CONTRACT THEORY, MEAN FIELD GAMES
& APPLICATION TO EPIDEMIC CONTROL

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Joint work with

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- A Probabilistic Approach to Extended Finite State Mean Field Games
  - Mathematics of Operations Research
- Finite-State Contract Theory with a Principal and a Field of Agents
  - Management Science
What is (Mathematical) Contract Theory?

A Crash Course on Principal - Agent problem

Hart & Holmström, Sannikov, Cvitanic-Zhang, Cvitanic-Possamaï-Touzi

- **Principal** proposes a **contract**
  - a payment stream \( r = (r_t)_{0 \leq t \leq T} \) (\( \mathcal{F} \)-predictable process)
  - a terminal payment \( \xi (\mathcal{F}_T, \text{square integrable random variable}) \)

- **Agent**
  - decides if he/she wants to work for the principal
  - incurs a (running) cost \( c : [0, T] \times E \times A \rightarrow \mathbb{R} \)
  - maximizes his/her expected utilities \( u : \mathbb{R} \rightarrow \mathbb{R} \) (resp. \( U : \mathbb{R} \rightarrow \mathbb{R} \)) of the continuous payments (resp. the terminal payment).
  - If an agent accepts the contract \((r, \xi)\), the agent’s expected total cost is

\[
J^{r, \xi}(\alpha) := \mathbb{E}_Q^{\alpha} \left[ \int_0^T \left[ c(t, X_t, \alpha_t) - u(r_t) \right] dt - U(\xi) \right].
\]
Principal’s Optimization Problem

Principal’s reward depends on
- the payments made to the agent

More notation
- $c_0 : [0, T] \times \mathcal{X} \to \mathbb{R}$ running cost function
- $C_0 : \mathcal{X} \to \mathbb{R}$ the terminal cost function resulting

Given that
- the agent chooses $\alpha = (\alpha_t)_{0 \leq t \leq T}$ as his/her control strategy,
- the contract offered by the principal is $(r, \xi)$

the principal’s expected total cost is:

$$J_0^\alpha(r, \xi) := \mathbb{E}^{Q(\alpha)} \left[ \int_0^T [c_0(t, X_t^\alpha) + r_t] dt + C_0(X_T^\alpha) + \xi \right].$$

Recast as a game problem:
- Stackelberg game!
- Moral Hazard $\implies$ weak formulation (martingale approach)
One Principal vs a Large Number of Agents

Elie, Mastrolia, Possamaï, R.C. Wang

- **Principal** chooses a **contract**
  - a payment stream \( r = (r_t)_{0 \leq t \leq T} \) (\( \mathcal{F} \)-predictable process)
  - a terminal payment \( \xi \) (\( \mathcal{F}_T \)-square integrable random variable)

- **Agents**
  - decide if they want to work for the principal
  - incur a (running) cost \( c : [0, T] \times E \times A \times S \to \mathbb{R} \)
  - maximize their utilities \( u : \mathbb{R} \to \mathbb{R} \) (resp. \( U : \mathbb{R} \to \mathbb{R} \)) of the continuous payments (resp. the terminal payment).
  - If an agent accepts the contract \( (r, \xi) \) and the statistical distribution of his/her state at time \( t \) is denoted \( p(t) \) for \( 0 \leq t \leq T \), the agent’s expected total cost

\[
J^{r,\xi}(\alpha, p) := \mathbb{E}^Q(\alpha, p) \left[ \int_0^T [c(t, X_t, \alpha_t, p(t)) - u(r_t)] \, dt - U(\xi) \right].
\]
Principal’ Optimal Contracting Problem

\[ V(\kappa) := \inf_{(r, \xi) \in C} \inf_{(\alpha, p) \in N(r, \xi)} \mathbb{E}^{Q(\alpha, p)} \left[ \int_0^T \left[ c_0(t, p(t)) + r_t \right] dt + C_0(p(T)) + \xi \right], \]

Principal minimizes her total expected cost assuming the agents reach a Nash equilibrium

\[ V(\kappa) := \inf_{(r, \xi) \in C} \inf_{\alpha \in MKV(r, \xi)} \mathbb{E}^{Q(\alpha, p)} \left[ \int_0^T \left[ c_0(t, p(t)) + r_t \right] dt + C_0(p(T)) + \xi \right], \]

Principal minimizes her total expected cost assuming the agents reach a cooperative equilibrium (central planner)
A Naive Model of Epidemic Containment

- Regulator tries to control the spread of a virus over a time period \([0, T]\)
- Jurisdiction consists of two counties \(A\) and \(B\)
- Each individual is either infected \((I)\) or healthy \((H)\), lives in county \(A\) or \(B\).
- State space is \(E = \{AI, AH, BI, BH\}\)
- \(\pi_{AI}, \pi_{AH}, \pi_{BI}, \pi_{BH}\) proportions of individuals in each state

State Evolution (spirit of SIR model for spread of disease)

1. Rate of contracting the virus depends on the proportion of infected individuals in the county so
   - Transition rate from state \(AH\) to state \(AI\) is \(\theta_A^- (\frac{\pi_{AI}}{\pi_{AI} + \pi_{AH}})\)
   - Transition rate from state \(BH\) to state \(BL\) is \(\theta_B^- (\frac{\pi_{BI}}{\pi_{BI} + \pi_{BH}})\).

