High-frequency analysis of stochastic PDEs with multiplicative noise

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Stochastic heat equation

$$\begin{aligned} \partial_t u(t,x) &= \tfrac{1}{2}\Delta u(t,x) + \sigma(t,x)\dot{W}(t,x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) &= 1, \qquad x \in \mathbb{R}^d. \end{aligned} \tag{SHE}$$

$\dot{W}$ Gaussian noise: white in time, white/colored in space

$$\mathbb{E}[\dot{W}(t,x)] = 0, \quad \mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)F(x-y),$$

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$$F(x) = \begin{cases} |x|^{-\alpha} \text{ with } \alpha \in (0,2) & \text{if } d \geq 2, \\ |x|^{-\alpha} \text{ with } \alpha \in (0,1) \quad \text{or} \quad \delta_0(x) & \text{if } d = 1. \end{cases}$$

$|x|^{-\alpha} =$ **Riesz kernel** $\quad \delta_0(x) =$ **space-time white noise**

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σ L^2-continuous and L^p-bounded predictable random field for all $p \geq 2$

- **Additive noise:** $\sigma \equiv$ const.
- **Multiplicative noise:** $\sigma(t,x) = \sigma_0(u(t,x))$ for some Lipschitz function $\sigma_0 \colon \mathbb{R} \to \mathbb{R}$

Some quick facts

Theorem (Dalang 1999)

Let $g(t,x) = (2\pi t)^{-d/2} \exp(-\frac{|x|^2}{2t})$ be the heat kernel.

1. If $\sigma(t,x) = \sigma_0(u(t,x))$ for some globally Lipschitz σ_0, then there exists a unique process u that is L^p-bounded for all $p > 0$ and satisfies for all (t,x),
$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\sigma_0(u(s,y))\, W(\mathrm{d}s, \mathrm{d}y) \quad \text{a.s.}$$
2. If $\sigma(t,x)$ is a predictable and L^p-bounded random field for all $p > 0$, then
$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\sigma(s,y)\, W(\mathrm{d}s, \mathrm{d}y)$$
is well-defined and L^p-bounded for all $p > 0$ as well.

In both cases, u is called the **mild solution** to (SHE).

Some quick facts

Theorem (Sanz-Solé & Sarrà 2002)

There exists a version of u which is

- $(\frac{1}{2} - \frac{\alpha}{4} - \epsilon)$-Hölder continuous in time and
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Remarks:

1. The correlation index α determines the **roughness** of the paths.
2. Regularity in time is always **worse** than that of Brownian motion.
3. For fixed x, $t \mapsto u(t,x)$ is **not** a semimartingale.
4. For fixed x, $t \mapsto u(t,x)$ is locally more like a fractional Brownian motion with Hurst index $H = \frac{1}{2} - \frac{\alpha}{4}$.

Problem formulation

Goal: Study limit theorems for normalized power variations

$$V_p^n(u,t) := \frac{1}{n}\sum_{i=1}^{[nt]}\left|\frac{u(\frac{i}{n},x)-u(\frac{i-1}{n},x)}{\tau_n}\right|^p, \qquad t\in(0,\infty)$$

Here:

- x fixed (e.g., $x=0$) and $p>0$;
- $\tau_n = C_\alpha n^{-(\frac{1}{2}-\frac{\alpha}{4})}$ is a normalizing factor

Motivation

Statistical problem: Assume we can observe the solution process

$$u(\tfrac{1}{n}, x), u(\tfrac{2}{n}, x), \ldots, u(\tfrac{[nT]}{n}, x)$$

at a fixed spatial point x with **high frequency** in time (i.e., n is large).

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Examples:

- The spatial correlation index α
- The "volatility" process $\sigma(t,x)$
- If we know the functional form of σ (e.g., $\sigma_0(x) = \lambda x$ (PAM)): the intensity parameter λ
- Can we decide (= statistically test) whether we have additive noise $\sigma_0 = const.$ or multiplicative noise $\sigma_0(x) = \lambda x$?

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Note: This is a **local** estimation problem (T fixed, **infill asymptotics**)!

Why normalized power variations?

Basic idea: For $u(t) := \int_0^t \sigma(s)\,\mathrm{d}B(s)$, it is well known that

$$\sum_{i=1}^{[nt]} (u(\tfrac{i}{n}) - u(\tfrac{i-1}{n}))^2 \xrightarrow{L^2} \int_0^t \sigma(s)^2\,\mathrm{d}s.$$

Rule of thumb:

Law of large numbers for $V_p^n(u,t) \longleftrightarrow$ Consistent estimators

Central limit theorem for $V_p^n(u,t) \longleftrightarrow$ Confidence bounds

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Related literature

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CLT only for SPDEs with additive noise / deterministic σ!
No CLT for multiplicative noise / stochastic volatility!

