MORE ON FOURIER SERIES.

The material in the lecture is based on the book by Muscalu and Schlag (Vol I).

1. Applications of the Fejer kernel

So far, we have worked with Fourier series in the context of continuous functions, but it is just as natural to consider Fourier series associated with L^p-functions on S^1, where $1 \leq p \leq \infty$. To understand in what sense the Fourier series converges toward $f \in L^p(S^1)$, we start with the following basic

Lemma 1.1. *Let $\{\Phi_N(x)\}$ be an approximate identity on S^1. Then we have*

$$\lim_{N\to\infty} \|\Phi_N * f - f\|_{L^p} \to 0$$

as $N \to \infty$ provided $f \in L^p(S^1)$, $1 \leq p < \infty$.

Proof. Given $\varepsilon > 0$ and $f \in L^p(S^1)$, $1 \leq p < \infty$, pick $g \in C^0(S^1)$ with

$$\|f - g\|_{L^p} < \frac{\varepsilon}{3}.$$

Further, pick N sufficiently large, such that

$$\|\Phi_N * g - g\|_{L^\infty} < \frac{\varepsilon}{3},$$

which then implies

$$\|\Phi_N * g - g\|_{L^p} < \frac{\varepsilon}{3}$$

by Holder's inequality. Then, write

$$\|\Phi_N * f - f\|_{L^p} \leq \|\Phi_N * f - \Phi_N * g\|_{L^p} + \|\Phi_N * g - g\|_{L^p} + \|f - g\|_{L^p}.$$

To conclude things, observe that by Minkowski's inequality, we have

$$\|\Phi_N * (f - g)\|_{L^p} \leq \|\Phi_N\|_{L^1} \leq \frac{\varepsilon}{3},$$

say, provided we assume (we as may) that $\sup_N \|\Phi_N\|_{L^1} = 1$. Since $\varepsilon > 0$ was arbitrary, we find that

$$\lim_{N\to\infty} \|\Phi_N * f - f\|_{L^p} = 0.$$

□

Since the family of Fejer kernels $\{K_N\}_{N\geq 1}$ is an approximate identity, this lemma yields the following fundamental

Proposition 1.2. *Trigonometric polynomials are dense in $L^p(S^1)$, $1 \leq p < \infty$. Moreover, the exponential functions $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ form an orthonormal basis for $L^2(S^1)$, and for any pair of functions $f, g \in L^2(S^1)$, we have Parseval's identity*

$$\int_{S^1} f(x)\overline{g(x)}\, dx = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{g}(n)}.$$

Thus the map $f \to \{\widehat{f}(n)\}_{n\in\mathbb{Z}}$ is an isometry from $L^2(S^1)$ onto $l^2(\mathbb{Z})$.

Proof. The preceding lemma implies

$$\lim_{N\to\infty} \|K_N * f - f\|_{L^p(S^1)} = 0$$

for any $f \in L^p(S^1), 1 \leq p < \infty$. Since $K_N * f$ is a trigonometric polynomial, the first assertion of the proposition follows. As to the second part, we know that the collection $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ forms an orthonormal set in the Hilbert space $L^2(S^1)$, and the first part implies it is a dense set, hence forms a basis. Then Bessel's inequality (here equality) gives the desired last relation. □

To conclude this section, we finally also mention the basic *Riemann-Lebesgue lemma*:

Lemma 1.3. *Let $f \in L^1(S^1)$. Then we have*

$$\lim_{n\to\infty} |\widehat{f}(n)| = 0.$$

Proof. Given $\varepsilon > 0$, pick a trigonometric polynomial g such that

$$\|g - f\|_{L^1} < \varepsilon.$$

This is possible according to the preceding proposition. Then if n_0 is the degree of g (which we define to be the maximum of the absolute values of the n occurring in g), we have

$$\widehat{g}(n) = 0\,\forall |n| > n_0.$$

Then for such n we have

$$|\widehat{f}(n)| \leq \|f - g\|_{L^1} + |\widehat{g}(n)| < \varepsilon + 0 = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. □

2. Back to partial Fourier sums; convergence with respect to different norms

It is natural to inquire whether given $f \in L^p(S^1)$, with $1 \leq p < \infty$, whether we have

$$\lim_{N\to\infty} \|S_N f - f\|_{L^p} = 0. \tag{2.1}$$

Also, note that we still have not established (or disproved) whether for general $f \in C^0(S^1)$ we have

$$\lim_{N\to\infty} \|S_N f - f\|_{L^\infty} = 0.$$

In light of Proposition 1.2, the case $p = 2$ here is straightforward: assuming $f \in L^2(S^1)$, we have

$$\|S_N f - f\|_{L^2}^2 = \sum_{|n|>N} |\widehat{f}(n)|^2,$$

and the right hand side converges to zero provided $N \to \infty$. However, the remaining cases of p are much harder to deal with. In fact, (2.1) is *wrong* if $p = 1$ and if $p = \infty$ (in which case we put $f \in C^0(S^1)$), but *is correct for all other p*. Thus the *pathological* behaviour of the Dirichlet kernel D_N describing S_N comes to the fore as a kind of *endpoint* phenomenon.

