ODE FALL 2010, LECTURE 2

1. Preparations for the Picard and Peano Theorems

We now want to study the general initial value problem

$$y' = f(x, y),\ y(\xi) = \eta \tag{1.1}$$

where $f(x, y)$ is a continuous function defined for $x \in I$, $y \in J$, with $\xi \in I$, $\eta \in J$, and as usual I, J two open intervals.

With this level of generality, we can no longer expect to find explicit solutions via simple algebraic tricks. Thus, we introduce abstract methods, thereby establishing existence of solutions in a 'function space'. Remarkably, the method used to establish the existence theorem of Picard may be used to actually compute the solutions numerically.

1.1. **Normed vector spaces.** We begin here by recalling the concept of a general vector space V over the field of real numbers $\mathbf{R}$ (alternatively, the field of complex numbers $\mathbf{C}$):

Definition 1.1. *A vector space V is a set (whose elements we call 'vectors'), equipped with an 'addition' operation $+ : V \times V \to V$, as well as a multiplication operation $\cdot : \mathbf{R} \times V \to V$ called 'scalar multiplication', such that*

(i) $(V, +)$ is an abelian group

(ii) the following compatibility relations between 'addition' and 'multiplication' are satisfied:

$$\lambda(v + w) = \lambda v + \lambda w,\ (\lambda, v, w) \in \mathbf{R} \times V \times V \text{ arbitrary}$$

$$(\lambda + \mu)v = \lambda v + \mu v,\ (\lambda, \mu, v) \in \mathbf{R} \times \mathbf{R} \times V$$

$$(\lambda\mu)v = \lambda(\mu v)$$

$$1 \cdot v = v$$

Important examples of vector spaces in our context are

Example: for $I \subset \mathbf{R}$ open and non-empty, the sets $C^k(I)$, $k \geq 0$, equipped with point wise addition and scalar multiplication (i. e. $(f + g)(x) = f(x) + g(x)$ etc), form a vector space (check!).

Exercise: let $I = [a, b]$ with $-\infty < a < b < \infty$. Show that $V := \{f : I \to \mathbf{R} | f \text{ Riemann integrable}\}$ forms a vector space.

Thus far, we have only introduced algebraic structures. We now turn our vector spaces into topological objects, by imposing some extra structure:

Definition 1.2. *Let V be a vector space over $\mathbf{R}$ or $\mathbf{C}$. We call a function*

$$|\cdot| : V \to \mathbf{R}_{\geq 0}$$

a norm, provided

$$(i)_N :\ |\mathbf{0}| = 0,\ |v| > 0 \text{ if } v \neq 0$$

$$(ii)_N :\ |\lambda v| = |\lambda||v|, \text{ where } |\lambda| \text{ refers to absolute value/modulus}$$

$$(iii)_N :\ (\text{triangle inequality})\ |v + w| \leq |v| + |w|$$

Exercise: show that if $(i)_N - (iii)_N$ are satisfied, then for any n vectors we have

$$|v_1 + v_2 + \ldots + v_n| \leq |v_1| + |v_2| + \ldots + |v_n|$$
$$\big||v| - |w|\big| \leq |v - w|$$

Given a normed vector space $(V, |\cdot|)$, we can introduce a 'distance function' on V via

$$\rho(v, w) := |v - w|$$

Then you should quickly check that the following three fundamental properties are satisfie, which imply that (V, ρ) becomes a *metric space*:

$(i)_\rho$: *we have* $\rho(v, w) > 0$ *if* $v \neq w$, *and* $\rho(v, v) = 0$ *for all* $v \in V$.

$(ii)_\rho$: *(symmetry)* $\rho(v, w) = \rho(w, v)$.

$(iii)_\rho$: *(triangle inequality)* $\rho(u, v) \leq \rho(u, w) + \rho(w, v)$.

