## LECTURE 3

### 1. The Peano existence theorem

As in last lecture we formulate the results for scalar valued equations. However, just as in last lecture, most of the results and proofs to follow, in particular Theorem 1.1 as well as Theorem 2.2, can be essentially immediately adapted to the case of first order systems of ODEs.

We again consider the general first order initial value problem

$$y' = f(x, y),\ \ y(\xi) = \eta, \tag{1.1}$$

but this time, we only assume that $f(x, y)$ is continuous, and no longer necessarily satisfies the Lipschitz condition $|f(x, y) - f(x, \bar{y})| \leq L|y - \bar{y}|$. Specifically, we make the

**Assumption**: *For $I = [\xi, \xi + a]$, $J = [\eta - b, \eta + b]$, we have $f(\cdot, \cdot) \in C^0(I \times J)$, $|f|_{C^0(I\times J)} \leq M$ for some $M > 0, a > 0, b > 0$.*

Then we have the following fundamental

**Theorem 1.1.** *(Peano' theorem) Under the preceding assumption, there exists a solution $y(x) \in C^1(\widetilde{I})$, with $\widetilde{I} = [\xi, \xi + \min\{a, \frac{b}{M+1}\}]$.*

*Remark* 1.2. The solution is not unique in general. We only need to recall the problem $y' = \sqrt{|y|}$.

*Remark* 1.3. We did not state the strongest possible conclusion as far as the domain of existence of the solution is concerned

As so often in analysis, the important thing to remember is not so much the theorem per se, but the technique of proof. In sharp contrast to the Picard theorem, here the proof will be *non-constructive*, and rely on a *compactness argument*. Both the Picard iteration and the compactness type argument are extremely important and recur in multiple contexts in the realm of ODE and PDE.

*Proof.* The idea is to reduce to the situation in Picard's theorem. For this, we begin by *mollifying* the function $f(x, y)$ with respect to the variable $y$. This means the following: pick a function

$$\chi(y) \in C^\infty(\mathbf{R})$$

such that $\chi \geq 0$ and $\chi(y) = 0$ provided $y \notin [1, 2]$. Furthermore, we may assume that

$$\int_{\mathbf{R}} \chi(y)\, dy = 1$$

By *re-scaling* the function $\chi(y)$, we introduce the functions

$$\chi_\varepsilon(y) := \varepsilon^{-1}\chi(\frac{y}{\varepsilon}),\ , \varepsilon > 0$$

Then note that $\text{supp}\chi_\varepsilon(y) \subset [\varepsilon, 2\varepsilon]$, while also $\int_{\mathbf{R}} \chi_\varepsilon(y)\, dy = 1$.
The *mollification* of $f$ is now given by the family of functions

$$f_\varepsilon(x, y) := f *_y \chi_\varepsilon(x, y) = \int_{\mathbf{R}} f(x, y - z)\chi_\varepsilon(z)\, dz \tag{1.2}$$

Note that we only mollify with respect to the $y$-variable. The above expression is only formal, and we need to make sure that it is well-defined. Thus we restrict to $y \in [\eta - \frac{bM}{M+1}, \eta + \frac{bM}{M+1}]$, and further $\varepsilon < \frac{b}{2(M+1)}$. In order to be able to invoke the version of Picard's theorem we proved last time, we need to extend $f_\varepsilon(x,y)$ to all values of $y \in \mathbf{R}$. This we can easily by extending $f_\varepsilon(x,y)$ as a constant beyond the values $y = \eta \pm \frac{bM}{M+1}$.

Then we have the following simple

**Lemma 1.4.** *Under the above restrictions on $y, \varepsilon > 0$, the expression (1.2) is well-defined. Furthermore, extending $f_\varepsilon(x,y)$ to all values of $y$ as above, we have the bounds*

$$|f_\varepsilon(x,y)|_{C^0(R)} \le M$$

*where $R = I \times \mathbf{R}$ as long as $\varepsilon < \frac{b}{2(M+1)}$; further, we have the Lipschitz bound*

$$|f_\varepsilon(x,y) - f_\varepsilon(x,\bar{y})| \le \frac{C}{\varepsilon} M |y - \bar{y}| \tag{1.3}$$

The proof of the lemma is left as a simple exercise.

