Contract Theory, Mean Field Games & Application to Epidemic Control

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CREDITS

Joint work with

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- *A Probabilistic Approach to Extended Finite State Mean Field Games*
  - Mathematics of Operations Research
- *Finite-State Contract Theory with a Principal and a Field of Agents*
  - Management Science

WHAT IS (MATHEMATICAL) CONTRACT THEORY?

A Crash Course on **Principal - Agent** problem

Hart & Holmström, **Sannikov**, **Cvitanic-Zhang**, **Cvitanic-Possamaï-Touzi**

- **Principal** proposes a **contract**
  - a payment stream $\boldsymbol{r} = (r_t)_{0 \leq t \leq T}$ ($\mathbb{F}-$predictable process)
  - a terminal payment ξ ($\mathcal{F}_T-$ square integrable random variable)
- **Agent**
  - decides if he/she wants to work for the principal
  - incurs a (running) cost $c : [0, T] \times E \times A \to \mathbb{R}$
  - maximizes his/her expected utilities $u : \mathbb{R} \to \mathbb{R}$ (resp. $U : \mathbb{R} \to \mathbb{R}$) of the continuous payments (resp. the terminal payment).
  - If an agent accepts the contract $(\boldsymbol{r}, \xi)$, the agent's expected total cost is

$$J^{\boldsymbol{r},\xi}(\boldsymbol{\alpha}) := \mathbb{E}^{\mathbb{Q}^{(\boldsymbol{\alpha})}}\left[\int_0^T [c(t, X_t, \alpha_t) - u(r_t)]dt - U(\xi)\right].$$

Principal's Optimization Problem

Principal's reward depends on

- the payments made to the agent

More notation

- $c_0 : [0, T] \times \mathcal{X} \to \mathbb{R}$ running cost function
- $C_0 : \mathcal{X} \to \mathbb{R}$ the terminal cost function resulting

Given that

- the agent chooses $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ as his/her control strategy,
- the contract offered by the principal is $(\boldsymbol{r}, \xi)$

the **principal's expected total cost** is:

$$J_0^{\boldsymbol{\alpha}}(\boldsymbol{r}, \xi) := \mathbb{E}^{\mathbb{Q}^{(\boldsymbol{\alpha})}} \left[\int_0^T [c_0(t, X_t^{\boldsymbol{\alpha}}) + r_t] dt + C_0(X_T^{\boldsymbol{\alpha}}) + \xi \right].$$

Recast as a game problem:

- **Stackelberg game !**
- **Moral Hazard** $\Longrightarrow$ weak formulation (martingale approach)

One Principal vs a Large Number of Agents

Elie, Mastrolia, Possamaï, R.C. Wang

- **Principal** chooses a **contract**
  - a payment stream $\boldsymbol{r} = (r_t)_{0 \leq t \leq T}$ ($\mathbb{F}-$predictable process)
  - a terminal payment ξ ($\mathcal{F}_T-$ square integrable random variable)
- **Agents**
  - decide if they want to work for the principal
  - incur a (running) cost $c : [0, T] \times E \times A \times \mathcal{S} \to \mathbb{R}$
  - maximize their utilities $u : \mathbb{R} \to \mathbb{R}$ (resp. $U : \mathbb{R} \to \mathbb{R}$) of the continuous payments (resp. the terminal payment).
  - If an agent accepts the contract $(\boldsymbol{r}, \xi)$ and the statistical distribution of his/her state at time t is denoted $p(t)$ for $0 \leq t \leq T$, the agent's expected total cost

$$J^{\boldsymbol{r},\xi}(\boldsymbol{\alpha}, \boldsymbol{p}) := \mathbb{E}^{\mathbb{Q}^{(\boldsymbol{\alpha},\boldsymbol{p})}} \left[\int_0^T [c(t, X_t, \alpha_t, p(t)) - u(r_t)]dt - U(\xi) \right].$$

Principal' Optimal Contracting Problem

Principal minimizes her total expected cost assuming the agents reach a Nash equilibrium

$$V(\kappa) := \inf_{(\boldsymbol{r},\xi)\in\mathcal{C}} \inf_{\substack{(\boldsymbol{\alpha},\boldsymbol{p})\in\mathcal{N}(\boldsymbol{r},\xi) \\ J^{\boldsymbol{r},\xi}(\boldsymbol{\alpha},\boldsymbol{p})\leq\kappa}} \mathbb{E}^{\mathbb{Q}^{(\boldsymbol{\alpha},\boldsymbol{p})}}\left[\int_0^T [c_0(t,p(t))+r_t]dt + C_0(p(T))+\xi\right],$$