Examples: (a) Let $V = \mathbf{R}^n$. Then we can define $|v| = \sqrt{\sum_{i=1}^n v_i^2}$, for $v = (v_1, v_2, \ldots, v_n)$, which is of course the usual Euclidean norm function. The triangle inequality $|v + w| \leq |v| + |w|$ here is equivalent to the *Cauchy-Schwarz inequality*

$$|v \cdot w| \leq |v||w|$$

(b) Now let $I \subset \mathbf{R}$ a compact interval, and let $V = C^0(I)$, i. e. the continuous real-valued functions. Set $|f| = \max_{x \in I} |f(x)|$ Carefully observe that this is always well-defined. Then $|\cdot|$ is a norm. To verify this, recall that a function $f \in C^0(I)$ equals the zero element iff $f(x) = 0$ for all $x \in I$. Then property $((i)_N$ is immediate, as is property $(ii)_N$. For property $(iii)_N$, let $f, g \in C^0(I)$ be given. Then for each $x \in I$, we have

$$|(f + g)(x)| \leq |f(x)| + |g(x)|$$

by the triangle inequality for $\mathbf{R}$, whence

$$\max_{x \in I} |f + g|(x) \leq \max_{x \in I} [|f(x)| + |g(x)|] \leq \max_{x \in I} |f(x)| + \max_{x \in I} |g(x)|$$

which is $(iii)_N$.

(c) Now let $I \subset \mathbf{R}$ a compact interval with non-empty interior, and define

$$|f| = \int_I |f(x)|\, dx$$

Exercise: show that this defines a norm on $C^0(I)$!

1.2. **Banach spaces.** Let $(V, |\cdot|)$ a normed vector space. Then given a sequence $\{x_n\}_{n \geq 1} \subset V$, we say that x_n converges to x, written as $x_n \to x$, provided

$$\lim_{n \to \infty} |x_n - x| = 0$$

One similarly defines convergence of a series $\sum_{i=1}^\infty x_i$ to some $x \in V$: this means that the sequence of partial sums $\sum_{n=1}^m x_n =: y_m$ converges to x in the sense from before.

In this context, we also have to introduce the concept of *Cauchy sequence*: a sequence $\{x_n\}_{n \geq 1} \subset V$ is called Cauchy, provided $\forall \varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbf{N}$ such that $\forall n, m \geq n_0(\varepsilon)$, we have

$$|x_n - x_m| < \varepsilon$$

Exercise: show that if $\{x_n\}_{n \geq 1} \to x$, then $\{x_n\}_{n \geq 1}$ is Cauchy!

In light of the preceding exercise, it is natural to ask whether every Cauchy sequence in a normed vector space actually converges. The general answer turns out to be no, and hence we make a

Definition 1.3. *We say a normed vector space is complete, provided every Cauchy sequence converges in* V.

Definition 1.4. *A complete normed vector space is called a Banach space*

Examples: (i) the Euclidean space $\mathbf{R}^n$ equipped with the standard Euclidean norm $|\cdot|$ is complete (check!).

(ii): Let I be a compact interval in $\mathbf{R}$, and consider $(C^0(I), |\cdot|)$, equipped with the supremum norm: $|f| = \max_{x\in I}|f(x)|$.

Theorem 1.5. *This is a Banach space*

Proof. Let $\{f_n\}_{n\geq 1} \subset C^0(I)$ a Cauchy sequence. Thus for every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbf{N}$ such that for $n, m \geq n_0(\varepsilon)$, we have

$$\max_{x\in I}|f_n(x) - f_m(x)| < \varepsilon \tag{1.2}$$

We need to show that $f_n \to f$ for some function $f \in C^0(I)$. Note that thanks to (1.2), we have that for each $x \in I$, the sequence $\{f_n(x)\}_{n\geq 1} \subset \mathbf{R}$ is Cauchy, and by completeness of $\mathbf{R}$, converges. Thus for each $x \in I$, we can find some $f(x) \in \mathbf{R}$, such that $f_n(x) \to f(x)$.

To complete the proof, we must show that f is actually a continuous function, and that indeed $f_n \to f$ in the sense of the normed vector space. Let $\varepsilon > 0$ be given, and let $x \in I$. Pick $n_0(\varepsilon)$ such that $\max_{y\in I}|f_n(y) - f_m(y)| < \frac{\varepsilon}{3}$ provided $n, m \geq n_0(\varepsilon)$. Then pick $\delta > 0$ such that

$$|f_{n_0}(x) - f_{n_0}(y)| < \frac{\varepsilon}{3}, \text{ provided } |x - y| < \delta$$

Then we have

$$|f(x) - f(y)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f(y) - f_{n_0}(y)|$$

Note that

$$|f(x) - f_{n_0}(x)| = |\lim_{n\to\infty} f_n(x) - f_{n_0}(x)| \leq \frac{\varepsilon}{3}$$

