0$.

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where we consider a periodic fluid in a box, or a fluid with homogeneous Dirichlet condition, and the initial data is given by $u|_{t=0} = u_0$.

We can use a dynamic pressure $\tilde{p}$ to obtain the α-Navier-Stokes model in rotational form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u - \bar{u} \times (\nabla \times u) = -\nabla \tilde{p} + f, \\ \operatorname{div} \bar{u} = 0, \\ \bar{u} = \left(I - \alpha^2 \Delta\right)^{-1} u. \end{cases}$$

Physical properties

It has been shown that this model possesses a lot of physical properties (conservation laws for energy and momentum), and it is also suitable for numerical simulation.

For instance, these equations possess a Kelvin-Noether circulation theorem and conserves helicity (see Foias-Holm-Titi 2001) which is defined on a volume V by:

$$\mathcal{H}(V)(t) = \int_V u(t,x) \cdot (\nabla \times u(t,x)) \, dx^3.$$

There are two equations on the vorticity $q := \nabla \times u$ and the helicity $\mathcal{H} := \langle u, \nabla \times u \rangle = \langle u, q \rangle$ which are given by

$$\begin{cases} \dfrac{\partial q}{\partial t} + \bar{u} \cdot \nabla q - q \cdot \nabla \bar{u} = \nu \Delta q + \nabla \times f, \\ \\ \dfrac{1}{2} \dfrac{dH(t)}{dt} = \displaystyle\int_V \nu \Delta u \cdot q \, dx^3 + \int_V (\nabla \times f) \cdot (\nabla \times u) \, dx^3. \end{cases}$$

This α-Navier-Stokes model must not be confused with Leray-α regularization of the Navier-Stokes, or a hyper-viscous Navier-Stokes equation (bi-Laplacian) since these last models do not provide the same physical properties.