Principal minimizes her total expected cost assuming the agents reach a cooperative equilibrium (central planner)

$$V(\kappa) := \inf_{(\boldsymbol{r},\xi)\in\mathcal{C}} \inf_{\substack{\boldsymbol{\alpha}\in\mathcal{MKV}(\boldsymbol{r},\xi) \\ J^{\boldsymbol{r},\xi}(\boldsymbol{\alpha})\leq\kappa}} \mathbb{E}^{\mathbb{Q}^{(\boldsymbol{\alpha},\boldsymbol{p})}}\left[\int_0^T [c_0(t,p(t))+r_t]dt + C_0(p(T))+\xi\right],$$

A Naive Model of Epidemic Containment

- Regulator tries to control the spread of a virus over a time period $[0, T]$
- Jurisdiction consists of two counties A and B
- Each individual is either infected (I) or healthy (H), lives in county A or B.
- State space is $E = \{AI, AH, BI, BH\}$
- $\pi_{AI}, \pi_{AH}, \pi_{BI}, \pi_{BH}$ proportions of individuals in each state

State Evolution (*spirit of SIR model for spread of disease*)

(1) Rate of contracting the virus depends on the proportion of infected individuals in the county so
   - transition rate from state AH to state AI is $\theta_A^-(\frac{\pi_{AI}}{\pi_{AI}+\pi_{AH}})$
   - transition rate from state BH to state BI is $\theta_B^-(\frac{\pi_{BI}}{\pi_{BI}+\pi_{BH}})$.

(2) Rate of recovery is a function of the proportion of healthy individuals in the county, so
   - transition rate from state AI to state AH is $\theta_A^+(\frac{\pi_{AH}}{\pi_{AI}+\pi_{AH}})$
   - transition rate from state BH to state BI is $\theta_B^+(\frac{\pi_{BH}}{\pi_{BI}+\pi_{BH}})$.

(3) Each individual can try to move to the other county: $\nu_I\alpha$ transition rates between the states AI and BI, and $\nu_H\alpha$ as the transition rates between the states AH and BH.

(4) Status of infection does not change when individual moves between counties.

θ_A^-, θ_B^-, θ_A^+ and θ_B^+ increasing differentiable functions from $[0, 1]$ to $\mathbb{R}^+$ characterize the quality of health care in counties A and B.

The Q-matrix of the Model

$$Q(t,\alpha,\pi) := \begin{array}{c} \\ \end{array} \begin{array}{cccc} AI & AH & BI & BH \end{array}$$

$$Q(t,\alpha,\pi) := \left[\begin{array}{cccc} \cdots & \theta_A^+\left(\frac{\pi_{AH}}{\pi_{AI}+\pi_{AH}}\right) & \nu_I\alpha & 0 \\ \theta_A^-\left(\frac{\pi_{AI}}{\pi_{AI}+\pi_{AH}}\right) & \cdots & 0 & \nu_H\alpha \\ \nu_I\alpha & 0 & \cdots & \theta_B^+\left(\frac{\pi_{BH}}{\pi_{BI}+\pi_{BH}}\right) \\ 0 & \nu_H\alpha & \theta_B^-\left(\frac{\pi_{BI}}{\pi_{BI}+\pi_{BH}}\right) & \cdots \end{array}\right] \begin{array}{c} AI \\ AH \\ BI \\ BH \end{array}$$

Explicit Epidemic Containment Model

Agent cost functions:

$$c_1(t, AI, \pi) = c_1(t, AH, \pi) := \phi_A \left(\frac{\pi_{AI}}{\pi_{AI} + \pi_{AH}} \right), \quad (1)$$

$$c_1(t, BI, \pi) = c_1(t, BH, \pi) := \phi_B \left(\frac{\pi_{BI}}{\pi_{BI} + \pi_{BH}} \right), \quad (2)$$

$$\gamma_{AI} = \gamma_{BI} := \gamma_I, \quad \gamma_{AH} = \gamma_{BH} := \gamma_H, \quad (3)$$

where ϕ_A and ϕ_B are two increasing mappings on $\mathbb{R}$.

Authority (Principal) running and terminal costs:

$$c_0(t, \pi) = \exp(\sigma_A \pi_{AI} + \sigma_B \pi_{BI}), \quad (4)$$

$$C_0(\pi) = \sigma_P \cdot (\pi_{AI} + \pi_{AH} - \pi_A^0)^2, \quad (5)$$

where π_A^0 is the population of county A at time 0.