We conclude that

$$|f(x) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

This establishes continuity of f. Next, we show that $f_n \to f$. Given $\varepsilon > 0$, pick $n_0(\varepsilon)$ as above. Then for $n \geq n_0(\varepsilon)$

$$\max_{x\in I}|f(x) - f_n(x)| = \max_{x\in I}|\lim_{m\to\infty} f_m(x) - f_n(x)| \leq \lim_{m\to\infty}\max_{x\in I}|f_m(x) - f_n(x)| < \varepsilon$$

□

(iii) The next example is quite different! Consider $(C^0(I), |\cdot|)$, where I is a compact interval with non-empty interior, and $|\cdot|$ as in example (c) above. Then we have

Theorem 1.6. $(C^0(I), |\cdot|)$ *is not complete.*

Proof. We need to exhibit a Cauchy sequence $\{f_n\}_{n\geq 2} \subset C^0(I)$ which does not converge to a function in $C^0(I)$. Here we may assume that $I = [0, 1]$(check!). We construct the sequence as follows:

let $f_n(x) = 0$ if $0 \leq x \leq \frac{1}{2} - \frac{1}{n}$, $f_n(x) = 1 - n(\frac{1}{2} - x)$ if $\frac{1}{2} - \frac{1}{n} < x \leq \frac{1}{2}$, and finally $f_n(x) = 1$ if $x > \frac{1}{2}$.

Exercise: verify that $\{f_n\}_{n\geq 2}$ does not converge to a limit function $f \in C^0(I)$!

Hint: assume there is $f \in C^0(I)$ such that $f_n \to f$, i. e. $\int_0^1 |f_n(x) - f(x)|\,dx \to 0$. Then we can find $\delta \in (0, \frac{1}{2})$ such that $|f(\frac{1}{2}) - f(x)| < \frac{1}{4}$ if $|x - \frac{1}{2}| < \delta$. Then show that for n sufficiently large,

$$\int_0^1 |f_n(x) - f(x)|\,dx \geq \int_{\frac{1}{2}-\delta}^{\frac{1}{2}+\delta} |f_n(x) - f(x)|\,dx \geq \frac{\delta}{4}$$

□

The last example (iii) shows that completeness is a delicate issue for infinite dimensional normed vector spaces.

1.3. **Contraction mappings on normed vector spaces; the Contraction Principle.** Let $(V, |.|)$ a normed vector space. Recall that a *linear map* $T : V \to V$ is one that satisfies

$$T(\lambda v + \mu w) = \lambda T(v) + \mu T(w)$$

for all $\lambda, \mu \in \mathbf{R}$ (or alternatively $\lambda, \mu \in \mathbf{C}$ if this is the underlying field) and all $v, w \in V$. Of course the norm is not necessary for this definition. However, the norm and the metric it defines on V via $\rho(v, w) = |v - w|$ allow us to define the concept of *continuous linear map.* We mention here without proof that a linear

$$T : V \to V$$

is continuous precisely if $|T(v)| \leq C|v|$ for some $C \in \mathbf{R}$.
Very important for the sequel will be

Definition 1.7. *We call a map $T : V \to V$ (not necessarily linear) a contraction map, provided there exists $\alpha \in [0, 1)$ with the property that*

$$|T(v) - T(w)| \leq \alpha |v - w| \tag{1.3}$$

Here it is key that the factor $\alpha < 1$.

Exercise: Give conditions on the matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix},\ a, b, c \in \mathbf{R}$$

that ensure that the map $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by left multiplication with A is a contraction!

With the concepts we have introduced, we now have the following fundamental theorem, which we state in the context we will need it in:

Theorem 1.8. *(Contraction Principle) Let $T : V \to V$ a contraction map (not necessarily linear), where $(V, |.|)$ is a complete normed vector space (i. e. a Banach space). Then T admits a unique fixed point, i. e. there is a unique $p \in V$ with the property that*

$$T(p) = p$$

Remark 1.9. This holds more generally for complete metric spaces and contraction mappings on them.

