LECTURE 1

1. Introduction

In this course, we shall be concerned with two general types of problems:

1.1. Scalar ODEs. . This is an equation of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0,\ y^{(k)}(x) := \frac{d^k y}{dx^k} \tag{1.1}$$

Here $F(\cdot, \ldots, \cdot)$ is a continuous function of $(n+2)$ scalar variables, and $y = y(x)$ is the unknown function. Our problem is to determine all functions $y(x)$ which satisfy (1.1); for this to make sense, we need to require that $y(x)$ is n times continuously differentiable, i. e. that each of the formal expressions

$$y'(x) = \frac{dy}{dx},\ y''(x) = \frac{d^2 y}{dx^2}, \ldots, y^{(n)}(x) = \frac{d^n y}{dx^n}$$

exists pointwise and is continuous. In that case, we call $y(x)$ a **solution** of (1.1). In this discussion, x may range over all of $\mathbf{R}$, or else over some non-empty interval $I \subset \mathbf{R}$. Later on, we may also consider complex valued x.

Notation: *We denote the set of all n times continuously differentiable functions on an interval x by $C^n(I)$.*

We say that problem (1.1) is in *implicit form.* Often, it is more conveninent when one has an equation of the more specialized form

$$y^{(n)} = f(x, y, y', y'', \ldots, y^{(n-1)}) \tag{1.2}$$

which is called *explicit.*

In both (1.1), (1.2), we call n the **degree** of the equation. As is natural to expect, the higher the degree, the more difficult the problem typically becomes.

1.2. First Order Systems of ODEs. Here the problem is to determine all tuples of functions

$$y_1(x), y_2(x), \ldots, y_n(x),$$

where again $x \in I \subset \mathbf{R}$, and such that we have

$$\begin{aligned} y_1'(x) &= f_1(x, y_1, \ldots, y_n) \\ y_2'(x) &= f_2(x, y_1, \ldots, y_n) \\ &\ldots \\ y_n'(x) &= f_n(x, y_1, \ldots, y_n) \end{aligned} \tag{1.3}$$

Here the functions $f_1(\cdot, \ldots, \cdot), \ldots, f_n(x, y_1, \ldots, y_n)$ are continuous functions of $(n+1)$ variables each. A n-tuple of functions $(y_1(x), y_2(x), \ldots, y_n(x)) \in C^1(I) \times \ldots \times C^1(I)$ is called a solution, provided (1.3) is identically satisfied.

Exercise: *Show that every problem of type* (1.2) *can be cast as a first order system with n variables.*

We shall sometimes switch back and forth between the systems and scalar point-of-view, since the former can sometimes simplify the latter for problems of type (1.2).

2. Examples of ODEs

Here we consider some significant examples that serve as motivation for the following developments. We pick them from important applications for the theory of ODEs.

2.1. **Biology.** *(a) Population Growth.* Here we consider a very simplistic model which you have encountered before, but whose importance cannot be over-emphasized for the theory of ODE's. As this example describes a temporal process, we switch from x to t. Let $y(t)$ denote a population of bacteria; assume that for A bacteriae there is one new bacterium per unit time, across the entire population, and independently of time. Then we infer

$$y(t + \triangle t) - y(t) = \frac{1}{A} y(t)\triangle t$$

We then idealize the discrete process of bacterial reproduction by a continuous one and pass to the limit $\triangle t \to 0$, which results in the model

$$y'(t) = cy(t),\ c = \frac{1}{A} \tag{2.1}$$

This is of course a very simple ODE and we can guess its solution from our highschool background. Nonetheless, we can observe here already some of the essential features of the general theory for first order scalar ODE. By trial and error or example, we infer the solutions $y(t) = \lambda e^{ct}$, where $\lambda \in \mathbf{R}$ is arbitrary. Of course to be meaningful for our model, we would pick $\lambda \geq 0$.
In order to remove the ambiguity due to λ, we impose what is called an

Initial condition. This means that we impose the value $y(t_0)$ for some $t_0 \in \mathbf{R}$, for example $y(0) = y_0$. Then we get a unique member of our family of solutions, $y(t) = y_0 e^{ct}$ solving the

Initial value problem:

$$y'(t) = cy(t),\ y(0) = y_0 \tag{2.2}$$

It is natural to ask whether there are other possible solutions, i. e. C^1-functions $y(t)$ solving (2.2). *This is not the case.* To see it, assume $y(t)$ is another such solution. Then multiply (2.1) by e^{-ct} and use Leibniz' rule to infer

$$[e^{-ct} y(t)]' = 0$$

whence $e^{-ct}y(t) = \mathit{const}$. But this constant is necessarily equal to y_0, due to the initial condition.