Results

Theorem (Law of large numbers; C. 2020a)

For every $p > 0$,

$$V_p^n(u,t) \stackrel{L^1}{\Longrightarrow} V_p(u,t) = \mu_p \int_0^t |\sigma(s,x)|^p \, \mathrm{d}s,$$

where $\mu_p = \mathbb{E}[|X|^p]$ for $X \sim N(0,1)$ and $\stackrel{L^1}{\Longrightarrow}$ denotes local uniform L^1-convergence.

Special cases:

Pospíšil & Tribe (2007), Swanson (2007), Liu & Tudor (2016), Bibinger & Trabs (2019), Cialenco & Huang (2019)

Central limit theorem

Two types of difficulty:

1. The solution process is rough:

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\sigma(s,y)\, W(\mathrm{d}s, \mathrm{d}y),$$

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2. **and** the volatility is rough (if we have multiplicative noise):

$$\sigma(t,x) = \sigma_0(u(t,x)) = \sigma_0\left(1 + \int_0^t \int_{\mathbb{R}^d} g(t-s, x-y)\sigma(s,y)\, W(\mathrm{d}s, \mathrm{d}y)\right).$$

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Conceptual findings: The roughness

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   Loosely speaking, the roughness of the solution process can be mitigated through iterated martingale approximations.

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   C. (2020b). High-frequency analysis of parabolic stochastic PDEs with multiplicative noise. Part I. Preprint under arXiv:1908.04145.
   C. (2020c). High-frequency analysis of parabolic stochastic PDEs with multiplicative noise. Part II. In preparation.

   The central limit theorem holds in some but fails in other cases!

Central limit theorem

Theorem (Central limit theorem, C. 2020a):

Let $p = 2$ or $p \geq 4$. Then, under additional hypotheses on the random field σ,

$$\sqrt{n}(V_p^n(u,t) - V_p(u,t)) \stackrel{\mathrm{d}}{\Longrightarrow} \mathcal{Z}_t = c_{p,\alpha} \int_0^t |\sigma(s,x)|^p \,\mathrm{d}B_s$$

where $c_{p,\alpha}$ is an explicit constant, $\stackrel{\mathrm{d}}{\Longrightarrow}$ denotes convergence in law in the local uniform topology, and B is a Brownian motion that is independent of W and σ.

Special cases ($\alpha = d = 1$, $\sigma \equiv 1$ and $p \in \{2,4\}$):

Bibinger & Trabs (2019, 2020), Cialenco & Huang (2020)

Restrictions

What are the "additional hypotheses"?

Among other things, we require $\sigma(t, x)$ be (essentially)

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Unfortunately:

This excludes the multiplicative case $\sigma(t,x) = \sigma_0(u(t,x))$, where σ_0 is a Lipschitz function (unless $\sigma = \text{const.}$)!

The multiplicative case (only $\alpha \leq 1$)

Theorem (C. 2020b,c)

Let $p \in 2\mathbb{N}$ and $\sigma(t,x) = \sigma_0(u(t,x))$, where σ_0 is smooth and has at most linear growth.

1. If $\alpha \in (0,1)$, then, as before,

$$\sqrt{n}(V_p^n(u,t) - V_p(u,t)) \overset{\mathrm{d}}{\Longrightarrow} \mathcal{Z}_t = c_{p,\alpha} \int_0^t |\sigma_0(u(s,x))|^p \,\mathrm{d}B_s.$$

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2. If $\alpha = 1$ (this includes the **white noise** case in $d = 1$), then

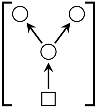
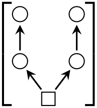
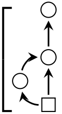
$$\sqrt{n}(V_p^n(u,t) - V_p(u,t)) \stackrel{\mathrm{d}}{\Longrightarrow} \mathcal{Z}_t = A_t + c_{p,\alpha} \int_0^t |\sigma_0(u(s,x))|^p \, \mathrm{d}B_s,$$

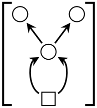
   where A_t is a finite variation process, adapted to W, and

$$A \equiv 0 \quad \Longleftrightarrow \quad p = 2 \quad \text{or} \quad \sigma_0 \equiv \text{const. (i.e., additive noise)}.$$

