THE FOURIER TRANSFORM ON $\mathbb{R}^n$

We now abandon the setting of S^1, and instead consider the unbounded case, i. e. the Fourier transform on $\mathbb{R}^n$. Note that the case $n = 1$ will help us later on elucidate certain issues left open for the Fourier transform on S^1.

1. The Schwartz space

First, we introduce a space of 'very nice functions' $\mathcal{S}(\mathbb{R}^n)$ on $\mathbb{R}^n$, which shall have the property that the Fourier transform maps $\mathcal{S}$ into itself. The definition is as follows:

Definition 1.1. *We denote by $\mathcal{S}(\mathbb{R}^n)$ the collection of all functions $f \in C^\infty(\mathbb{R}^n)$ with the property that*

$$\sup_{x\in\mathbb{R}^n} \left|(1 + |x|^N)\partial_x^\alpha f(x)\right| < \infty$$

for any $N \in \mathbf{N}$ and any $\alpha \in \mathbf{N}^n$.

Example: The function $f(x) = e^{-|x|^2}$, $|x|^2 = \sum_{j=1}^n x_j^2$, is in $\mathcal{S}(\mathbb{R}^n)$.

Clearly $\mathcal{S}(\mathbb{R}^n)$ forms an algebra (vector space invariant under multiplication of functions) which is invariant under differentiation as well as multiplication with any polynomial $P(x)$. It will serve as a natural setting for the Fourier transform on $\mathbb{R}^n$.

2. The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$.

We formally define the Fourier transform of any function $f(x)$ on $\mathbb{R}^n$ by means of the formula

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx,\ \xi \in \mathbb{R}^n.$$

It is easy to see that the integral on the right converges absolutely provided $f \in \mathcal{S}(\mathbb{R}^n)$. More importantly, we have

Proposition 2.1. *The Fourier transform maps $\mathcal{S}(\mathbb{R}^n)$ into itself.*

Proof. Observe that

$$(i\partial_\xi)^\alpha \widehat{f}(\xi) = \int_{\mathbb{R}^n} (2\pi x)^\alpha f(x)e^{-2\pi i x\cdot\xi}\,dx$$

where we use the notation $x^\alpha := \prod_{j=1}^n x_j^{\alpha_j}$, and further

$$\begin{aligned}(2\pi i\xi)^\beta(i\partial_\xi)^\alpha \widehat{f}(\xi) &= \int_{\mathbb{R}^n} (2\pi x)^\alpha f(x)(2\pi i\xi)^\beta e^{-2\pi i x\cdot\xi}\,dx\\ &= \int_{\mathbb{R}^n} (2\pi x)^\alpha f(x)(-\partial_x)^\beta e^{-2\pi i x\cdot\xi}\,dx\\ &= \int_{\mathbb{R}^n} (\partial_x)^\beta[(2\pi x)^\alpha f(x)]e^{-2\pi i x\cdot\xi}\,dx\end{aligned}$$

By assumption we have

$$\sup_{x\in\mathbb{R}^n} (1 + |x|)^{n+1}\left|(\partial_x)^\beta[(2\pi x)^\alpha f(x)]\right| < +\infty,$$

and so we find that

$$\sup_{\xi\in\mathbb{R}^n} \left|(2\pi i\xi)^\beta(i\partial_\xi)^\alpha \widehat{f}(\xi)\right| < +\infty$$

□

We shall now see that in fact the Fourier transform gives a bijection of $\mathcal{S}(\mathbb{R}^n)$ into itself, with an explicit inverse. In fact, define

$$\check{f}(x) := \int_{\mathbb{R}^n} f(\xi) e^{2\pi i x\cdot\xi}\, dx$$

Then, if $f \in \mathcal{S}(\mathbb{R}^n)$, we have the following

Theorem 2.2. *(Fourier inversion)*

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi = f(x).$$

Proof. To begin with, observe that since $\widehat{f}(\xi) \in \mathcal{S}(\mathbb{R}^n)$, the preceding integral converges absolutely. It appears natural to expand

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(y) e^{-2\pi i y\cdot\xi}\, dy$$

and switch the order of integration between y and ξ. However, we then encounter the integral

$$\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi}\, d\xi,$$

which does not converge. To remedy the situation, we introduce a damping factor, i. e. consider

$$\widehat{f}_\varepsilon(\xi) := \int_{\mathbb{R}^n} f(y) e^{-2\pi i y\cdot\xi - \varepsilon|\xi|^2}\, dy = \widehat{f}(\xi) e^{-\varepsilon|\xi|^2},\ \varepsilon > 0.$$

Then using the dominated convergence theorem we have

$$\int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \widehat{f}_\varepsilon(\xi) e^{2\pi i x\cdot\xi}\, d\xi,$$

and it is in the latter integral that we will interchange the order of integration. This is now allowed since the integral converges absolutely with respect to ξ, y. Thus write

$$\int_{\mathbb{R}^n} \widehat{f}_\varepsilon(\xi) e^{2\pi i x\cdot\xi}\, d\xi = \int_{\mathbb{R}^n} f(y) \big(\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi - \varepsilon|\xi|^2}\, d\xi \big)\, dy$$

