CONVERGENCE OF FOURIER SERIES.

The material in the lecture is based on the book by Muscalu and Schlag (Vol I).

1. Introduction

Let $f(x)$ be a continuous function on the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$. To such a function, we can associate its Fourier coefficients:

$$\widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\,dx,\ n \in \mathbb{Z}.$$

For example, assume that $f(x)$ is of the form of a *trigonometric polynomial*, meaning

$$f(x) = \sum_{|n|=0}^{m} a_n e^{2\pi inx}.$$

Then using the crucial *orthogonality relations*

$$\int_0^1 e^{2\pi i(n-m)x}\,dx = \delta_{nm},\ n, m \in \mathbb{Z},$$

we find that

$$\widehat{f}(n) = a_n,$$

and so we have that the corresponding *Fourier series*, given *formally* by

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi inx},$$

in this case co-incides with f.

However, the Fourier series can be associated to much more general functions. In fact, we make

Definition 1.1. *Given any L^1-function on S^1, we define its Fourier series to be given by the formal series*

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{2\pi inx},\ \widehat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}\,dx.$$

More generally, if μ is a measure on S^1, we define its Fourier series correspondingly:

$$\sum_{n\in\mathbb{Z}} \widehat{\mu}(n)e^{2\pi inx},\ \widehat{\mu}(n) = \int_0^1 e^{-2\pi inx}\,d\mu.$$

It is then a very natural and extremely important question *how to recover a function f* (or measure μ) from its Fourier series. In particular, does the Fourier series converge to the function (or measure) in a suitable sense? Put more succinctly, one may ask in what sense the sequence of *partial Fourier sums*

$$S_N f(x) := \sum_{|n|\le N} \widehat{f}(n)e^{2\pi inx},\ N \in \mathbb{Z}.$$

converges toward $f(x)$ as $N \to +\infty$.

When taken in the most literal, pointwise, sense, this is a hard question which was only fully settled in the 1960s!

2. Convolution

Working with Fourier series naturally leads to the convolution operation of two functions. We generally define

Definition 2.1. *Given $f, g \in L^1(S^1)$, we define*

$$(f * g)(x) := \int_{S^1} f(x-y)g(y)\,dy$$

This is an L^1-function via Fubini's theorem:

$$\|f * g\|_{L^1} \leq \|f\|_{L^1}\|g\|_{L^1}$$

In particular, $L^1(S^1)$ becomes an algebra under this operation.

It is then natural to investigate the relation between Fourier coefficients and convolutions. We have

Lemma 2.2. *Let $f, g \in L^1(S^1)$. Then*

$$\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)\,\forall n \in \mathbb{Z}.$$

Proof. This is again an elementary consequence of Fubini's theorem: we have

$$\begin{aligned}\widehat{f * g}(n) &= \int_0^1 e^{-2\pi inx}\big(\int_0^1 f(x-y)g(y)\,dy\big)\,dx\\ &= \int_0^1\int_0^1 e^{-2\pi in(x-y)}f(x-y)e^{-2\pi iny}g(y)\,dy\,dx\\ &= \big(\int_0^1 f(x)e^{-2\pi inx}\,dx\big)\big(\int_0^1 g(y)e^{-2\pi iny}\,dy\big)\end{aligned}$$

□

At this point it is natural to enquire whether multiplication on the level of f, g (provided this is well-defined) translates into convolution for the Fourier coefficients, provided this is suitable defined.
In fact, such a statement can be made rigorous, if one *restricts* f, g to a suitable sub space of the continuous functions $C^0(S^1)$.

Definition 2.3. *One defines the Wiener algebra $\mathcal{A}(S^1)$ to consist of all functions $f \in C^0(S^1)$ with the property that*

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty$$

We shall see later that the functions with this property form an algebra, and that if $f, g \in \mathcal{A}(S^1)$, then indeed we have

$$\widehat{fg}(n) = \sum_{m\in\mathbb{Z}} \widehat{f}(m)\widehat{g}(n-m),\ \forall n \in \mathbb{Z}. \tag{2.1}$$

3. The Dirichlet kernel

We now come back the question of convergence of the Fourier series. Thus consider the partial Fourier sum

$$\begin{aligned}\sum_{|n|\leq N}\widehat{f}(n)e^{2\pi inx} &= \sum_{|n|\leq N}\int_0^1 e^{2\pi in(x-y)}f(y)\,dy\\ &= \int_0^1 e^{-2\pi iN(x-y)}\frac{e^{2\pi i(2N+1)(x-y)}-1}{e^{2\pi i(x-y)}-1}f(y)\,dy\\ &= \int_0^1 \frac{\sin[(2N+1)\pi(x-y)]}{\sin[\pi(x-y)]}f(y)\,dy\\ &= D_N * f(x)\end{aligned}$$