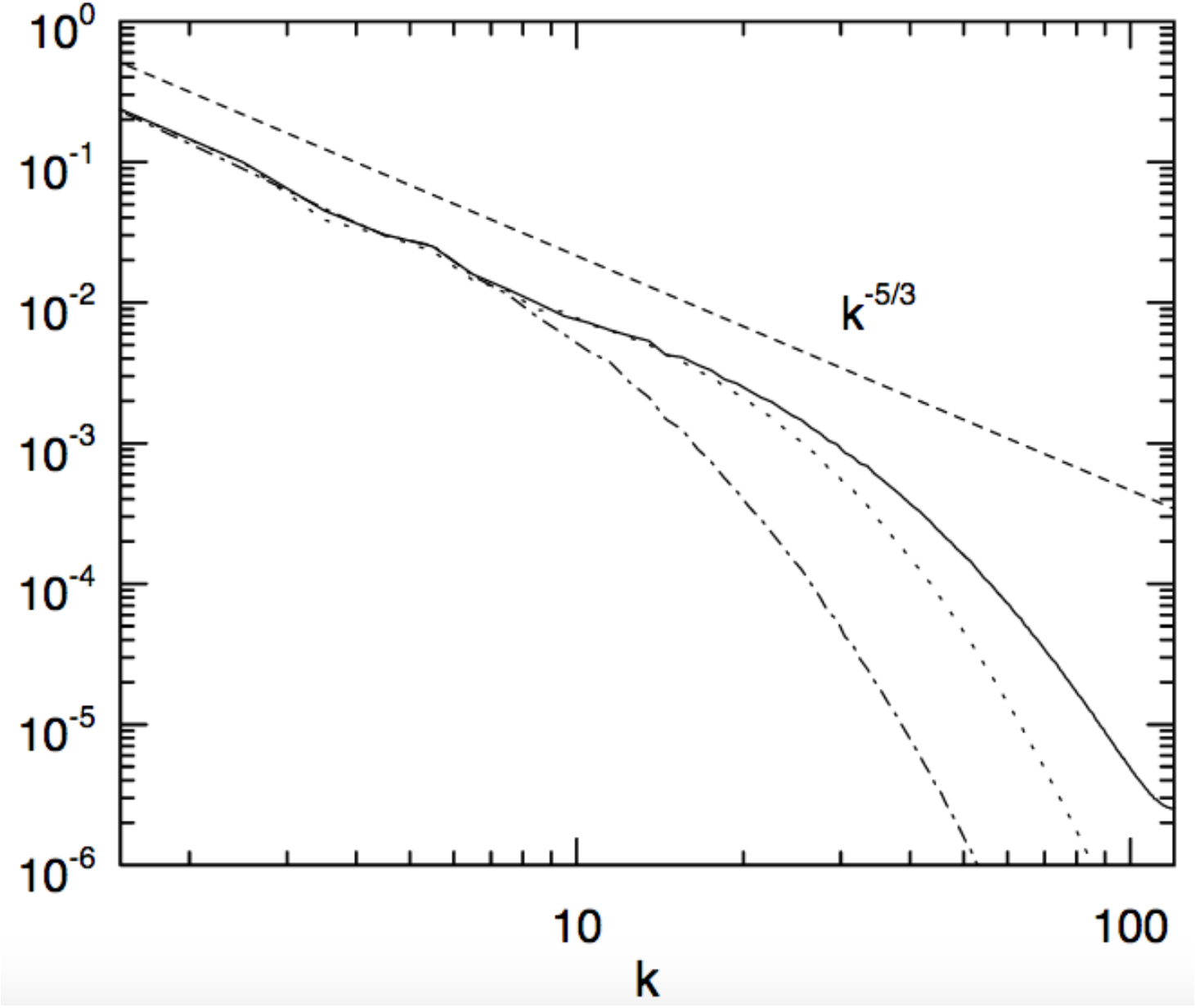

Figure: Energy spectrum $E(k)$ versus the wave number k. α-Navier-Stokes for $\alpha = 0$ (solid line),1/32 (dotted line) and 1/8 (dotted-dash line). Credits: Chen-Holm-Margolin-Zhang.