We shall now aim to at least partially understand when (2.1) is valid. First, a crucial result coming essentially from functional analysis:

Proposition 2.1. *Let $1 \leq p < \infty$. Then we have*

$$\lim_{N\to\infty} \|S_N f - f\|_{L^p(S^1)} = 0$$

for all $f \in L^p(S^1)$ if and only if

$$\sup_N \|S_N\|_{p\to p} = \sup_N \big(\sup_{\|f\|_{L^p} \leq 1} \|S_N f\|_{L^p} \big) < \infty$$

Similarly, we have

$$\lim_{N\to\infty} \|S_N f - f\|_{L^\infty(S^1)} = 0$$

for all $f \in C^0(S^1)$ if and only if

$$\sup_N \|S_N\|_{\infty\to\infty} = \sup_N \big(\sup_{\|f\|_{L^\infty} \leq 1} \|S_N f\|_{L^\infty} \big) < \infty$$

Proof. (*if*) This is the easy part. Assume for example that for $1 \leq p < \infty$ we have

$$\sup_N \|S_N\|_{p\to p} =: M < \infty.$$

Then given $f \in L^p(S^1)$ and $\varepsilon > 0$, pick (see preceding proposition) $g \in C^\alpha$, $\alpha > 0$, with

$$\|f - g\|_{L^p} < \frac{\varepsilon}{2(M+1)}.$$

Then pick N large enough such that

$$\|S_N g - g\|_{L^p} < \frac{\varepsilon}{2}.$$

Finally, we get for such N

$$\begin{aligned}\|S_N f - f\|_{L^p} &\leq \|S_N g - g\|_{L^p} + \|S_N(f-g)\|_{L^p} + \|f-g\|_{L^p}\\ &< \frac{\varepsilon}{2} + \frac{M\varepsilon}{2(M+1)} + \frac{\varepsilon}{2(M+1)} < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the claim follows.

(*only if*) We prove this by contradiction, using a constructive argument. Thus, taking the case $1 \leq p < \infty$, say, that we have

$$\lim_{N\to\infty} \|S_N f - f\|_{L^p(S^1)} = 0 \tag{2.2}$$

for all $f \in L^p(S^1)$, but that

$$\sup_N \|S_N\|_{p\to p} = +\infty.$$

In particular, for each $l \geq 1$, there is a $N_l \in \mathbf{N}$ with

$$\|\|S_{N_l}\|_{p\to p} > 2^l,$$

which in particular means that there is a function f_l, which we may take to be a trigonometric polynomial, such that

$$\|S_{N_l} f_l\|_{L^p} > 2^l,\ \|f_l\|_{L^p} = 1.$$

We now use these functions f_l to construct a 'very bad' function $f(x) \in L^p(S^1)$ which violates (2.2), thus contradicting our assumption.

In fact, we shall set

$$f(x) = \sum_{l=1}^{\infty} l^{-2} e^{2\pi i M_l x} f_l(x)$$

where the integers M_l are chosen to satisfy the following requirements:

- $M_l - N_l \to \infty$ as $l \to \infty$, in particular, $M_l \to \infty$ as $l \to \infty$.
- Setting for $j \geq 2$

  $$g_l := \sum_{j=1}^{l-1} j^{-2} e^{2\pi i M_j x} f_j(x),$$

  we have $S_{M_l - N_l - 1} g_l = g_l$. This simply means that for every frequency n occurring in one of the trigonometric polynomials $f_j, j \leq l-1$ (a finite list), we have

  $$|n + M_j| < M_l - N_l.$$

- $M_l + n \geq 0$ for any frequency occurring in f_l.
- $S_{M_l+N_l} f = S_{M_l+N_l} g_{l+1}$.

Then we conclude that

$$\begin{aligned} S_{M_l+N_l}f - S_{M_l-N_l-1}f &= S_{M_l+N_l}g_{l+1} - S_{M_l-N_l-1}g_{l+1} \\ &= S_{M_l+N_l}g_l - S_{M_l-N_l-1}g_l \\ &+ (S_{M_l+N_l} - S_{M_l-N_l-1})(l^{-2}e^{2\pi i M_l x} f_l(x)) \\ &= l^{-2}e^{2\pi i M_l x} S_{N_l} f_l(x). \end{aligned}$$