Here D_N is the so-called *Dirichlet kernel*, given by

$$D_N(x) = \frac{\sin[(2N+1)\pi x]}{\sin[\pi x]}.$$

Observe that this kernel gets larger and larger at the origin, the larger N gets. On the other hand, due to its highly oscillatory behavior, its integral is equal to 1:

$$\int_0^1 D_N(x)\,dx = \int_0^1 \Big(\sum_{|n|\leq N} e^{2\pi inx}\Big)\,dx = 1.$$

The question of convergence of the partial Fourier series can thus be re-phrased as to whether

$$\lim_{N\to\infty} D_N * f(x) = f(x)$$

in the pointwise sense, say. Unfortunately, due to the highly singular behavior of the Dirichlet kernel, this is in general not true, even for continuous functions! Nonetheless, *imposing a bit of regularity on f*, it will be possible to show convergence.

4. Convergence of the Fourier series for Holder continuous functions

Let us first introduce the class of Holder continuous functions on S^1:

Definition 4.1. *Let $\alpha \in (0,1)$. We say that $f \in C^\alpha(S^1)$, provided*

$$[f]_\alpha := \sup_{x,y\in[0,1]} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < \infty$$

Thus we introduce a little extra regularity over the mere requirement of continuity in this definition. It turns out that this already suffices to conclude convergence of the partial Fourier sums to the original function:

Theorem 4.2. *Let $f \in C^\alpha(S^1)$. Then we have*

$$\lim_{N\to\infty} S_N f(x) = \lim_{N\to\infty} (D_N * f)(x) = f(x)\,\forall x \in S^1.$$

Proof. To begin with, write

$$S_N f(x) - f(x) = \int_0^1 [f(x-y) - f(x)] D_N(y)\,dy$$

The strategy now is to split this integral into two: when $|y|$ is very small, then $D_N(y)$ gets large (in the L^∞-sense), but we gain from the additional Holder regularity of f. On the other hand, if y is not too small, we take advantage of the rapid oscillations of $D_N(y)$ to effectively carry out an integration by parts, which also gains smallness.
Specifically, write

$$\begin{aligned}\int_0^1 [f(x-y)-f(x)]D_N(y)\,dy &= \int_{-\frac{1}{2}}^{\frac{1}{2}} [f(x-y)-f(x)]D_N(y)\,dy \\ &= \int_{|y|\leq\delta} [f(x-y)-f(x)]D_N(y)\,dy \\ &+ \int_{\frac{1}{2}>|y|>\delta} [f(x-y)-f(x)]D_N(y)\,dy \\ &:= A + B.\end{aligned}$$

Here we will pick δ at the end small in a way depending on N.

The estimate for A. We get

$$|A| \leq [f]_\alpha \int_{|y|<\delta} y^\alpha |D_N(y)|\,dy \leq C[f]_\alpha \int_{|y|<\delta} y^{\alpha-1}\,dy \leq C_\alpha [f]_\alpha \delta^\alpha$$

The estimate for B. Here we write

$$B = \int_{\frac{1}{2}>|y|>\delta} [f(x-y) - f(x)] D_N(y)\, dy = \int_{\frac{1}{2}>|y|>\delta} h_x(y) \sin[(2N+1)\pi y]\, dy$$
$$= -\int_{\frac{1}{2}>|y|>\delta} h_x(y) \sin[(2N+1)\pi (y + \frac{1}{2N+1})]\, dy$$

where we have introduced the notation

$$h_x(y) = \frac{f(x-y) - f(x)}{\sin(\pi y)}$$

Observe that the second integral expression above can also be written as

$$-\int_{\frac{1}{2}>|z-\frac{1}{2N+1}|>\delta} h_x(z - \frac{1}{2N+1}) \sin[(2N+1)\pi z]\, dz$$
$$= -\int_{\frac{1}{2}>|z|>\delta} h_x(z - \frac{1}{2N+1}) \sin[(2N+1)\pi z]\, dz$$
$$+ \int_{[\delta, \delta+\frac{1}{2N+1}]} h_x(z - \frac{1}{2N+1}) \sin[(2N+1)\pi z]\, dz$$
$$- \int_{[-\delta, -\delta+\frac{1}{2N+1}]} h_x(z - \frac{1}{2N+1}) \sin[(2N+1)\pi z]\, dz$$