We can now apply Picard's theorem as proved last time, to the initial value problems

$$y' = f_\varepsilon(x,y),\ y(\xi) = \eta, \tag{1.4}$$

Note that here we use $L = \frac{C}{\varepsilon}M$. We obtain unique $C^1$-regular solutions $y_\varepsilon(x)$ on $I = [\xi, \xi + a]$. It is then intuitively clear that one way to potentially obtain a solution to the original problem (1.1) is to let $\varepsilon \to 0$ and look for a limit to $y_\varepsilon(x)$. The key to extract such a limit comes from a crucial *compactness property* of the family of functions $\{y_\varepsilon(x)\}_{\varepsilon>0}$. To prepare this, we first state the simple

**Lemma 1.5.** *For $x \in \widetilde{I}$ (defined in the statement of theorem 1.1), we have the uniform bounds*

$$|y_\varepsilon(x) - \eta| \le \frac{bM}{M+1},\ |y_\varepsilon'(x)| \le M$$

The proof is immediate; for example, using (1.4), we get

$$|y_\varepsilon(x) - \eta| \le \int_\xi^x M\,ds \le M|x - \xi|$$

from which the first bound follows. The second bound is immediate.

The preceding lemma implies that the family of functions $\{y_\varepsilon(x)\}_\varepsilon > 0$ is *uniformly continuous* on $\widetilde{I}$. The desired compactness is then a consequence of the following very general fact:

**Proposition 1.6.** *(Arzela-Ascoli) Let $K$ be a compact metric space, and $A \subset C^0(K)$ a non-empty subset. Then $A$ is pre-compact precisely if the following two conditions are satisfied:*

*(i) $A$ is bounded: $\exists M \in \mathbf{R}$ such that $\max_{x\in K}|f(x)| \le M\ \forall f \in A$.*

*(ii) $A$ is uniformly continuous: $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ such that $\forall x, y \in K$ with $\rho(x,y) < \delta_\varepsilon$, we have*

$$|f(x) - f(y)| < \varepsilon,\ f \in A$$

Assuming this for now, we finish the proof of Peano's theorem as follows: Letting $A = \{y_\varepsilon\}_{\varepsilon>0} \subset C^0(\widetilde{I})$, Lemma 1.5 implies (check!) that both conditions of Proposition 1.6 are satisfied. In particular, choosing any sequence $\{\varepsilon_j\}_{j\ge1}$ with $\varepsilon_j \to 0$, we can select a subsequence $\{\varepsilon_{j_k}\}_{k\ge1} \subset \{\varepsilon_j\}_{j\ge1}$ with the property that $\{y_{\varepsilon_{j_k}}\}_{k\ge1}$ converges with respect to $|\cdot|_{C^0(\widetilde{I})}$ to some $y_* \in C^0(\widetilde{I})$. For simplicity, we again label the subsequence as $\{\varepsilon_j\}_{j\ge1}$.

**Claim**: *The function $y_*(x)$ solves (1.1) on $\widetilde{I}$.*

*Proof.* (Claim) First it is clear that $y_\varepsilon(\xi) = \eta\ \forall \varepsilon > 0$, so that clearly also $y_*(\xi) = \eta$. To see that $y_*' = f(x, y_*)$ on $\widetilde{I}$, we observe that

$$\lim_{j\to\infty} f_{\varepsilon_j}(x, y_{\varepsilon_j}(x)) = f(x, y_*(x)),\ x \in \widetilde{I} \tag{1.5}$$

uniformly with respect to $x \in \widetilde{I}$. This in turn follows in 2 steps: first,

$$|f_{\varepsilon_j}(x, y_{\varepsilon_j}) - f_{\varepsilon_j}(x, y_*)| \leq \int_{\mathbf{R}} |f(x, y_{\varepsilon_j} - z) - f(x, y_* - z)| \chi_{\varepsilon_j}(z)\, dz \leq \max_{z \in [\varepsilon_j, 2\varepsilon_j]} \{|f(x, y_{\varepsilon_j} - z) - f(x, y_* - z)|\}$$

whence by locally uniform continuity of $f(\cdot, \cdot)$, we get

$$\lim_{j \to \infty} |f_{\varepsilon_j}(x, y_{\varepsilon_j}) - f_{\varepsilon_j}(x, y_*)| = 0$$

uniformly with respect to $x \in \widetilde{I}$.