Thus the initial value problem (2.2) admits a unique solution globally in time. Further, this solution depends continuously on the initial data, in the following restrictive sense:

For given y_0, t_0, and $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\widetilde{y}_0$ with $|\widetilde{y}_0 - y_0| < \delta$, denoting the solution with initial condition $y_0, \widetilde{y}_0$ by $y(t), \widetilde{y}(t)$, we have

$$|y(t_0) - \widetilde{y}(t_0)| < \varepsilon$$

If an ODE satisfies the three requirements of *(i) existence of a solution, (ii) uniqueness of such solution with the given initial condition, and (iii) continuous dependence of solution on initial condition*, then we call it **well-posed**.

Remark 2.1. We shall soon see that even some harmless looking ODEs may not be well-posed in this sense, because uniqueness may fail. Nonetheless, we shall be able to establish local existence for a broad class of initial value problems of the type $y' = f(x, y)$, $y(t_0) = y_0$, where we only assume $f(\cdot, \cdot)$ to be continuous.

(b): Stunted population growth; the logistic equation. The preceding model is overly simplified since it assumes that the population can grow arbitrarily. More realistically, there is some ceiling beyond which the

population cannot grow. Calling M this ceiling, a simple model modifying (2.1) to reflect such an upper bound is

$$y'(t) = cy(t)[M - y(t)],\ c, M > 0 \tag{2.3}$$

We shall soon see without effort that the solutions display the following behavior:

2.2. **Celestial Mechanics, Physics.** The next examples are much more sophisticated, and belong to the realm of systems of ODEs. Indeed, they are what motivated the field of *dynamical systems.*

(a): Two-body problem. We imagine the sun of mass m_2 fixed at the coordinate origin of $\mathbf{R}^2$, and the planet of mass $m_1 << m_2$ moving in that plane (this is not a restriction for the two-body problem) with coordinates $\mathbf{x}(t) =< x_1(t), x_2(t) >$. The force on the planet as exerted by the sun is given by the vector

$$\mathbf{F} = -Gm_1 m_2 \frac{\mathbf{x}}{|\mathbf{x}|^3},\ |\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$$

and by Newton's law of acceleration, this equals $m_1 \ddot{\mathbf{x}} = m_1 \frac{d^2\mathbf{x}}{dt^2}$. We therefore obtain the vector-valued(!) ODE

$$\ddot{\mathbf{x}} = -Gm_2 \frac{\mathbf{x}}{|\mathbf{x}|^3}, \tag{2.4}$$

At first sight this does not fall into either category (1.1) or (1.3), however, we can turn it into a first order system by introducing the additional variables

$$y_1 = \dot{x}_1,\ y_2 = \dot{x}_2,$$

by means of which we get

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{y}_1 &= -c \frac{x_1}{(x_1^2 + x_2^2)^{\frac{3}{2}}} \\ \dot{x}_2 &= y_2 \\ \dot{y}_2 &= -c \frac{x_2}{(x_1^2 + x_2^2)^{\frac{3}{2}}} \end{aligned} \tag{2.5}$$

where we put $c = Gm_2$. Here the solutions are quite nice, the planet moves on a conic section with one focus co-inciding with the origin. By solving this problem explicitly, Newton demonstrated Kepler's laws, which had been obtained by sifting through numerical data by Kepler!

(b): Three-body problem. Now assume that instead we have three planets P_1, P_2, P_3 with masses $m_{1,2,3}$, which all attract one another. Writing the coordinates of P_i as $\mathbf{x}_i =< x_i, y_i >$, $i = 1, 2, 3$, where we assume that the three planets move in the same plane (which is a special situation), we get the system of ODEs(in very compact notation!)

$$\ddot{\mathbf{x}}_i = -\sum_{j \neq i} Gm_j \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} \tag{2.6}$$

As a very simple but tedious exercise, you can write this out as a first order system, which requires the use of 12 variables. In effect, one can reduce the complexity by noticing that the center of mass doesn't move, which still leaves one with 8 variables to deal with. Here closed-form solutions as for the two-body problem are the exception, and the general solutions are extremely complicated. If you are interested in this kind of topic, the book 'Lectures on Celestial Mechanics' by Siegel and Moser is recommended.