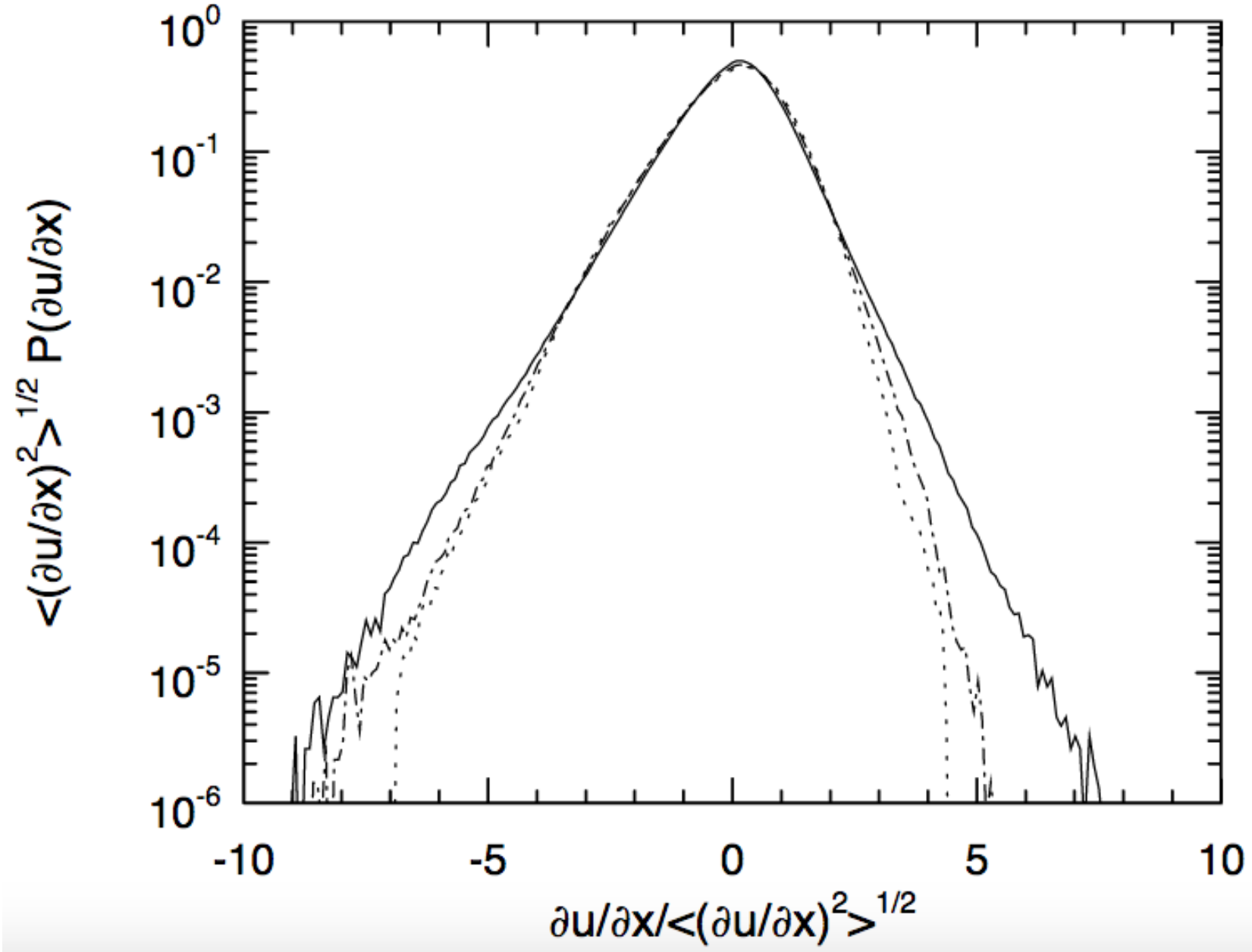

Figure: Normalized PDFs of the longitudinal velocity gradient $\partial u/\partial x$.
α-Navier-Stokes for $\alpha = 0$ (solid line),1/32 (dotted line) and 1/8 (dotted-dash line).
Credits: Chen-Holm-Margolin-Zhang.