Second, we have

$$\lim_{\varepsilon \to 0} |f_\varepsilon(x, y_*(x)) - f(x, y_*(x))| = 0,\ x \in \widetilde{I}$$

uniformly with respect to $x \in \widetilde{I}$. To see this, write for small $\varepsilon$

$$f_\varepsilon(x, y_*(x)) - f(x, y_*(x)) = \int_{\mathbf{R}} [f(x, y_* - z) - f(x, y_*)] \chi_\varepsilon(z)\, dz$$

provided $x \in \widetilde{I}$; here we of course use that $\int \chi(z)\, dz = 1$. The continuity of $f(\cdot, \cdot)$ together with non-negativity of $\chi_\varepsilon(z)$ (!) imply

$$\limsup_{\varepsilon \to 0} |\int_{\mathbf{R}} [f(x, y_* - z) - f(x, y_*)] \chi_\varepsilon(z)\, dz| \leq \limsup_{\varepsilon \to 0} \max_{z \in [\varepsilon, 2\varepsilon]} |f(x, y_* - z) - f(x, y_*)| \int \chi_\varepsilon(z)\, dz = 0$$

The two preceding steps imply the validity of (1.5). Using it, we infer

$$y_*(x) = \eta + \int_\xi^x f(t, y_*(t))\, dt,$$

which implies $y_*' = f(x, y_*)$. □

This completes the proof of Peano's theorem, up to the Arzela-Ascoli theorem. □

1.1. **Proof of Arzela-Ascoli.** Here we prove the sufficiency of conditions (i), (ii) in Arzela-Ascoli, leaving the necessity as an exercise. Thus, assuming (i), (ii), we show that the set $A \subset C^0(K)$ is pre-compact. This is equivalent to the statement that for every sequence $\{f_n\}_{n \geq 1} \subset A$, there exists a sub-sequence $\{f_{n_k}\}_{k \geq 1} \subset \{f_n\}_{n \geq 1}$ converging in $C^0(K)$.

Thus let $\{f_n\}_{n \geq 1} \subset A$ be given. To construct a converging sub-sequence, we shall use a Cantor diagonal procedure to find a sub-sequence converging on a countable dense subset of $K$. We first pick such a subset. Put $\varepsilon_k = \frac{1}{k}$, $k \in \mathbf{N}$. By compactness of $K$, for each $k \in \mathbf{N}$, we can cover $K$ by finitely many discs $D^1_{\varepsilon_k}, D^2_{\varepsilon_k}, \ldots, D^{j_k}_{\varepsilon_k}$ of radius $\varepsilon_k$, centered at $p^1_k, p^2_k, \ldots, p^{j_k}_k$, respectively. By re-labeling, we put $\cup_{k=1}^\infty \cup_{l=1}^{j_k} p^l_k = \{p_k\}_{k \geq 1}$. By property (i), we can pick a subsequence $\{f_{n_{11}}, f_{n_{12}}, \ldots, f_{n_{1l}}, \ldots\} \subset \{f_n\}_{n \geq 1}$, such that $\{f_{n_{1l}}(p^1)\}_{l \geq 1}$ converges. Next, pick $\{f_{n_{2l}}\}_{l \geq 1} \subset \{f_{n_{1l}}\}_{l \geq 1}$ such that also $\{f_{n_{2l}}(p^2)\}_{l \geq 1}$ converges. Inductively, pick $\{f_{n_{rl}}\}_{l \geq 1} \subset \{f_{n_{(r-1)l}}\}_{l \geq 1}$ such that

$$\{f_{n_{rl}}(p_j)\}_{l \geq 1},\ j = 1, 2, \ldots, r$$

all converge. Then the *diagonal sequence* $\{f_{n_{rr}}\}_{r \geq 1}$ has the property (check!) that

$$\{f_{n_{rr}}(p_j)\}_{r \geq 1}$$

converges *for all* $j \geq 1$. Now we conclude via the following

**Claim**: *The sequence* $\{f_{n_{rr}}\}_{r \geq 1}$ *converges uniformly to some* $f \in C^0(K)$.

*Proof.* (Claim) We first prove that $\{f_{n_{rr}}(x)\}_{r\geq 1}$ converges for each $x\in K$. Given $x\in K$, pick a sequence $\{p_{j_k}\}_{k\geq 1}\subset\{p_k\}_{k\geq 1}$ with $\lim_{k\to\infty}p_{j_k}=x$. Given $\varepsilon>0$, pick $k_0\in\mathbf{N}$ such that $|f_n(p_{j_k})-f_n(x)|<\frac{\varepsilon}{2}$ $\forall k\geq k_0$, $\forall n\geq 1$; this is possible on account of (ii). Then