2.3. **Engineering.** *The Van der Pol's equation.*

This is the following second order differential equation

$$\ddot{x} = -x + (1 - \dot{x}^2)\dot{x}$$

which comes up in the theory of electrical circuits. Here we again think of x as a function of time t. We can interpret this as a first order system with two unknowns $v(t) = \dot{x}(t)$, $x(t)$, as follows:

$$\dot{x} = v$$

$$\dot{v} = -x + (1 - v^2)v$$

To visualize what the solutions look like, we can plot the 'trajectory' of the points $(x(t), v(t))$ in the two-dimensional (x, v)-plane, the 'phase-space' of the equation. Given an initial condition $x(0) = x_0$, $v(0) = v_0$, we then obtain a curve on which the values of $(x(t), v(t))$ lie. The remarkable about the solutions to this equation is that no matter how we choose the initial data away from the unique fixed point $(0, 0)$, the solution always approaches a fixed 'limiting cycle', corresponding to a unique periodic solution! This is an instance of the famous Poincare-Bendixson theorem, which we will consider in this course.

3. Three fundamental types of first order scalar ODEs

We now begin with our development of the theory, starting with explicitly solvable general classes of problems. We will soon leave this realm and introduce abstract/qualitative methods which allow conclusions without explicitly solving the equations. However, we first have some fun with explicitly solvable models, in the spirit of the 17/18th century.

3.1. **Separable first order ODEs.** . This refers to the following type of equation:

$$y'(x) = g(y)f(x) \tag{3.1}$$

As usual x is the independent variable, while y the dependent one. The reason for the name comes from the natural urge to separate the variables via formal manipulation, as follows:

$$\frac{dy}{dx} = g(y)f(x) \longrightarrow \frac{dy}{g(y)} = f(x)dx,$$

and then integrate

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

The expression on the right will equal some function $G(y) + C_1$, at least as long as $g(y)$ is 'sufficiently well-behaved', and the right hand equals $F(x) + C_2$. If we then assume that $G(y)$ can be inverted, i. e. that G^{-1} exists, we can in principle determine all solutions via $y(x) = G^{-1}(F(x) + C)$ where $C = C_2 - C_1$ is an arbitrary constant. Of course, this is not mathematics, and needs to be turned into such:

Theorem 3.1. *Let $f(x)$, $g(y)$ be C^0-functions on some non-empty open intervals $I, J \subset \mathbf{R}$ respectively. Assume that $g(y) \neq 0$ for $y \in J$, and let $\eta \in J$, $\xi \in I$ be given. Then there exists a unique $C^1(\widetilde{I})$-solution for some nonempty open interval $\widetilde{I} \subset I$, $\xi \in \widetilde{I}$, to the problem*

$$y'(x) = g(y)f(x),\ y(\xi) = \eta, \tag{3.2}$$

which is given by the expression

$$y(x) = G^{-1} \circ F(x),\ G(y) = \int_\eta^y \frac{d\widetilde{y}}{g(\widetilde{y})},\ F(x) = \int_\xi^x f(x)\,dx$$

In particular, the theorem asserts that G^{-1} etc are meaningful and of class C^1.

Proof. Define $G(y) = \int_\eta^y \frac{d\widetilde{y}}{g(\widetilde{y})}$ for $y \in J$. This is well-defined and C^1 by the fundamental theorem of calculus with $G(\eta) = 0$, and further $G'(y) \neq 0$ for $y \in J$. By a basic theorem of analysis, there is an inverse function $H : G(J) \to J$ with the properties that

$$H \circ G = id_J,\ G \circ H = id_{G(J)}$$

Next, define $F(x) = \int_\xi^x f(x)\,dx$, which is $C^1(I)$. By shrinking I to an open non-empty interval $\widetilde{I}$, still containing ξ, we can ensure that (check!) $F(\widetilde{I}) \subset G(J)$. Now consider the function

$$y(x) := H \circ F(x),\, x \in \widetilde{I}$$

Claim 1: *$y(x)$ solves* (3.2). Indeed, first observe that by the chain rule

$$H'(y) = \frac{1}{G'(H(y))} = g(H(y)),\, y \in G(J),$$

whence again by the chain rule

$$y'(x) = H'(F(x))F'(x) = g(H(F(x)))F'(x) = g(y(x))f(x),\, x \in \widetilde{I}$$

Next, we obviously have $y(\xi) = \eta$.