Proof. (i) Existence of fixed point. To find such a p, pick any $x_0 \in V$ and consider the sequence

$$x_n := T^n(x_0)$$

We claim that this is a *Cauchy sequence.* To see this, pick any $\varepsilon > 0$, and let $n > m$ be natural numbers. Then we have

$$|T^n(x_0) - T^m(x_0)| \leq \alpha^m |T^{n-m}(x_0) - x_0|$$

by applying (1.3) m times, and further

$$|T^{n-m}(x_0) - x_0| \leq \sum_{i=0}^{n-m-1} |T^{n-m-i}(x_0) - T^{n-m-i-1}(x_0)| \leq |T(x_0) - x_0| \Big(\sum_{i=0}^{n-m-1} \alpha^{n-m-i-1} \Big)$$

Combining the preceding inequalities and using $\sum_{i=0}^{n-m-1} \alpha^{n-m-i-1} < \frac{1}{1-\alpha}$, we get

$$|T^n(x_0) - T^m(x_0)| \leq \frac{\alpha^m}{1 - \alpha} |T(x_0) - x_0|$$

Thus if we choose m large enough that $\frac{\alpha^m}{1-\alpha} |T(x_0) - x_0| < \varepsilon$, we have

$$|T^n(x_0) - T^m(x_0)| < \varepsilon$$

Since $(V, |\cdot|)$ is complete, we have that $\{T^n(x_0)\}_{n \geq 1}$, being Cauchy, converges to some $p \in V$. But then it is easy to see that $T(p) = p$ is a fixed point.
(ii): uniqueness. Now assume that $q \in V$ is another fixed point for T, i. e. $T(q) = q$. Then we have

$$|p - q| = |T(p) - T(q)| \leq \alpha |p - q|$$

Since $\alpha \in [0, 1)$, this forces $|p - q| = 0$, whence $p = q$. $\square$

2. The Picard Existence Theorem

We now apply the machinery developed in the previous section to prove the famous

Theorem 2.1. *(Picard Existence and Uniqueness Theorem) Consider the initial value problem*

$$y' = f(x, y),\ y(\xi) = \eta \tag{2.1}$$

Here it is assumed that $f(\cdot, \cdot)$ is continuous on $[\xi, \xi + a] \times \mathbf{R}$ where $a > 0$, and furthermore satisfies a Lipschitz condition with respect to y:

$$|f(x, y) - f(x, \bar{y})| \leq L|y - \bar{y}| \tag{2.2}$$

for some $L \in \mathbf{R}_{\geq 0}$; here all $x \in [\xi, \xi + a]$, $y, \bar{y} \in \mathbf{R}$ are allowed. Then (2.1) admits precisely one C^1-solution $y(x)$ on $J = [\xi, \xi + a]$.

Proof. **(Step 1)** *Formulation as a fixed point problem.* Observe that if $y(x)$ is a C^1-solution of (2.1), then by the fundamental theorem of calculus, we have

$$y(x) = \eta + \int_\xi^x f(t, y(t))\, dt \tag{2.3}$$

Conversely, assume that $y(x) \in C^0(J)$ satisfies equation (2.3). By our assumptions on $f(\cdot, \cdot)$, and again using the fundamental theorem of calculus, we infer $y(x) \in C^1(J)$, and by differentiating we recover (2.1).
We conclude from this that in order to solve (2.1), it suffices to solve the *integral equation* (2.3). The advantage of changing the point of view in this fashion is that integration is in some sense better than differentiation, since we gain smoothness by integrating. This will play out in the next step.

(Step 2) *Introduction of a Banach space, verifying contraction property.* Now pick some $b \in (0, a]$, and consider the space $C^0(I_b)$, where $I_b = [\xi, \xi + b]$, equipped with the norm $|f| := \max_{x \in I_b} |f(x)|$. By theorem 1.5, this is a Banach space, i. e. it is complete. Now define the map

$$T : C^0(I_b) \to C^0(I_b),\ T(y) = \eta + \int_\xi^x f(t, y(t))\, dt,$$

where of course now x is confined to I_b. Now use

Lemma 2.2. *The map T is a contraction on $C^0(I_b), |\cdot|)$ for $b \leq \min\{a, \frac{1}{2L}\}$.*

To prove this lemma, let $y, \bar{y} \in C^0(I_b)$. Then for $x \in I_b$, we have

$$T(y)(x) - T(\bar{y})(x) = \int_\xi^x \left[f(t, y(t)) - f(t, \bar{y}(t))\right] dt$$

Using (2.2) and well-known properties of the Riemann integral, we obtain

$$|T(y)(x) - T(\bar{y})(x)| \leq \int_\xi^x L|y(t) - \bar{y}(t)|\, dt \leq Lb \max_{t \in I_b} |y(t) - \bar{y}(t)|$$

Taking the maximum over $x \in I_b$, we get

$$|T(y) - T(\bar{y})| \leq Lb|y - \bar{y}| \leq \frac{1}{2}|y - \bar{y}|,$$

which proves the lemma.