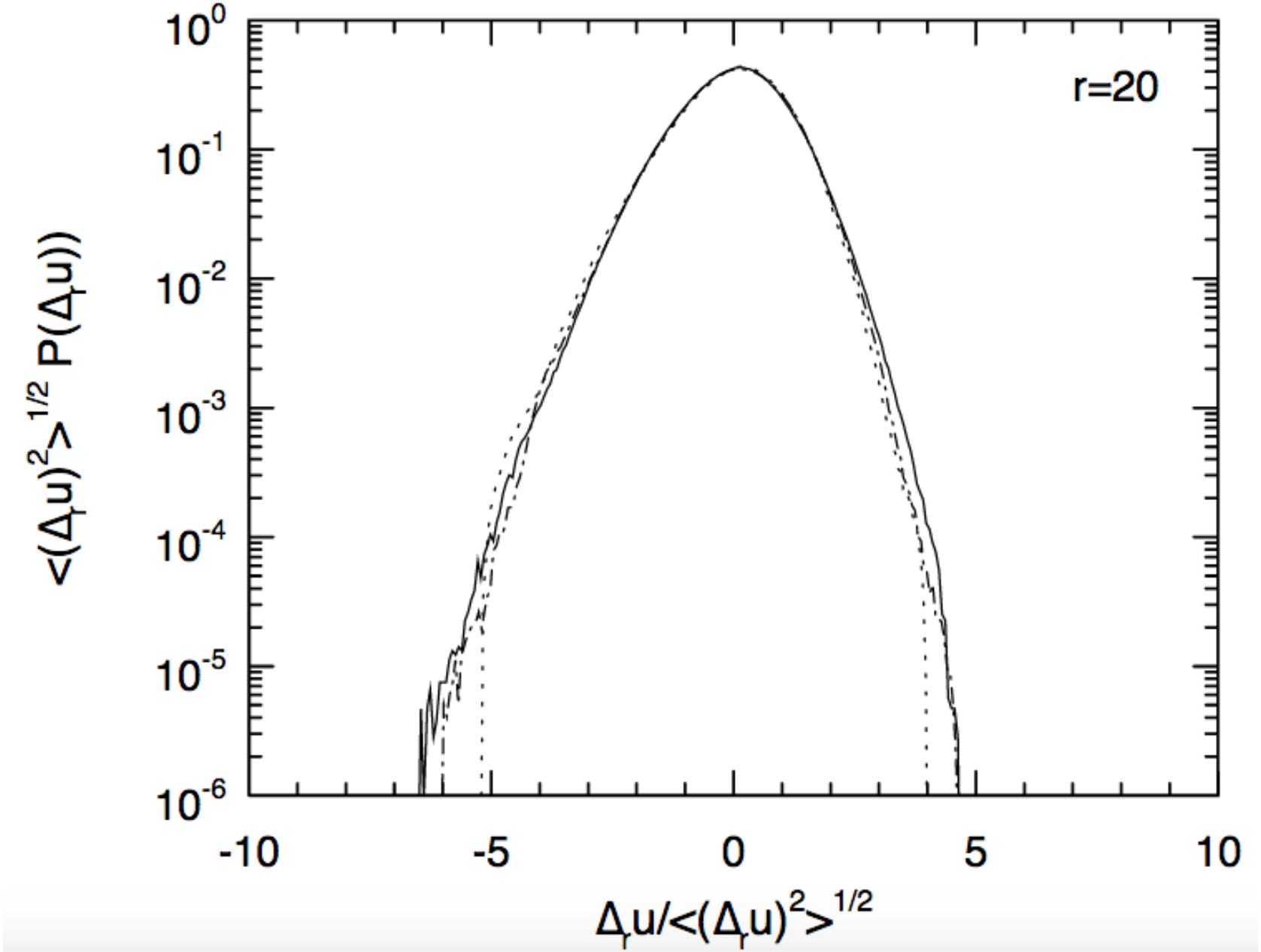

Figure: Normalized PDFs of the velocity increment in the inertial range. α-Navier-Stokes for $\alpha = 0$ (solid line),1/32 (dotted line) and 1/8 (dotted-dash line). Credits: Chen-Holm-Margolin-Zhang.