Claim 2: *$y(x)$ as defined above is the only solution of* (3.2) *for $x \in \widetilde{I}$, the latter as above.* To see this, assume that $z(x)$ is also a solution defined on $\widetilde{I}$; in particular, $z(\widetilde{I}) \subset J$ (else the differential equation does not make sense). Then from $z'(x) = g(z(x))f(x)$ we obtain for $x \in \widetilde{I}$

$$\int_\xi^x \frac{z'(\widetilde{x})}{g(z(\widetilde{x}))}\,d\widetilde{x} = \int_\xi^x f(\widetilde{x})\,d\widetilde{x}$$

But the integral on the left equals

$$\int_\xi^x \frac{z'(\widetilde{x})}{g(z(\widetilde{x}))}\,d\widetilde{x} = \int_\eta^{z(x)} \frac{d\widetilde{y}}{g(\widetilde{y})}$$

via substituting $z(\widetilde{x}) = \widetilde{y}$, whence since $x \in \widetilde{I}$, we must have $z(x) = H(F(x))$. □

Corollary 3.2. *Let $f(x)$, $g(y)$ be C^0-functions on some non-empty open intervals $I, J \subset \mathbf{R}$ respectively. Assume that $g(\eta) \neq 0$, $\eta \in J$, and let $\xi \in I$ be given. Then there exists a unique $C^1(\widetilde{I})$-solution for some nonempty open interval $\widetilde{I} \subset I$, $\xi \in \widetilde{I}$, to the problem*

$$y'(x) = g(y)f(x),\, y(\xi) = \eta, \tag{3.3}$$

Next, assume $y(x) \in C^1(I)$ solves

$$y'(x) = g(y)f(x),\, y(\xi) = \eta, \tag{3.4}$$

in particular $g(y(x)) \in J$ for all $x \in I$, and further assume that $g(y(x)) \neq 0$ for all $x \in I$. Then $y(x)$ is the only solution of (3.4).

Proof. For the first part, assuming $g(\eta) \neq 0$, there exists on open interval $\widetilde{J}$ containing η and such that $g(y) \neq 0$ for all $y \in \widetilde{J}$. Then replace J by $\widetilde{J}$, g by $g|_{\widetilde{J}}$, and apply the preceding theorem to obtain a solution on some open interval $\widetilde{I}$ containing ξ. To get uniqueness, assume $z(x)$ is another solution on $\widetilde{I}$, which however doesn't necessarily map into $\widetilde{J}$(but into J, of course). Then the set of points $x \in \widetilde{I}$ for which $y(x) = z(x)$ is (relatively)closed(continuity of y, z), and it is also open. In fact, if $y(x_*) = z(x_*)$ for some $x_* \in \widetilde{I}$, then for x sufficiently close to x_* we still have $z(x) \in \widetilde{J}$ since $y(x_*) \in \widetilde{J}$, and hence by the argument above we get $z(x) = y(x)$ for x sufficiently close to x_*. Since $\widetilde{I}$ is connected, we have $z(x) = y(x)$.
For the second part of the corollary, if $\widetilde{y}$ is another C^1-solution of (3.4), then as before the set $A := \{x \in I \,|\, y(x) = \widetilde{y}(x)\}$ is closed as well as open (first part of corollary), hence equals all of I. □