$$\begin{aligned}\limsup_{k,l\to\infty}|f_{n_{kk}}(x)-f_{n_{ll}}(x)|\leq&\limsup_{k,l\to\infty}|f_{n_{kk}}(p_{j_{k_0}})-f_{n_{ll}}(p_{j_{k_0}})|\\&+\limsup_{k\to\infty}|f_{n_{kk}}(x)-f_{n_{kk}}(p_{j_{k_0}})|\\&+\limsup_{l\to\infty}|f_{n_{ll}}(x)-f_{n_{ll}}(p_{j_{k_0}})|\\&\leq\varepsilon\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, the sequence $\{f_{n_{kk}}(x)\}_{k\geq 1}$ is Cauchy. Define $f(x):=\lim_{k\to\infty}f_{n_{kk}}(x)$. In order to complete the proof of the Claim, it suffices to show that this limit is uniform in $x$(why?).

Given $\varepsilon>0$, pick a finite subset $B:=\{p_{j_k}\}_{k=1}^{L_\varepsilon}\subset\cup_{k=1}^{\infty}p_k$ with the property that $\forall x\in K$ $\exists p_{j_l}\in B$ with $\rho(x,p_{j_l})<\delta_{\frac{\varepsilon}{3}}$ with $\delta_\varepsilon$ as in (ii). Then for any $x\in K$ we have

$$\begin{aligned}|f(x)-f_{n_{kk}}(x)|\leq&\limsup_{r\to\infty}|f_{n_{rr}}(p_{j_l})-f_{n_{kk}}(p_{j_l})|\\&+\limsup_{r\to\infty}|f_{n_{rr}}(p_{j_l})-f_{n_{rr}}(x)|\\&+|f_{n_{kk}}(p_{j_l})-f_{n_{kk}}(x)|\end{aligned}$$

Since $\{f_{n_{kk}}(p_{j_l})\}_{k\geq 1}$ converges, we can pick $k_0$ sufficiently large such that $\limsup_{r\to\infty}|f_{n_{rr}}(p_{j_l})-f_{n_{kk}}(p_{j_l})|<\frac{\varepsilon}{3}$ for $k\geq k_0$, $l=1,2,\ldots,L_\varepsilon$. Hence $|f(x)-f_{n_{kk}}(x)|\leq\varepsilon$ for all $k\geq k_0$, uniformly in $x\in K$. □

## 2. The structure of the set of solutions

The treatment here follows essentially Hormander, 'Lectures on Nonlinear Hyperbolic Differential Equations', Springer, p. 5-6. We already know that uniqueness of solutions for (1.1) fails in general under the condition $f\in C^0$. It is then natural to ask what the totality of solutions for the problem (1.1) looks like. In particular, given some $x_0\in\widetilde{I}$, we can ask what the set $\{y(x_0)|\,y(x)\text{ solves (1.1)}\}$ looks like.

**Example** Consider (*) $y'=\sqrt{|y|}$, $y(0)=0$. Here we know that the non-zero solutions (without initial condition) are either of the form $y(x)=\pm\frac{1}{4}(\pm x+C)^2$, $\pm x\geq -C$, where $C\in\mathbf{R}$ is arbitrary, or else have graphs that arise by joining part of the graph of $y(x)=0$ with one or two of the preceding graphs. In particular, for $x_0\geq 0$, say, the solutions $y(x)$ with the initial condition $y(0)=0$ satisfy

$$y(x_0)=\frac{1}{4}(x_0+C)^2,\ -x_0\leq C\leq 0$$

Thus we get $\{y(x_0)|\,y(x)\text{ satisfies (*)}\}=[0,\frac{x_0^2}{4}]$, a closed interval.

It turns out that the preceding example gives the generic behavior. In the sequel, we shall want to get rid of the added requirement that $y\in J$, and in fact assume that $f\in C^0(I\times\mathbb{R})$. Then, letting $J,M,\widetilde{I}$ be defined as before(for some $b>0$), we claim that as long as we work on $\widetilde{I}$, any solution of (1.1) (without the requirement that $y(x)\in J$) on $\widetilde{I}$ will in fact take values in $J$, i. e. we have an *a priori bound* on solutions defined on $\widetilde{I}$:

**Lemma 2.1.** *Assume that $y\in C^1(\widetilde{I},\mathbb{R})$ solves* (1.1). *Then we have*

$$y(x)\in J^o$$

*for each $x\in\widetilde{I}$.*

*Proof.* If not, then there is a solution $y(x)$ which attains the value $\eta\pm b$ for some $x_1\in\widetilde{I}$. By continuity, we may assume that $y(x_1)=\eta\pm b, y(x)\in J^o$ for $x\in[\xi,x_1)$. But then

$$y(x_1)=\eta+\int_{\xi}^{x_1}f(s,y(s))\,ds,$$

and

$$\Big|\int_{\xi}^{x_1} f(s, y(s))\,ds\Big| \le M \cdot \frac{b}{M+1} < b$$

and so $y(x_1) \in J^o$, a contradiction. □

In the following, we now assume $f \in C^0(I \times \mathbb{R})$, and we fix some $b > 0$ and construct $J, M, \widetilde{I}$ as in the preceding, so that we know that at least one solution will exist on $\widetilde{I}$. Such a solution will necessarily take values in $J^o$.

**Theorem 2.2.** *Under the same assumptions as for theorem 1.1, for any $x_0 \in \widetilde{I}$, the set*

$$J_{x_0} := \{y(x_0) |\, y(x) \text{ solves } (1.1) \text{ on } \widetilde{I}\}$$

*is a compact and connected.*

*Proof.* The compactness follows from the boundedness of $J_{x_0}$, in turn a consequence of the preceding lemma, as well as the closedness of $J_{x_0}$. To see the latter, assume that $y_n(x)$, $n = 1, 2, \ldots$, solve(1.1), and that $y_n(x_0) \to y_* \in \mathbf{R}$. By uniform continuity and boundedness, as in the proof of Peano's theorem, for the $y_n(x)$, we can then extract a subsequence $\{y_{n_k}(x)\}$, which converges uniformly on $\widetilde{I}$ to some limit $y_*(x)$. This limit solves (1.1), as follows as usual by passing to the integral equation. In particular, we have $y_n(x_0) \to y_*(x_0)$, whence $J_{x_0}$ is closed.
In order to complete the proof of the theorem, we have to show that $J_{x_0}$ is connected. this we do via contradiction: assume $J_{x_0}$ is not connected. Then we can write

$$J_{x_0} = J_1 \cup J_2, \tag{2.1}$$

where both $J_{1,2}$ are compact, and we have $\operatorname{dist}(J_1, J_2) = 2\delta > 0$. Pick two solutions $y_{1,2}(x)$ of (1.1) with the property that $y_1(x_0) \in J_1, y_2(x_0) \in J_2$. The idea then is to *deform* $y_1(x)$ *into* $y_2(x)$, and thereby construct a solution of (1.1) with $y(x_0)$ in neither $J_1$ nor $J_2$, contradicting (2.1). The technical complication here comes from the fact that we cannot use Peano's existence theorem to construct this deformation, on account of the lack of uniqueness of solutions; instead, we shall invoke the Picard theorem, via slick modification of $f(x, y)$:

First, we pick a sequence $f_j(x, y)$ which is $C^\infty$ with respect to $y$, and such that

$$f_j(x, y) \to f(x, y)$$

uniformly for $x \in \widetilde{I}$, $y \in [\eta - \frac{bM}{M+1}, \eta + \frac{bM}{M+1}]$, as in the proof of Theorem 1.1. Next, introduce the auxiliary functions

$$f_j^1(x, y) := f_j(x, y) + f(x, y_1(x)) - f_j(x, y_1(x))$$
$$f_j^2(x, y) := f_j(x, y) + f(x, y_2(x)) - f_j(x, y_2(x))$$

Observe that we then still have

$$y_1'(x) = f_j^1(x, y_1(x)),\ y_2'(x) = f_j^2(x, y_2(x)),$$

but the functions $f_j^{1,2}(x, y)$ are now $C^\infty$ smooth with respect to $y$. Also, note that

$$\lim_{j\to\infty} f_j^{1,2}(x, y) = f(x, y)$$

for $x, y$ as above.