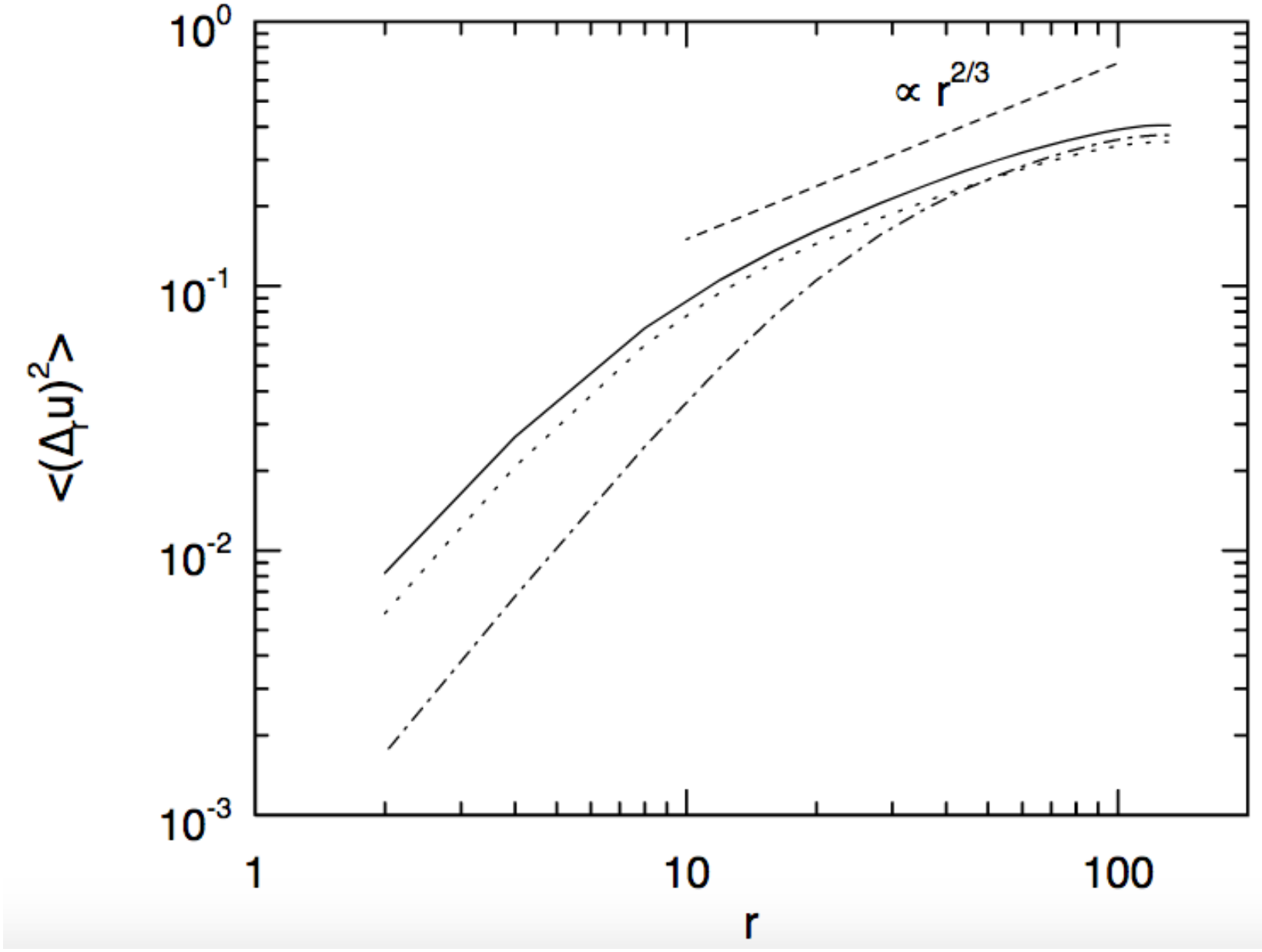

Figure: The second order structure function as a function of r. α-Navier-Stokes for $\alpha = 0$ (solid line),1/32 (dotted line) and 1/8 (dotted-dash line). Credits: Chen-Holm-Margolin-Zhang.