(Step 3) *Application of Contraction Principle, construction of local solution.* We now apply theorem 1.8 on I_b with $b = \min\{a, \frac{1}{2L}\}$. We obtain a unique $y \in C^0(I_b)$ with the property that

$$T(y)(x) = y(x),\ x \in I_b$$

In case that $b < a$, we then repeat the preceding process on $[b, c]$ where $c = \min\{a, 2b\}$, and $y(b)$ is given by the solution on I_b. A finite number of these steps, extending the solution to $[2b, 3b]$ etc gives the solution on $J = [\xi, \xi + a]$. $\square$

3. Refinements of Picard

In many applications, the condition that $f(x,y)$ is Lipschitz continuous *uniformly in* y, i. e. that there exists a constant L such that for all $(x,y),(x,\bar{y})$ with $x\in[\xi,\xi+a]$ (but no restriction on $y,\bar{y}$!) we have

$$|f(x,y)-f(x,\bar{y})|\leq L|y-\bar{y}|$$

is often not satisfied. However, we can easily generalize the preceding theorem. First

Definition 3.1. *Let $f(x,y)$ defined for $x\in[\xi,\xi+a]$ and $y\in\mathbb{R}$. Then we say that $f(\cdot,\cdot)$ is* **locally Lipschitz** *with respect to y provided for each $y\in\mathbb{R}$, there exists a $\delta=\delta(y)>0$ and a $L=L(y)$ with the property*

$$|f(x,y_1)-f(x,y_2)|\leq L|y_1-y_2|$$

for all $y_{1,2}\in[y-\delta,y+\delta]$, and $x\in[\xi,\xi+a]$ arbitrary.

Theorem 3.2. *Let $f(x,y)\in C^0([\xi,\xi+a]\times\mathbb{R})$ be locally Lipschitz with respect to y. Then the initial value problem*

$$y'=f(x,y),\ y(\xi)=\eta$$

admits a unique C^1-solution on some interval $I=[\xi,\xi+b]$ with $b>0$ depending on the Lipschitz continuity of f at $y=\eta$ (i. e. the numbers δ,L), as well as $M=\max_{x\in[\xi,\xi+a]}|f(x,\eta)|$.

Proof. The idea is to apply Theorem 2.1 for a modified f. Thus pick $\delta=\delta(\eta)>0$ as in the above definition, and introduce the function

$$\widetilde{f}(x,y):=\chi(y)f(x,y)$$

where we pick $\chi\in C^0(\mathbb{R})$ to be such that $\chi(y)=1$ for $|y-\eta|<\frac{\delta}{2}$ and $\chi(y)=0$ for $|y-\eta|\geq\delta$. For example, one may pick χ to be piecewise linear. In particular, we can ensure (check!) that

$$|\chi(y_1)-\chi(y_2)|\leq\frac{2}{\delta}|y_1-y_2|,\ 0\leq\chi(y)\leq 1$$

Then it is easily verified that the function $\widetilde{f}(x,y)$ satisfies a *uniform Lipschitz bound* with respect to y, i. e. there exists a $\widetilde{L}$ (verify this and compute it in terms of δ,L,M !) such that we have

$$|\widetilde{f}(x,y_1)-\widetilde{f}(x,y_2)|\leq\widetilde{L}|y_1-y_2|\,\forall x\in[\xi,\xi+a],\ y_{1,2}\in\mathbb{R}$$

By Theorem 2.1, there exists a unique solution $\widetilde{y}\in C^1([\xi,\xi+a])$ with

$$\widetilde{y}'=\widetilde{f}(x,\widetilde{y}),\ \widetilde{y}(\xi)=\eta$$

To finish the proof, we observe that by the mean value theorem, there is some $x_*\in[\xi,x]$ with

$$\begin{aligned}|\widetilde{y}(x)-\eta|=|\widetilde{y}'(x_*)(x-\xi)|=|\widetilde{f}(x_*,\widetilde{y}(x_*))||x-\xi|&\leq(|f(x_*,\eta)|+\widetilde{L}\delta)|x-\xi|\\&\leq(M+\delta\widetilde{L})|x-\xi|\end{aligned}$$