The asymptotic drift process

We only consider $d = \alpha = 1$ (white noise). Then

$A_t = A$ [] $(t) + A$ [] $(t) + A$ [] (t)

$+ A$ [] $(t) + A$ [] (t)

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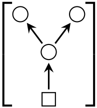
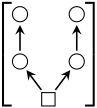
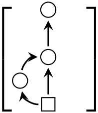
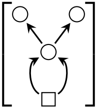
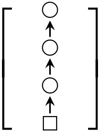
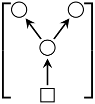
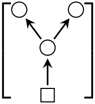
$A_t = A$  $(t) + A$  $(t) + A$  (t)

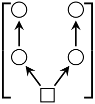
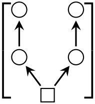
$+ A$  $(t) + A$  (t)

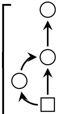
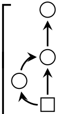
Observation

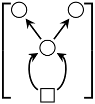
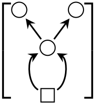
The asymptotic drift terms in the case $\alpha = 1$ (and also in the cases $\alpha \in (1, 2)$) are most conveniently indexed by rooted binary directed acyclic graphs (DAGs). This is closely related to the graphical notation used by M. Hairer and others for renormalization terms appearing in the analysis of singular SPDEs.

What are these terms?

A [] $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^5\sigma_0'')(u(s,x))\,\mathrm{d}s \quad \times \quad C$ [],

A [] $(t) = \frac{1}{2}\int_0^t a_6(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^6(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$ [],

A [] $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$ [],

A [] $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$ [],

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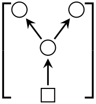
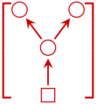
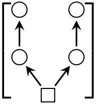
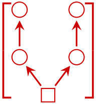
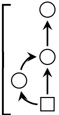
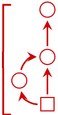
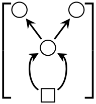
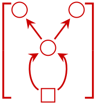
A [] $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^5\sigma_0'')(u(s,x))\,\mathrm{d}s \quad \times \quad C$ [],

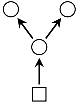
A [] $(t) = \frac{1}{2}\int_0^t a_6(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^6(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$ [],

A [] $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$ [],

A [] $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$ [],

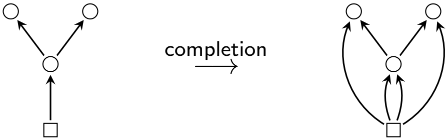
A [] $(t) = \int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \quad \times \quad C$ []

The constants

Example: 

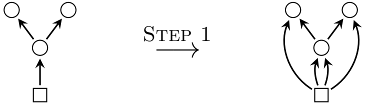
STEP 1: Complete the graph

$=$ add an edge from the root to any vertex with only one incoming edge



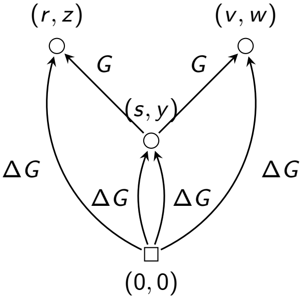
The constants

Example:



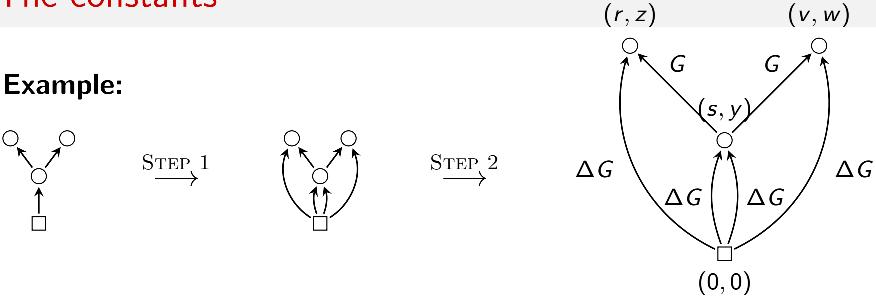
STEP 2: Label the graph:

- Attach $(0,0)$ to $\square$ and integration variables to all $\circ$'s
- Attach ΔG to all edges starting in $\square$ and G to all other edges



The constants

Example:

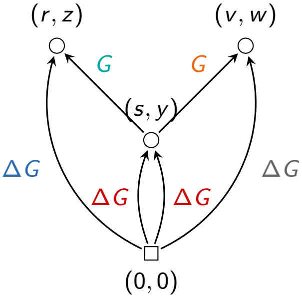


STEP 3: Associate an iterated convolution to this labeled graph! Let

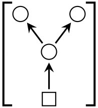
$$G(t,x) := C\,g(t,x)\mathbf{1}_{\{t>0\}} = C(2\pi t)e^{-\frac{|x|^2}{2t}}\mathbf{1}_{\{t>0\}},$$
$$\Delta G(t,x) := G(t,x) - G(t-1,x),$$

where $C = \sqrt{\frac{2}{\pi}}$ for $d = \alpha = 1$.