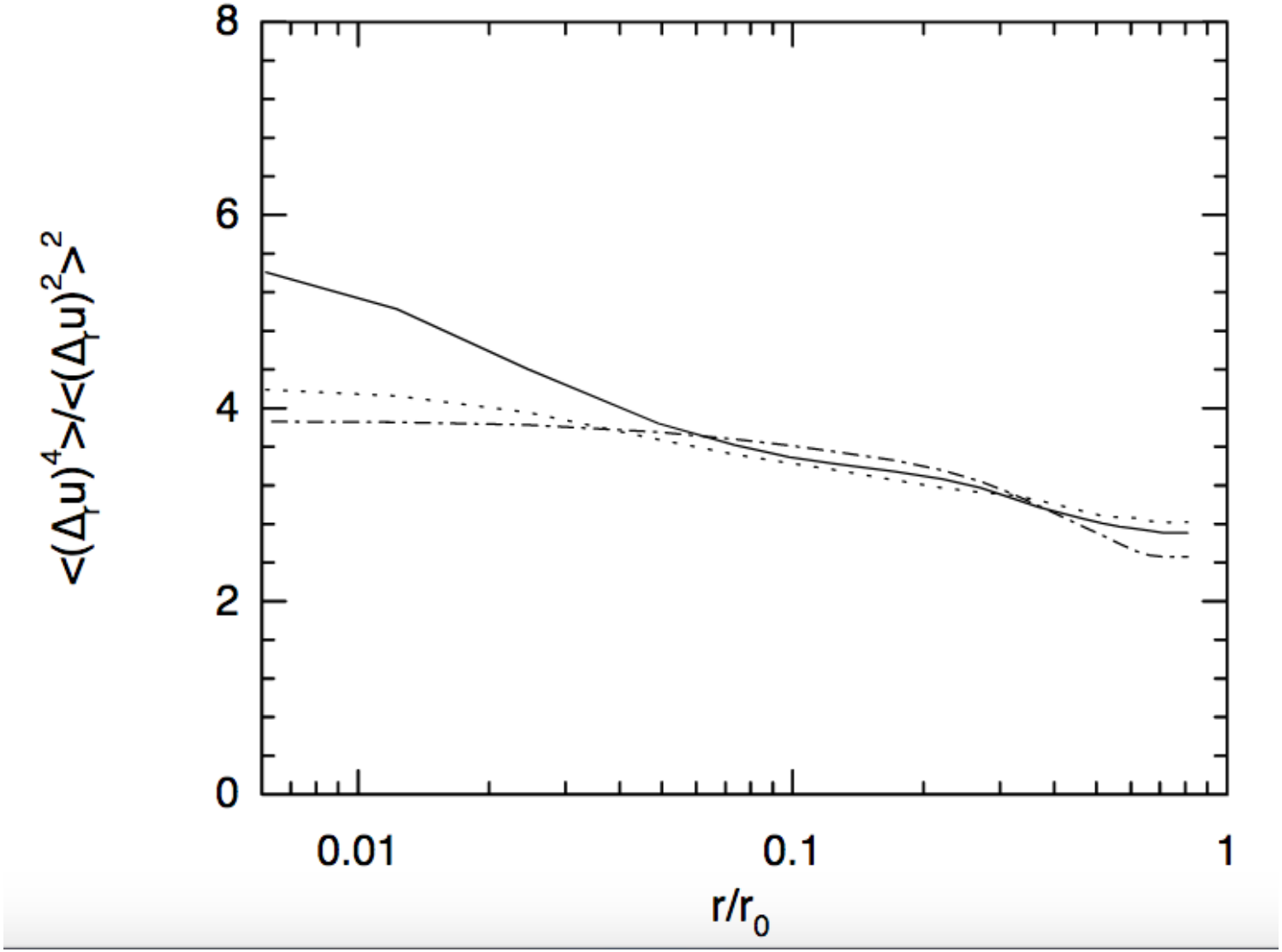

Figure: The flatness, $\langle(\Delta_r u)^4\rangle/\langle(\Delta_r u)^2\rangle^2$ as a function of separation r/r_0. α-Navier-Stokes for $\alpha = 0$ (solid line),1/32 (dotted line) and 1/8 (dotted-dash line). Credits: Chen-Holm-Margolin-Zhang.

The α-Navier-Stokes model for the filtered variable !

Choosing the filtered variable $\bar{u}$ as key variable, we can write the stochastic α-Navier-Stokes equation in the abstract form

$$\begin{cases} d\bar{u} + \left(\nu A\bar{u} + (I + \alpha^2 A)^{-1}\widetilde{B}(\bar{u}, \bar{u} + \alpha^2 A\bar{u})\right) dt = d\xi \quad \text{in } \mathrm{H}, \\ \bar{u}(0) = \bar{u}_0 \in \mathrm{H}, \end{cases} \tag{1}$$

where ξ is a noise term, and with the property that if $\alpha = 0$, this is equivalent to stochastic Navier-Stokes

$$\begin{cases} du + (\nu Au + B(u, u))\, dt = d\xi \quad \text{in } \mathrm{H}, \\ u(0) = u_0 \in \mathrm{H}. \end{cases} \tag{2}$$

Previous work

- Caraballo-Real-Taniguchi (2006)

→ have proven the existence and uniqueness of variational solutions

→ with multiplicative noise $G(u)dW$ assuming G Lipschitz in H-S space.

→ different abstract formulation

→ only estimation of the fourth moments in the random space $\mathrm{L}^4(\Omega)$

- Deugoue and Sango (2009)

→ have relaxed the Lipschitz condition (but conserving the sub-linearity)

→ existence of weak solutions

→ uniqueness is satisfied by assuming again some Lipschitz conditions

→ estimations for all the moments of the solution, but the noise only appears in a finite number of modes

- G. and Manca (2020)

→ existence and uniqueness of strong solution

→ moments of any order for an additive infinite-dimensional space-time noise

→ existence and uniqueness of invariant measure

→ exponential moments for the invariant measure

Hypothesis (1)

We assume that the operator Q and A satisfies the following trace class condition

$$\mathrm{Tr}[Q^*(I + \alpha^2 A)Q] < \infty.$$

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The next assumption will be useful when we shall study the uniqueness of an invariant measure for the semigroup associated to the solution of the SPDE.

Essentially, it allows to use the Bismut-Elworthy formula and derive a Strong Feller-type property.

Hypothesis (2)

The operator $Q : K \to \mathrm{H}$ is invertible and $D(A^{3/2}) \subset D(Q^{-1})$.

Existence and uniqueness of solution

Theorem (G.&Manca)

Assume that the linear operator Q satisfies Hypothesis (1).

Let u_0 be a random variable with values on $(\mathrm{V}, \mathcal{B}(\mathrm{V}))$ such that for some $k \geq 1$ it holds

$$\mathbb{E}\left[\left(|u_0|_2^2 + |\nabla u_0|_2^2\right)^k\right] < \infty.$$

Then there exists a unique strong solution $u(t), t \geq 0$ of problem with initial value $u(0) = u_0$.