To apply this, we consider the example from before

Example 1: consider the logistic equation (2.3). Here we have $f(t) = c$ (with t replacing x) and $g(y) = y(M-y)$. We can apply the preceding corollary as long as the solution $y(t)$ stays away from $y = 0$ or $y = M$. Assuming our initial condition at time $t = 0$ are chosen to be neither 0 nor M, we can compute

$$\int \frac{dy}{y(M-y)} = \int c\,dt = ct + C_0$$

for some constant C_0 which will be chosen to match the initial condition. This yields

$$\frac{1}{M}\log\left[\left|\frac{y}{M-y}\right|\right] = ct + C_0$$

If the data $y(0) = y_0 < M$, we then get the solution

$$y(t) = \frac{M}{1+e^{-M(ct+C_0)}}$$

where $C_0 = -\frac{1}{M}\log(\frac{M}{y_0} - 1)$. On the other hand, when $y(0) = y_0 > M$, we have

$$y(t) = \frac{M}{1-e^{-M(ct+C_0)}},\ C_0 = -\frac{1}{M}\log(1-\frac{M}{y_0})$$

As these solutions satisfy $y(t)(M - y(t)) \neq 0$ for all $t \in I$, the corollary implies that they are the unique solutions of the corresponding initial value problem. What happens if $y_0 = M$? An obvious solution is $y(t) = M$, the constant one.

Exercise: *show that this is the only solution with $y_0 = M$.*

Theorem 3.1 and corollary are not quite satisfactory since they do not give us a result for the case when $g(\eta) = 0$. Of course in this case, (3.2) has a solution $y(x) = \eta$, but it is not immediately clear that that is the only solution. Indeed, the following example shows that one needs to be cautious:

Example 2: Consider $y' = \sqrt{|y|}$, i. e. $f(x) = 1$, $g(y) = \sqrt{|y|}$. Impose the initial condition $y(0) = 0$. Then $y(x) = 0$ is a solution, but it's not the only solution; in fact, there are infinitely many more! To see this, we first seek more general solutions of the ODE without the given initial condition. One easily finds the family $y(x) = \frac{1}{4}(x + C)^2$, where $C \in \mathbf{R}$ is arbitrary, but it needs to be assumed that $x > -C$ (check!). But then we can construct more general solutions to the preceding initial value problem, as follows: for $C < 0$ arbitrary, define $y_C(x)$ by

$$y_C(x) = 0 \text{ if } x \leq -C,\ y_C(x) = \frac{1}{4}(x + C)^2 \text{ if } x > -C.$$

Then check that this is a C^1-solution to our initial value problem.
An inspection of the graph of $g(y)$ suggests that this may be related to the cusp at $y = 0$. Indeed, we can easily see that imposing sufficient smoothness on $g(y)$ ensures uniqueness of solutions:

Theorem 3.3. *Let η as in* (3.2) *be such that $g(\eta) = 0$ and $g(y) \neq 0$ for y sufficiently close but not equal to η, and assume for any $\alpha > 0$ the integrals $\int_\eta^{\eta\pm\alpha} \frac{d\widetilde{y}}{g(\widetilde{y})}$ diverge. Then problem* (3.2) *admits only the constant solution $y(x) = \eta$.*

Proof. Assume that there is a solution $y(x)$ different from the constant one. Without loss of generality, we may assume that there is some $x_1 > x_0 \geq \xi$ such that $y(x_0) = \eta, y(x) > \eta, g(y(x)) \neq 0$ for $x \in (x_0, x_1]$. Then pick $x_1 > x_2 > x_0$, and recall the identity

$$\int_{y(x_2)}^{y(x_1)} \frac{d\widetilde{y}}{g(\widetilde{y})} = \int_{x_2}^{x_1} f(\widetilde{x})\, d\widetilde{x},$$

since the arguments of the proof of theorem 3.1 apply between x_1, x_2. But then letting $x_2 \to x_0$ we obtain contradiction since the left hand integral diverges while the right hand one doesn't. $\square$

Using the preceding theorem, we can re-capture the uniqueness statement for example 1 above , but of course the theorem does not apply to example 2 (check!).

The different phenomena exhibited by these examples will recur next lecture when we prove the much more general existence theorems of Picard and Peano.

Example 3: Here we consider a model problem which in some sense displays the reverse phenomenon than the preceding one, namely $g(y)$ *vanishes too rapidly*:

$$y' = y^2,\ y(1) = c \neq 0$$

Using separation of variables, we obtain

$$\frac{1}{y} = -x + C$$

where the initial condition implies $C = 1 + c^{-1}$, whence we find the solution

$$y(x) = \frac{1}{1 + c^{-1} - x}$$

This function 'blows up', i. e. becomes infinite, when $x \to 1 + c^{-1}$. Thus even though this ODE looks perfectly fine with smooth (infinitely differentiable) $g(y)$, the solution cannot exist on all of $\mathbb{R}$.