2. Rate of recovery is a function of the proportion of healthy individuals in the county, so
   - Transition rate from state \(AI\) to state \(AH\) is \(\theta_A^+ (\frac{\pi_{AH}}{\pi_{AI} + \pi_{AH}})\)
   - Transition rate from state \(BH\) to state \(BL\) is \(\theta_B^+ (\frac{\pi_{BH}}{\pi_{BI} + \pi_{BH}})\).

3. Each individual can try to move to the other county: \(\nu_I\alpha\) transition rates between the states \(AI\) and \(BI\), and \(\nu_H\alpha\) as the transition rates between the states \(AH\) and \(BH\).

4. Status of infection does not change when individual moves between counties.

\(\theta_A^-, \theta_B^-, \theta_A^+\) and \(\theta_B^+\) increasing differentiable functions from \([0, 1]\) to \(\mathbb{R}^+\) characterize the quality of health care in counties \(A\) and \(B\).
The Q-matrix of the Model

\[ Q(t, \alpha, \pi) := \begin{bmatrix}
A_l & \theta^+ \left( \frac{\pi_{AH}}{\pi_{AI} + \pi_{AH}} \right) & B_l & B_l \\
\theta^- \left( \frac{\pi_{AI}}{\pi_{AI} + \pi_{AH}} \right) & \nu_l \alpha & 0 & 0 \\
0 & 0 & \nu_l \alpha & \theta^- \left( \frac{\pi_{BI}}{\pi_{BI} + \pi_{BH}} \right) \\
\nu_H \alpha & \theta^+ \left( \frac{\pi_{BH}}{\pi_{BI} + \pi_{BH}} \right) & \ldots & \ldots
\end{bmatrix} \]
EXPLICIT EPIDEMIC CONTAINMENT MODEL

Agent cost functions:

\[ c_1(t, A_I, \pi) = c_1(t, A_H, \pi) := \phi_A \left( \frac{\pi_{A_I}}{\pi_{A_I} + \pi_{A_H}} \right), \]

\[ c_1(t, B_I, \pi) = c_1(t, B_H, \pi) := \phi_B \left( \frac{\pi_{B_I}}{\pi_{B_I} + \pi_{B_H}} \right), \]

\[ \gamma_{AI} = \gamma_{BI} := \gamma_I, \quad \gamma_{AH} = \gamma_{BH} := \gamma_H, \]

where \( \phi_A \) and \( \phi_B \) are two increasing mappings on \( \mathbb{R} \).

Authority (Principal) running and terminal costs:

\[ c_0(t, \pi) = \exp(\sigma_A \pi_{A_I} + \sigma_B \pi_{B_I}), \]

\[ C_0(\pi) = \sigma_P \cdot (\pi_{A_I} + \pi_{A_H} - \pi_A^0)^2, \]

where \( \pi_A^0 \) is the population of county \( A \) at time 0.

Objectives of the authority:

▶ Trade-off between the control of the epidemic and population planning
▶ Minimization of the infection rate of both counties
▶ \( \sigma_A, \sigma_B \) and \( \sigma_P \) reflects the relative importance the authority attributes to each of these objectives.

Equilibrium behavior of the individuals

▶ Tendency to move away from the county with higher infection rate and poorer health care
▶ May result in overpopulation of the other county. Therefore, the authority also wishes to maintain the population of both counties at a steady level.
Infection Rates

**Figure:** Evolution of the infection rates with and without authority’s intervention.
MOVEMENT OF POPULATIONS

**Figure:** Evolution of the populations’ sizes in both counties, with and without the intervention of the regulator.
**Efforts to Move**

**Figure:** Evolution of the efforts made by the individuals to move from one county to the other in the presence of the regulator.
**Figure:** Evolution of the infection rates with and without authority's intervention.
**Regulator Control over the Sizes of the Populations**

**Figure:** Regulator control over the sizes of the county populations: effect of the parameter $\sigma_P$ on the terminal value of the population of county A.
**Infection Rates**

![Graphs showing infection rates with and without authority's intervention for different parameters.](image)

**Figure:** Evolution of infection rate with and without authority's intervention for two different sets of parameters: $\nu_I = 0.2$ and $\nu_H = 0.04$ (left), $\nu_I = 0.04$ and $\nu_H = 0.2$ (right).