By Fubini's theorem, we can factorise

$$\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi - \varepsilon|\xi|^2}\, d\xi = \prod_{j=1}^{n} \big(\int_{-\infty}^{\infty} e^{2\pi i(x_j-y_j)\cdot\xi_j - \varepsilon\xi_j^2}\, d\xi_j \big)$$

Then we have

Lemma 2.3. *We have the relation*

$$\int_{-\infty}^{\infty} e^{2\pi i a\cdot\xi - \varepsilon\xi^2}\, d\xi = \sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{\pi^2 a^2}{\varepsilon}}$$

Proof. First, we have

$$\int_{-\infty}^{\infty} e^{2\pi i a\cdot\xi - \varepsilon\xi^2}\, d\xi = e^{-\frac{\pi^2 a^2}{\varepsilon}} \int_{-\infty}^{\infty} e^{-\varepsilon(\xi - \frac{i\pi a}{\varepsilon})^2}\, d\xi$$

Cauchy's theorem from complex analysis allows us to shift the contour in the last integral, thus

$$\int_{-\infty}^{\infty} e^{-\varepsilon(\xi - \frac{i\pi a}{\varepsilon})^2}\, d\xi = \int_{-\infty}^{\infty} e^{-\varepsilon\xi^2}\, d\xi = \sqrt{\frac{\pi}{\varepsilon}}.$$

$\square$

The lemma allows us to conclude that

$$\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi - \varepsilon|\xi|^2}\, d\xi = (\frac{\pi}{\varepsilon})^{\frac{n}{2}} e^{-\frac{\pi^2|x-y|^2}{\varepsilon}}$$

Then we have the next

Lemma 2.4. *The family of functions $\phi_\varepsilon(x) := (\frac{\pi}{\varepsilon})^{\frac{n}{2}} e^{-\frac{\pi^2|x|^2}{\varepsilon}}$, $\varepsilon > 0$, forms an approximate identity.*

This lemma is straightforward to check, for example, observe that

$$\int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\varepsilon}} e^{-\frac{\pi^2 a^2}{\varepsilon}}\, da = 1$$

via direct change of variables, and then from Fubini

$$\int_{\mathbb{R}^n} \left(\frac{\pi}{\varepsilon}\right)^{\frac{n}{2}} e^{-\frac{\pi^2 |x|^2}{\varepsilon}}\, dx = 1.$$

Using the preceding lemma, we finally infer

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f(y) \Big(\int_{\mathbb{R}^n} e^{2\pi i (x-y)\cdot\xi - \varepsilon |\xi|^2}\, d\xi \Big)\, dy = f(x).$$

□

3. Plancherel's theorem and the Fourier transform on $L^2(\mathbb{R}^n)$

A very important consequence of the Fourier inversion theorem is the analogue of the isometric property of the Fourier transform on S^1, called *Plancherel's theorem* in the present context:

Theorem 3.1. *(Plancherel) The Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ is an L^2-isometry: more precisely, for any $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, d\xi$$

Proof. First, we observe that for any $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\widehat{g}(x)\, dx &= \int_{\mathbb{R}^n} f(x) \Big(\int_{\mathbb{R}^n} g(y) e^{-2\pi i y\cdot x}\, dy \Big)\, dx \\ &= \int_{\mathbb{R}^n} g(y) \Big(\int_{\mathbb{R}^n} f(x) e^{-2\pi i y\cdot x}\, dx \Big)\, dy \\ &= \int_{\mathbb{R}^n} g(y)\widehat{f}(y)\, dy,\end{aligned}$$

where the interchange of integrations is guaranteed by Fubini's theorem. Next, invoking Fourier inversion, we find

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx &= \int_{\mathbb{R}^n} f(x)\overline{\check{\widehat{g}}(x)}\, dx \\ &= \int_{\mathbb{R}^n} f(x)\widehat{\overline{\widehat{g}(x)}}\, dx \\ &= \int_{\mathbb{R}^n} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, d\xi\end{aligned}$$

where for the last line we invoked the preceding observation. □

Plancherel's theorem is extremely important, as it allows us amongst other things to *naturally extend* the Fourier transform to $L^2(\mathbb{R}^n)$. Observe that while this was trivial on S^1, this is far from trivial on $\mathbb{R}^n$, since the integral

$$\int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx$$

does not necessarily converge in the absolute sense if $f \in L^2(\mathbb{R}^n)$! However, given such f, we simply pick a sequence $\{f_n\}_{n\geq 1} \subset \mathcal{S}(\mathbb{R}^n)$ with

$$f_n \longrightarrow f$$

in the $L^2(\mathbb{R}^n)$-sense as $n \to \infty$. Then we know from Plancherel's theorem that $\{\widehat{f_n}\}_{n\geq 1}$ forms a Cauchy-sequence in $L^2(\mathbb{R}^n)$, and hence we can define

$$\widehat{f} := \lim_{n\to\infty} \widehat{f_n},$$

using the fact that $L^2(\mathbb{R}^n)$ is complete.