It follows that

$$|\widetilde{y}(x)-\eta|\leq\frac{\delta}{2}$$

provided

$$|x-\xi|\leq\frac{\delta}{2(M+\delta\widetilde{L})}$$

Then defining $b:=\min\{a,\frac{\delta}{2(M+\delta\widetilde{L})}\}$, we have that $\widetilde{f}(x,\widetilde{y}(x))=f(x,\widetilde{y}(x))$, and in particular, we find

$$\widetilde{y}'(x)=f(x,\widetilde{y}(x)),\ \widetilde{y}(\xi)=\eta$$

for $x\in[\xi,\xi+b]$, i. e. $\widetilde{y}(x)$ is the desired solution on $[\xi,\xi+b]$. We omit the uniqueness proof. □

An important application of the preceding theorem is a *continuation criterion* that will make its appearance later on. This says that if the function $f(x,y)$ is locally Lipschitz with respect to y and *does not escape to infinity*, then the solution can be continued:

Theorem 3.3. *Let $f(x,y) \in C^0([\xi, \xi+a] \times \mathbb{R})$ be locally Lipschitz with respect to y. Assume that there is a C^1-solution $y(x) \in C^1([\xi, \xi+b))$ for some $0 < b < a$, i. e. $y' = f(x,y)$, and that there exists a* **compact set** *$K \subset \mathbb{R}$ and a sequence $x_n \to \xi + b$, $n \geq 1$, with*

$$y(x_n) \in K$$

for all $n \geq 1$. Then there exists a $b_1 > b$ and a unique solution $y_1(x) \in C^1([\xi, \xi+b_1)$ with $y_1(x) = y(x)$ provided $x \in [\xi, \xi+b)$.

Proof. Write

$$K \subset \cup_{y \in K}(y - \frac{\delta(y)}{2}, y + \frac{\delta(y)}{2})$$

with $\delta(y)$ as in the definition of 'locally Lipschitz'. By compactness, we can find a finite set $\{y_1, y_2, \ldots, y_m\} \subset K$ with

$$K \subset \cup_{i=1}^m (y_i - \frac{\delta(y_i)}{2}, y_i + \frac{\delta(y_i)}{2})$$

From this one concludes that there exist $\delta_* > 0$ (namely $\delta_* = \min_{i=1,\ldots,m} \frac{\delta(y_i)}{2}$), as well as L_* (namely $L_* = \max_{i=1,\ldots,m} L(y_i)$) such that we have

$$|f(x,y_1) - f(x,y_2)| \leq L_* |y_1 - y_2|$$

provided that $y_{1,2} \in [y - \delta_*, y + \delta_*]$ for some $y \in K$ and $x \in [\xi, \xi+a]$. Now put

$$M_* := \max_{x \in [\xi, \xi+a], y \in K} |f(x,y)|, \; \delta_1 := \frac{\delta_*}{2(M_* + \delta_* \widetilde{L}_*)}$$

where $\widetilde{L}_*$ is computed from δ_*, L_*, M_*, as in the proof of Theorem 3.2. Finally, pick $n_* \geq 1$ large enough with $|\xi + b - x_{n_*}| < \frac{\delta_1}{2}$. Then applying Theorem 3.2 to the initial value problem

$$\widetilde{y}' = f(x, \widetilde{y}), \; \widetilde{y}(x_{n_*}) = y(x_{n_*}),$$

we can extend $y(x)$ to the interval $[\xi, \xi + b + \frac{\delta_1}{2}]$, as desired. □

Corollary 3.4. *If the solution $y(x)$ cannot be extended beyond $\xi + b$, then we necessarily have*

$$\lim_{x \to \xi + b} |y(x)| = +\infty$$

Corollary 3.5. *Assume that there exists a compact set $K \subset \mathbb{R}$ such that the solution $y(x) \in K$ whenever it is defined. Then the solution $y(x)$ exists on all of $[\xi, \xi+a]$.*

Finally, we remark that all the preceding versions of Picard's theorem hold verbatim for first order systems of ODEs.