Objectives of the authority:

- Trade-off between the control of the epidemic and population planning
- Minimization of the infection rate of both counties
- σ_A, σ_B and σ_P reflects the relative importance the authority attributes to each of these objectives.

Equilibrium behavior of the individuals

- Tendency to move away from the county with higher infection rate and poorer health care
- May result in overpopulation of the other county. Therefore, the authority also wishes to maintain the population of both counties at a steady level.

INFECTION RATES

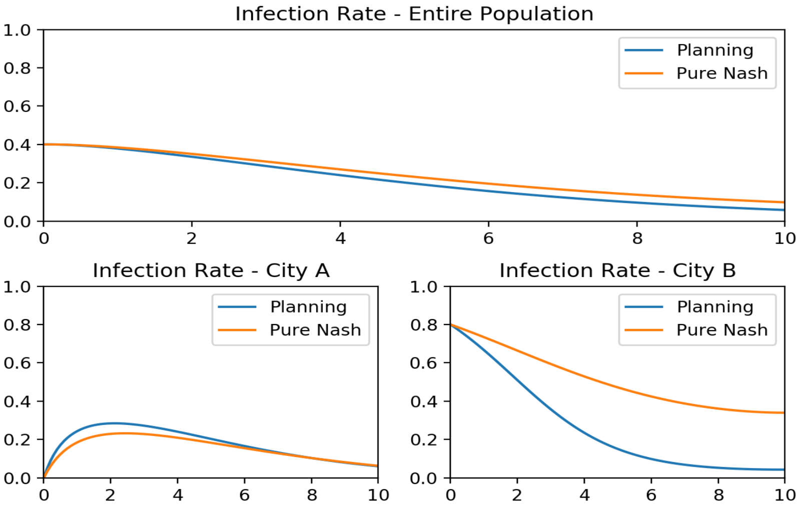

FIGURE: Evolution of the infection rates with and without authority's intervention.

Movement of Populations

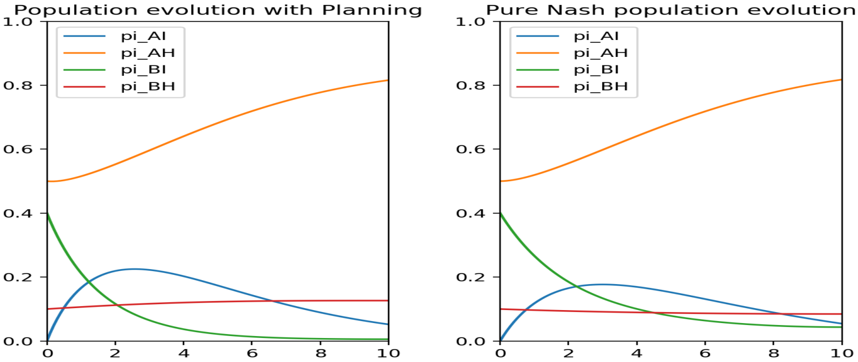

FIGURE: Evolution of the populations' sizes in both counties, with and without the intervention of the regulator.

Efforts to Move

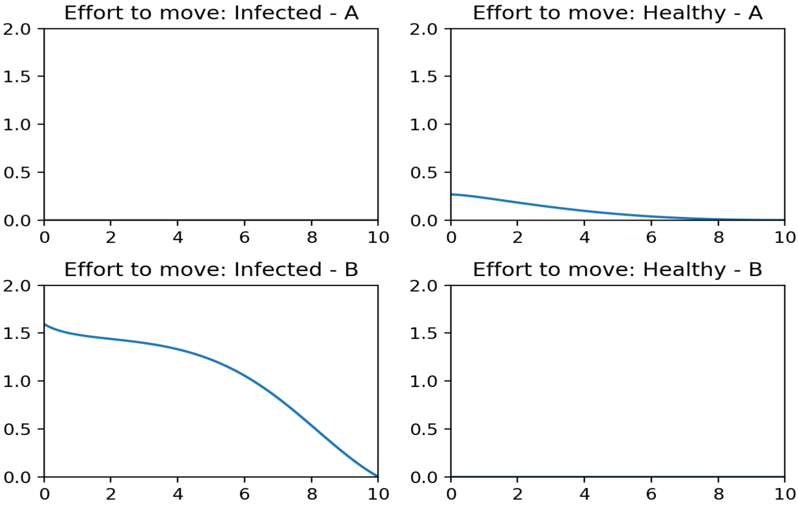

Figure: Evolution of the efforts made by the individuals to move from one county to the other in the presence of the regulator.