Moreover, there exists a constant $c > 0$, depending only on k, Q such that for any $T > 0$ it holds

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left(|u(t)|_2^2\right)^k\right] \leq c\left(\mathbb{E}\left[\left(|u_0|_2^2 + \alpha^2|\nabla u_0|_2^2\right)^k\right] + T\right),$$

and

$$\mathbb{E}\left[\int_0^T \left(|u(s)|_2^2 + \alpha^2|\nabla u(s)|_2^2\right)^{k-1}\left(|\nabla u(s)|_2^2 + \alpha^2|Au(s)|_2^2\right)\mathrm{d}s\right]$$
$$\leq c\left(\mathbb{E}\left[\left(|u_0|_2^2 + \alpha^2|\nabla u_0|_2^2\right)^k\right] + T\right).$$

Lemma

Let $u_0 \in \mathrm{V}$ and assume that Hypothesis (1) holds.

Then, for any $n \in \mathbb{N}$, $T > 0$ there exists a solution $u_n \in \mathrm{L}^2([0,T]; D(A))$.

Moreover, for any $k \in \mathbb{N}^*$ there exists a constant $c = c(k, Q) > 0$ such that for any $n \in \mathbb{N}^*$, $t \geq 0$

$$\mathbb{E}\left[\left(|u_n(t)|_2^2 + \alpha^2 |\nabla u_n(t)|_2^2\right)^k\right] + k\nu\mathbb{E}\left[\int_0^t \left(|u_n|_2^2 + \alpha^2|\nabla u_n|_2^2\right)^{k-1}\left(|\nabla u_n|_2^2 + \alpha^2|Au_n|_2^2\right)\mathrm{d}s\right] \leq \mathbb{E}\left[\left(|u_0|_2^2 + \alpha^2|\nabla u_0|_2^2\right)^k\right] + ct.$$

Lemma

Let $u_0 \in \mathrm{V}$ and assume that Hypothesis (1) holds.

Then for any $k \in \mathbb{N}$ there exists $c = c(k, Q) > 0$ such that for any $T > 0$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(|u_n|_2^2 + \alpha^2|\nabla u_n|_2^2\right)^k\right] \leq c\mathbb{E}\left[\left(|u_0|_2^2 + \alpha^2|\nabla u_0|_2^2\right)^k\right] + cT.$$

Exponential moments

The following result will be used to derive some concentration properties for the invariant measure.

Proposition

Let $u_0 \in \mathrm{V}$ and assume that Hypothesis (1) holds.

For any $\varepsilon > 0$ such that $-\nu + 2\lambda_1^{-1}\varepsilon \mathrm{Tr}[Q^*(I + \alpha^2 A)Q] < 0$ there exists $K_\varepsilon > 0$, independent by u_0, such that for any $n \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E}\left[e^{\varepsilon\left(|u_n(t)|_2^2 + \alpha^2 |\nabla u_n(t)|_2^2\right)}\right] \\
&+ \varepsilon\left(\nu - 2\lambda_1^{-1}\varepsilon \mathrm{Tr}[Q^*(I + \alpha^2 A)Q]\right) \\
&\times \int_0^t \mathbb{E}\left[e^{\varepsilon\left(|u_n(s)|_2^2 + \alpha^2 |\nabla u_n(s)|_2^2\right)}\left(|\nabla u_n(s)|_2^2 + \alpha^2 |A u_n(s)|_2^2\right)\right] \mathrm{d}s \\
&\qquad \leq \mathbb{E}\left[e^{\varepsilon\left(|u_0|_2^2 + \alpha^2 |\nabla u_0|_2^2\right)}\right] + K_\varepsilon t.
\end{aligned}$$

Remark

Following a classic strategy we shall show a Strong Feller type property by using the Bismut-Elworthy formula. Indeed, to apply it we need $\ker(Q) = \{0\}$ *and that* $\int_0^T |Q^{-1}\eta_m^h(t,x)|_2^2 \mathrm{d}t$ *is defined.*

Since $\int_0^T |A^{3/2}\eta_m^h(t,x)|_2^2 \mathrm{d}t$ *is bounded, in order to have* $\int_0^T |Q^{-1}\eta_m^h(t,x)|_2^2 \mathrm{d}t$ *defined it is sufficient to have* $D(A^{3/2}) \subset D(Q^{-1})$. *If we set* $Q = A^{-\frac{1}{2}(1+\varepsilon)}$ *this condition is fulfilled if* $\frac{1}{2}(1+\varepsilon) \leq 3/2$, *that is* $\varepsilon \leq 2$.

Moreover, Hypothesis (1) reads

$$\mathrm{Tr}[A^{-\varepsilon}] < \infty.$$

That is, $\varepsilon > d/2$, *where* d *is the dimension of* $\mathcal{U}$.