Thus

$$\|S_{N_l+M_l}f - S_{M_l-N_l-1}f\|_{L^p} = l^{-2}\|S_{N_l}f_l\|_{L^p} > 2^l l^{-2}$$

which diverges as $l \to \infty$. On the other hand, since $M_l - N_l \to \infty$, we have

$$\begin{aligned} \limsup_{l\to\infty} \|S_{N_l+M_l}f - S_{M_l-N_l-1}f\|_{L^p} &\leq \limsup_{l\to\infty} \|S_{N_l+M_l}f - f\|_{L^p} + \limsup_{l\to\infty} \|S_{M_l-N_l-1}f - f\|_{L^p} \\ &= 0, \end{aligned}$$

a contradiction. □

We can now answer whether (2.1) holds, at least in the cases $p = 1, \infty$. In fact, the answer is negative:

Theorem 2.2. *There is $f \in L^1(S^1)$ with*

$$\lim_{N\to\infty} \|S_N f - f\|_{L^1} \neq 0$$

Also, there is a $f \in C^0(S^1)$ with

$$\lim_{N\to\infty} \|S_N f - f\|_{L^\infty} \neq 0.$$

Proof. By an exercise for you, we have

$$\|D_N\|_{L^1(S^1)} \geq c \log N$$

for a suitable $c > 0$. This implies that (with D_N denoting the Dirichlet kernel)

$$\sup_{\|f\|_{L^\infty}\leq 1} |D_N * f(0)| = \|D_N\|_{L^1} \geq c \log N,$$

and hence

$$\|S_N\|_{L^\infty \to L^\infty} \geq \sup_{\|f\|_{L^\infty}\leq 1} |D_N * f(0)| \geq c \log N.$$

Thus the preceding proposition yields the result for a $f \in C^0(S^1)$.

Next, with K_M the Fejer kernel, recall that $\|K_M\|_{L^1} = 1$, and so

$$\|D_N * K_M\|_{L^1} = \|S_N(K_M)\|_{L^1} \leq \|S_N\|_{1\to 1}.$$

On the other hand, since $\{K_M\}_{M\geq 1}$ is an approximate identity, we have

$$\lim_{M\to\infty} \|D_N * K_M\|_{L^1} = \|D_N\|_{L^1} \geq c \log N.$$

So again the preceding proposition implies the existence of a $f \in L^1(S^1)$ with the desired property. □

It is a quite remarkable fact which we shall soon understand in the context of the Hilbert transform that *one does have convergence*

$$\lim_{N\to\infty} \|S_N f - f\|_{L^p} = 0$$

for all $f \in L^p(S^1)$, provided $1 < p < \infty$.

3. Regularity and decay of Fourier coefficients

A key aspect of the Fourier transform is that it translates differentiation into multiplication: if $f \in C^1(S^1)$, then

$$\widehat{f'}(n) = \int_0^1 f'(x)e^{-2\pi inx}\,dx = 2\pi in \int_0^1 f(x)e^{-2\pi inx}\,dx = 2\pi in\widehat{f}(n).$$

In particular, if we know *a priori* that $|\widehat{f'}(n)| \leq C$ for all n, then we can infer the following decay behaviour for $\widehat{f}(n)$:

$$|\widehat{f}(n)| \leq \frac{C}{2\pi n}.$$

Conversely, making restrictions on the support of the Fourier transform, we may infer control over f' from control over f. Such results go by the name of *Bernstein's theorem*:

Theorem 3.1. *Assume $\widehat{f}(k) = 0$ for all $|k| > n$. Then we have for any $1 \leq p \leq \infty$ the inequality*

$$\|f'\|_{L^p} \leq Cn\|f\|_{L^p}$$

for a suitable constant C.

Proof. This follows with the aid of the kernel of de la Vallee Poussin, which is defined by

$$V_N(x) := (1 + e^{2\pi iNx} + e^{-2\pi iNx})K_N(x).$$

Recall that the Fejer kernel $K_N(x) = \sum_{|k|<N}(1 - \frac{|k|}{N})e^{2\pi ikx}$. Then observe that for example if $0 \leq j < N$, we have (by reading off the coefficient of the exponential $e^{2\pi iix}$)

$$\widehat{V_N}(j) = 1 - \frac{j}{N} + (1 - \frac{N-j}{N}) = 1,$$

and similarly for $-N < j < 0$. It follows that for an f as in the theorem, we have

$$\widehat{V_n * f}(j) = \widehat{V_n}(j)\widehat{f}(j) = \widehat{f}(j)\,\forall j.$$

Furthermore, we have

$$\|V_n'(x)\|_{L^1(S^1)} = \|\frac{d}{dx}\big[(1 + e^{2\pi inx} + e^{-2\pi inx})K_n(x)\big]\|_{L^1(S^1)} \leq Cn.$$

Finally, we infer that

$$\|f'\|_{L^p} = \|\frac{d}{dx}(V_n * f)\|_{L^p} = \|V_n' * f\|_{L^p} \leq \|V_n'\|_{L^1}\|f\|_{L^p} \leq Cn\|f\|_{L^p}.$$

□