Evolution of the Infection Rates

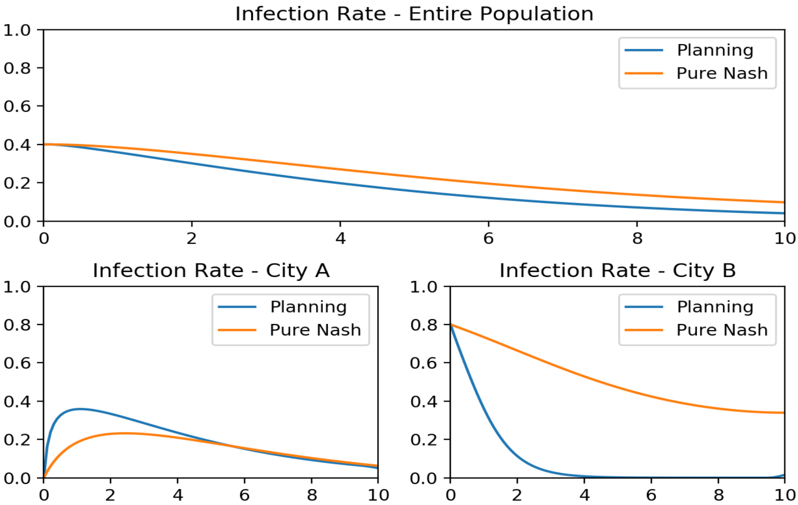

Figure: Evolution of the infection rates with and without authority's intervention.

Regulator Control over the Sizes of the Populations

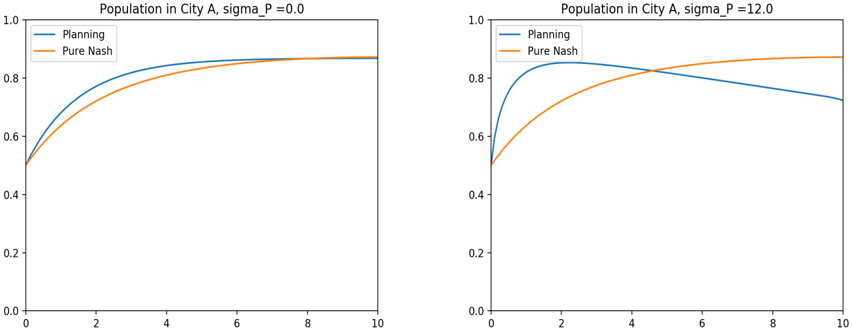

Figure: Regulator control over the sizes of the county populations: effect of the parameter σ_P on the terminal value of the population of county A.

INFECTION RATES

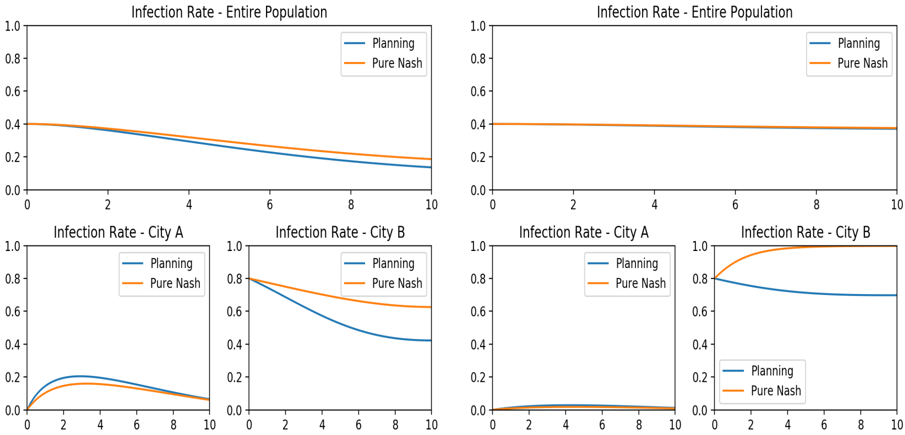

FIGURE: Evolution of infection rate with and without authority's intervention for two different sets of parameters: $\nu_I = 0.2$ and $\nu_H = 0.04$ (left), $\nu_I = 0.04$ and $\nu_H = 0.2$ (right).