Then, if $\dim \mathcal{U} = 2$ *or* $\dim \mathcal{U} = 3$, *a covariance operator* Q *of the form* $Q = A^{-\frac{1}{2}(1+\varepsilon)}$ *satisfies Hypothesis (1) and (2) i.e.* $D(A^{3/2}) \subset D(Q^{-1})$ *whenever* $\varepsilon \in]d/2, 2]$.

Theorem (G.&Manca)

Le us assume that Hypothesis (1) and (2) hold.

Then, there exists a unique invariant probability measure μ for the transition semigroup P_t, $t \geq 0$.

Moreover $\mu(D(A)) = 1$ and for any $\varepsilon > 0$ sastisfying $-\nu + 2\varepsilon\lambda_1^{-1}\mathrm{Tr}[Q^*(I + \alpha^2 A)Q] < 0$ there exists $K_\varepsilon > 0$ such that

$$\int_V e^{\varepsilon\left(|x|_2^2 + \alpha^2 |\nabla x|_2^2\right)} \left(|\nabla x|_2^2 + \alpha^2 |Ax|_2^2\right) \mu(dx) \leq \frac{K_\varepsilon}{\varepsilon(\nu - \varepsilon\lambda_1^{-1}\mathrm{Tr}[Q^*(I + \alpha^2 A)Q])}.$$

Systems of equations - Applications

In biological context for red/white blood cells, lysosomes, liposomes and virus.
$\rightarrow$ Come tomorrow for the mathematical challenges raised by the Covid-19 crisis.

$$\begin{cases} \partial_t u + \nu A u + (I + \alpha^2 A)^{-1} \widetilde{B}(u, u + \alpha^2 A u) = \mathcal{P}\left(\dfrac{\delta E(\phi)}{\delta \phi} \nabla \phi \right), \\ \operatorname{div}(u) = 0, \\ \phi_t + u \cdot \nabla \phi = -\gamma \dfrac{\delta E(\phi)}{\delta \phi}, \end{cases}$$

where

$$E(\phi) = \mathcal{E}_\varepsilon(\phi) + \frac{1}{2} M_1 (\mathcal{A}(\phi) - a)^2 + \frac{1}{2} M_2 (\mathcal{B}_\varepsilon(\phi) - b)^2,$$

with

$$\mathcal{E}_\varepsilon(\phi) = \frac{k}{2\varepsilon} \int_{\mathcal{U}} \left(\varepsilon \Delta \phi + \frac{\phi}{\varepsilon} (1 - \phi^2) \right)^2 \mathrm{d}x,$$

$$\mathcal{A}(\phi) = \int_{\mathcal{U}} \phi \, \mathrm{d}x,$$

$$\mathcal{B}_\varepsilon(\phi) = \int_{\mathcal{U}} \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \frac{1}{4\varepsilon} (\phi^2 - 1)^2 \right) \mathrm{d}x.$$

Theorem (G.&Manca)

Let $T > 0$ and $(u_0, \phi_0) \in D(A) \times \mathrm{L}^2(Q)$ with $\phi_0 = -1$ on ∂Q. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a filtered complete probability space, with two cylindrical Wiener processes on $\mathrm{L}^2(Q)$ and assume that Hypothesis (1) holds.

Then there exists a unique couple of stochastic processes $(u, \phi) \in \mathrm{L}^2([0, T]; D(A)) \times \mathrm{L}^2([0, T]; \mathrm{H}^2(Q))$ which is solution of the stochastic system.

Moreover, for any $k \in \mathbb{N}^*$ there exists a constant $c = c(k, T, u_0, \phi_0) > 0$ such that

$$\sup_{0\le t\le T} \mathbb{E}\left[\left(|u(t)|_2^2 + \alpha^2|\nabla u(t)|_2^2 + E(\phi(t))\right)^k\right] \le c,$$

and

$$\mathbb{E}\left[\int_0^T \left(|u|_2^2 + \alpha^2|\nabla u|_2^2 + E(\phi)\right)^{k-1} \left(\nu(|\nabla u|_2^2 + \alpha^2|Au|_2^2) + \gamma\left|\frac{\delta E(\phi)}{\delta\phi}\right|_2^2\right) \mathrm{d}s\right] \le c.$$

Thanks for your attention !