To obtain the deformation of $y_1(x)$ into $y_2(x)$, we now consider, for each $\lambda \in [0, 1]$ the auxiliary problems

$$y_{j,\lambda}' = \lambda f_j^1(x, y_{j,\lambda}) + (1 - \lambda) f_j^2(x, y_{j,\lambda}), y_{j,\lambda}(\xi) = \eta, \tag{2.2}$$

By Picard'd theorem, there is a unique solution $y_{j,\lambda}$ on $\widetilde{I}$, and of course when $\lambda = 0$, $y_{j,\lambda}(x) = y_2(x)$, while when $\lambda = 1$, $y_{j,\lambda}(x) = y_1(x)$ (bad notation...). Specializing to $x = x_0$, we see that as $\lambda$ traces out $[0, 1]$, $y_{j,\lambda}(x_0)$ connects $y_1(x_0) \in J_1$ to $y_2(x_0) \in J_2$. To be more precise, we need

**Exercise**: verify that $y_{j,\lambda}$ depends continuously on $\lambda$ (for fixed $j$).

Since we have $\mathrm{dist}(J_1, J_2) = 2\delta > 0$, for each $j \in \mathbf{N}$ we can find $\lambda_j \in (0,1)$ such that

$$\mathrm{dist}(y_{j,\lambda_j}(x_0), J_1 \cup J_2) \geq \delta$$

Using the same compactness argument as in the proof of Peano's theorem, the set $\{y_{j,\lambda_j}(x)\}_{j\geq 1} \subset C^0(\widetilde{I})$ is compact, whence we can extract a uniformly converging subsequence, which converges to some $y_*(x) \in C^0(\widetilde{I})$. Then

**Exercise**: verify that $y_*(x)$ is $C^1$ and solves (1.1).

We then clearly also have $\mathrm{dist}(y_*(x_0), J_1 \cup J_2) \geq \delta$, and this contradicts (2.1). □

We can refine the preceding theorem specifically for scalar ODEs a bit as follows: we know that

$$J_{x_0} = [a(x_0), b(x_0)],\ x_0 \in J_{x_0},$$

for some functions $a(x_0), b(x_0)$. The next theorem says that these functions are *themselves solutions of* (1.1). Specifically:

**Theorem 2.3.** *Both the upper limit $a(x_0)$ and $b(x_0)$, $x_0 \in \widetilde{I}$, solve* (1.1)*; we call them the maximal, resp. the minimal solution.*

*Proof.* We first note that whenever $y_1(x), y_2(x)$ solve (1.1), so does

$$y_*(x) := \max\{y_1(x), y_2(x)\}$$

To see this, distinguish between points $x$ where $y_1(x) \neq y_2(x)$ and those where $y_1(x) = y_2(x)$. In the first case, there is a neighborhood of $x$ where $\max\{y_1(x), y_2(x)\} = y_1(x)$ or $y_2(x)$, whence the statement is clear. If we have $y_1(x) = y_2(x)$, then by differentiability of $y_{1,2}(x)$, we have

$$y_1(x+t) = y_1(x) + tf(x, y_1(x)) + o(t),\ y_2(x+t) = y_2(x) + tf(x, y(x)) + o(t)$$

from which we infer

$$y_*(x+t) = y_*(x) + tf(x, y_*(x)) + o(t)$$

This implies differentiability of $y_*(\widetilde{x})$ at $\widetilde{x} = x$, and $y_*'(x) = f(x, y_*(x))$.
Now for each $\widetilde{x} \in \widetilde{I}$, by definition of $b(\widetilde{x})$ we can choose a solution $y_{\widetilde{x}}(x)$ of (1.1) with $b(\widetilde{x}) = y_{\widetilde{x}}(\widetilde{x})$. Then pick a countable dense subset

$$\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_k, \ldots \subset \widetilde{I}$$

and consider the sequence of functions

$$y_N(x) := \max_{j=1,2,\ldots,N}\{y_{\widetilde{x}_j}(x)\}$$

According to the preceding paragraph applied inductively, we see that $y_N(x)$ solves (1.1). Furthermore, since as in the proof of Theorem 1.1 the set $\{y_N(x)\}_{N\geq 1} \subset C^0(\widetilde{I})$ is compact, we may select a subsequence $\{y_{N_k}(x)\}_{k\geq 1}$ which converges to some $y_*(x)$ (where as usual we restrict $x \in \widetilde{I}$) solving (1.1). But since $y_N(\widetilde{x}_j) = b(\widetilde{x}_j)$ for $j \leq N$, we necessarily obtain

$$y_*(\widetilde{x}_j) = b(\widetilde{x}_j)\ \forall j \geq 1,$$

and by density of $\{\widetilde{x}_j\}_{j\geq 1} \subset \widetilde{I}$, we infer that $y_*(x) = b(x)$ for all $x \in \widetilde{I}$.

**Exercise**: *Verify this last step by proving the continuity of the function $x_0 \to b(x_0)$.*

The argument for $a(x)$ is similar. □