The constants



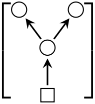
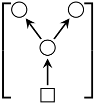
Then

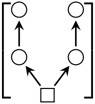
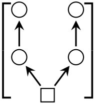
C [] $:= \int_{(0,\infty)\times\mathbb{R}} \int_{(0,\infty)\times\mathbb{R}} \int_{(0,\infty)\times\mathbb{R}} (\Delta G(s-0, y-0))^2$

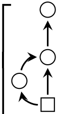
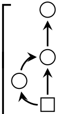
$$\times \Delta G(r-0, z-0) \Delta G(v-0, w-0) G(r-s, z-y)$$

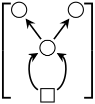
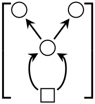
$$\times G(v-s, w-y) \, \mathrm{d}s \, \mathrm{d}y \, \mathrm{d}r \, \mathrm{d}z \, \mathrm{d}v \, \mathrm{d}w$$

What are these terms?

A  $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^5\sigma_0'')(u(s,x))\,\mathrm{d}s \quad \times \quad C$  ,

A  $(t) = \frac{1}{2}\int_0^t a_6(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^6(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$  ,

A  $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$  ,

A  $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$  ,

A  $(t) = \int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \quad \times \quad C$ 

The finite variation processes

Every f of, say, polynomial growth can be expanded in terms of **generalized Hermite polynomials**:

$$f(x) = \sum_{n=0}^{\infty} a_n(v, f) H_n(v, x),$$

where, for $v > 0$ and $n \in \mathbb{N}_0$,

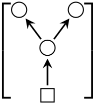
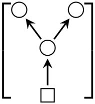
$$H_n(v, x) := v^{\frac{n}{2}} H_n(\tfrac{x}{\sqrt{v}}), \quad H_n(x) := \frac{(-1)^n}{n!} e^{\frac{x^2}{2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-\frac{x^2}{2}}$$

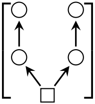
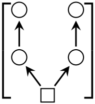
and thus,

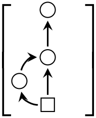
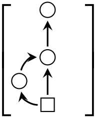
$$a_n(v, f) = \frac{n!}{v^{n/2}} \mathbb{E}[f(\sqrt{v}Z) H_n(Z)], \quad Z \sim N(0, 1).$$

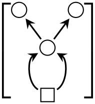
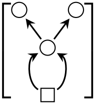
In particular, for $p \in 2\mathbb{N}$,

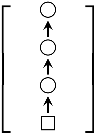
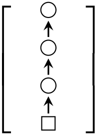
$$a_n(v, |\cdot|^p) = \begin{cases} 0 & n \text{ odd}, \\ \frac{p!}{((p-n)/2)!} (\frac{v}{2})^{(p-n)/2} & n \text{ even}. \end{cases}$$

A  $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^5\sigma_0'')(u(s,x))\,\mathrm{d}s \quad \times \quad C$  ,

A  $(t) = \frac{1}{2}\int_0^t a_6(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^6(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$  ,

A  $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$  ,

A  $(t) = \frac{1}{2}\int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \times C$  ,

A  $(t) = \int_0^t a_4(\sigma_0^2(u(s,x)), |\cdot|^p)(\sigma_0^4(\sigma_0')^2)(u(s,x))\,\mathrm{d}s \quad \times \quad C$ 

This explains:

$$A \equiv 0 \qquad \Longleftrightarrow \qquad p = 2 \quad \text{or} \quad \sigma_0 = \text{const.}$$

References

Main references:

- C. Chong (2020a). High-frequency analysis of parabolic stochastic PDEs. *The Annals of Statistics*, 48(2):1143–1167.
- C. Chong (2020b). High-frequency analysis of parabolic stochastic PDEs with multiplicative noise. Part I. Preprint under arXiv:1908.04145.
- C. Chong (2020c). High-frequency analysis of parabolic stochastic PDEs with multiplicative noise. Part II. In preparation.

References

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Thank you!