3.2. **Homogeneous and related.** We call the differential equatin

$$y' = f(\frac{y}{x}) \tag{3.5}$$

homogeneous. Here again x is the independent variable, and we seek the dependent variable y in terms of x. One solves this by reducing it to the separable case via a trick: introduce the new dependent variable $u := \frac{y}{x}$. Then we have

$$u' = \frac{y'}{x} - \frac{y}{x^2},$$

whence we obtain the equation

$$u' = \frac{1}{x}[f(u) - u],$$

which is separable with $f(x) = \frac{1}{x}$, $g(u) = f(u) - u$.

A slightly more complicated type of equation which can be reduced to the homogeneous and separable cases is the following:

$$y' = f(\frac{ax + by + c}{\alpha x + \beta y + \gamma}), \tag{3.6}$$

where $a, b, c, \alpha, \beta, \gamma$ are real constants, and we exclude those cases where the fractional expression is not defined or vanishes identically.

(i) First assume that

$$\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} = 0$$

Then if say $(a, b) \neq 0$ in $\mathbf{R}^2$, we can write this as

$$y' = h(ax + by),$$

where

$$h(z) = f(\frac{z + c}{\frac{\alpha}{a} z + \gamma})$$

Next, introducing the new dependent variable $z = ax + by$, we have

$$z' = a + bh(z),$$

which is separable. The case $(\alpha, \beta) \neq 0$ is handled analogously.

(ii) Now assume that

$$\begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} \neq 0$$

Then we can find (x_0, y_0) with the property that

$$\begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} c \\ \gamma \end{bmatrix}$$

Then we have

$$f(\frac{ax+by+c}{\alpha x+\beta y+\gamma}) = f(\frac{a\widetilde{x}+b\widetilde{y}}{\alpha\widetilde{x}+\beta\widetilde{y}})$$

where we introduced $\widetilde{x} = x + x_0$, $\widetilde{y} = y + y_0$. From here we deduce the equation

$$\widetilde{y}' = f(\frac{a+b\frac{\widetilde{y}}{\widetilde{x}}}{\alpha+\beta\frac{\widetilde{y}}{\widetilde{x}}}),$$

which is homogeneous.

3.3. **Linear ODEs.** . These are problems of the form

$$y'(x) + g(x)y(x) = h(x), y(\xi) = \eta,\ g \in C^0(\mathbb{R}),\ h \in C^0(\mathbb{R})$$

We distinguish between two cases:

The homogeneous case: $h(x) = 0$. The key idea is to turn this into a directly integrable ODE by multiplying by an *integrating factor*, which in this case equals $e^{\int_\xi^x g(s)\,ds}$. Indeed, we then see that the problem becomes

$$\frac{d}{dx}[y(x)e^{\int_\xi^x g(s)\,ds}] = 0,$$

whence we have

$$y(x) = \lambda e^{-\int_\xi^x g(s)\,ds}$$

for a suitable constant $\lambda \in \mathbb{R}$. The initial condition reveals that $\lambda = \eta$.

The in-homogeneous case: $h(x) \neq 0$. We take advantage of the same multiplying factor to obtain

$$\frac{d}{dx}[y(x)e^{\int_\xi^x g(s)\,ds}] = e^{\int_\xi^x g(s)\,ds}h(x)$$

Upon integrating, we obtain

$$y(x)e^{\int_\xi^x g(s)\,ds}] - \eta = \int_\xi^x e^{\int_\xi^{\widetilde{x}} g(s)\,ds}h(\widetilde{x})\,d\widetilde{x}$$

Thus we obtain the following representation of the solution:

$$y(x) = \eta e^{-\int_\xi^x g(s)\,ds} + e^{-\int_\xi^x g(s)\,ds}\int_\xi^x e^{\int_\xi^{\widetilde{x}} g(s)\,ds}h(\widetilde{x})\,d\widetilde{x}$$

Note that the first expression on the right is a solution of the homogeneous problem with the given initial condition, while the second more complicated expression is a special solution of the in-homogeneous problem with vanishing initial condition.