4. The Fourier transform on more general Lebesgue spaces; interpolation

To begin with, starting with the formal definition

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx,$$

it is clear that assuming only $f \in L^1(\mathbb{R}^n)$ suffices to define this integral in the point wise sense, as this integral then converges absolutely. More precisely, we have the simple

Proposition 4.1. *Given $f \in L^1(\mathbb{R}^n)$, the function*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx$$

is in $C_0^0(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$ vanishing at infinity, i. e. $\lim_{|\xi|\to\infty} |\widehat{f}(\xi)| = 0$. Moreover, we have the point wise bound

$$\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}$$

Proof. To see the continuity of $\widehat{f}(\xi)$, observe that

$$\lim_{\xi\to\xi_*} \widehat{f}(\xi) = \widehat{f}(\xi_*)$$

on account of Lebesgue's theorem of dominated convergence for all functions

$$x \to f(x)e^{-2\pi i x\cdot\xi}$$

are dominated in absolute value by $|f(x)| \in L^1(\mathbb{R}^n)$. To see convergence to zero at infinity, approximate f in $L^1(\mathbb{R}^n)$ by functions in $\mathcal{S}(\mathbb{R}^n)$ and use Proposition 2.1 above. The inequality is also clear by moving absolute values inside the integral. □

The preceding proposition and Plancherel's theorem allow us to define the Fourier transform on $L^1(\mathbb{R}^n)$, with image in $L^\infty(\mathbb{R}^n)$, as well as on $L^2(\mathbb{R}^n)$, with image in $L^2(\mathbb{R}^n)$. It is then eminently natural to enquire whether we can define in a natural way the Fourier transform on $L^p(\mathbb{R}^n)$ for more general p, with image in some $L^q(\mathbb{R}^n)$.

In fact, in Harmonic Analysis one very often encounters a more general prototype of this sort of question: one is given an operator T, which is defined naturally on some vector space of functions V on $\mathbb{R}^n$, say, (in our case this would be $\mathcal{S}(\mathbb{R}^n)$), and it is known that we have the inequalities

$$\|Tf\|_{L^{q_1}} \leq C_1\|f\|_{L^{p_1}},\ \|Tf\|_{L^{q_2}} \leq C_1\|f\|_{L^{p_2}},\ f \in V. \tag{4.1}$$

for two pairs of Lebesgue indices $(p_1, q_1), (p_2, q_2)$. In particular, T can be extended continuously as an operator between L^{p_1} and L^{q_1}, as well as an operator between L^{p_2} and L^{q_2}. (Thus in our situation, we would have $p_1 = 1, q_1 = \infty$, $p_2 = q_2 = 2$, $C_1 = C_2 = 1$.)
Then we raise the natural

Question: *Does* (4.1) *imply that T can be extended to more general L^p-spaces, and in particular, those p* **between** *p_1 and p_2?*

This is a typical question about *interpolation*, and a general answer, which will in particular clarify the situation for the Fourier transform, is furnished by the following very useful

Theorem 4.2. *(Riesz-Thorin) Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and assume (X, μ) is a measure space. Assume that T is a linear operator defined on all simple functions on X and taking values in the measurable functions on X, which satisfies*

$$\|T(f)\|_{L^{q_0}(X)} \leq A_0\|f\|_{L^{p_0}(X)},\ \|T(f)\|_{L^{q_1}(X)} \leq A_1\|f\|_{L^{p_1}(X)}.$$

for all simple functions f. Then for any $p \in [p_0, p_1]$, with $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$, $\theta \in [0,1]$, we have

$$\|T(f)\|_{L^q} \leq A_0^\theta A_1^{1-\theta} \|f\|_{L^p},$$

where q is defined via $\frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$.

Before proving this, we see how it applies to the Fourier transform: From before we know that $p_0 = 1, q_0 = \infty$ works, as well as $p_2 = q_2 = 2$. Then for $p \in [1,2]$, we have

$$\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2} \to \theta = \frac{2}{p} - 1,$$

and so

$$\frac{1}{q} = \frac{\theta}{\infty} + \frac{1-\theta}{2} = 1 - \frac{1}{p},$$

and hence we have the

Corollary 4.3. *The Fourier transform on $\mathbb{R}^n$ transforms L^p into $L^{p'}$ (with p' the Holder dual exponent) provided $1 \leq p \leq 2$, and moreover, we have*

$$\|\widehat{f}\|_{L^{p'}} \leq \|f\|_{L^p}, \, p \in [1,2].$$