Then we add all these terms to infer

$$2B = \int_{\frac{1}{2}>|y|>\delta} [h_x(y) - h_x(y - \frac{1}{2N+1})] \sin[(2N+1)\pi y]\, dy$$
$$+ \int_{[\delta, \delta+\frac{1}{2N+1}]} h_x(z - \frac{1}{2N+1}) \sin[(2N+1)\pi z]\, dz$$
$$- \int_{[-\delta, -\delta+\frac{1}{2N+1}]} h_x(z - \frac{1}{2N+1}) \sin[(2N+1)\pi z]\, dz.$$

Now we estimate each of these terms separately: first, using

$$|h_x(y) - h_x(z)| \le C|y|^{-1}|y-z|^{\alpha}[f]_{\alpha} + C(\min\{y, z\})^{-2}|y-z|\|f\|_{L^\infty}$$

for a suitable $y_* \in (y, z)$, we get (assuming $\delta \gg N^{-1}$)

$$|h_x(y) - h_x(y - \frac{1}{2N+1})| \le C[f]_{\alpha}\delta^{-1}N^{-\alpha} + C\|f\|_{L^\infty}\delta^{-2}N^{-1}$$

and so using the trivial bound for the integral

$$\Big|\int_{\frac{1}{2}>|y|>\delta} [h_x(y) - h_x(y - \frac{1}{2N+1})] \sin[(2N+1)\pi y]\, dy\Big|$$
$$\le C[f]_{\alpha}\delta^{-1}N^{-\alpha} + C\|f\|_{L^\infty}\delta^{-2}N^{-1}$$

For the remaining two integrals above, we use

$$|h_x(z - \frac{1}{2N+1})| \le C\delta^{-1}\|f\|_{L^\infty}$$

provided $|z| > \delta \gg N^{-1}$, and so

$$\begin{aligned}&\Big|\int_{[\delta,\delta+\frac{1}{2N+1}]} h_x(z-\frac{1}{2N+1})\sin[(2N+1)\pi z]\,dz\Big|\\&+\Big|\int_{[-\delta,-\delta+\frac{1}{2N+1}]} h_x(z-\frac{1}{2N+1})\sin[(2N+1)\pi z]\,dz\Big|\\&\leq C\delta^{-1}N^{-1}\|f\|_{L^\infty}\end{aligned}$$

In summary, one obtains

$$B \leq C(f,\alpha)[\delta^{-1}N^{-\alpha} + \delta^{-2}N^{-1}]$$

Finally, making the choice $\delta = N^{-\frac{\alpha}{3}}$, say, we get

$$A + B \leq C(f,\alpha)[N^{-\frac{\alpha^2}{3}} + N^{-\frac{2}{3}\alpha} + N^{-1+\frac{2\alpha}{3}}]$$

and so we indeed conclude that in the limit $N \to \infty$ we get

$$\lim_{N\to\infty}[S_N f(x) - f(x)] = 0\,\forall x \in S^1.$$

□

5. Convergence results for $C^0(S^1)$; the Fejer kernel

It turns out that the preceding convergence result very much fails if extended to all of $C^0(S^1)$. In fact, one can construct continuous functions whose Fourier series diverges on a prescribed set of measure zero (but this is non-elementary). In fact, a priori it is not even clear *how to re-construct* f in pointwise fashion from its Fourier coefficients $\widehat{f}(n)$. However there is a beautiful result due to Fejer which asserts that indeed one can re-cover $f \in C^0(S^1)$ from its Fourier coefficients, provided one *averages over the partial Fourier sums.*

Specifically, given $f \in C^0(S^1)$, introduce the *Cesaro means*

$$\sigma_N f(x) := \frac{1}{N}\sum_{n=0}^{N-1} S_n f(x)$$

Recalling the definition of S_n, we can also spell this out as follows:

$$\begin{aligned}\sigma_N f(x) &= \frac{1}{N}\sum_{n=0}^{N-1}\int_0^1 (\sum_{|k|\leq n} e^{2\pi i k(x-y)}) f(y)\,dy\\&= \frac{1}{N}\int_0^1 [\sum_{|n|=0}^{N-1}(N-|n|)e^{2\pi i n(x-y)}] f(y)\,dy\end{aligned}$$

Now the kernel function $K_N(x) := \frac{1}{N}\sum_{|n|=0}^{N-1}(N-|n|)e^{2\pi i n x}$ has the remarkable property that it can be expressed as a perfect square:

$$\begin{aligned}\sum_{|n|=0}^{N-1}(N-|n|)e^{2\pi i n x} &= \big[e^{2\pi i\frac{N-1}{2}x} + e^{2\pi i\frac{N-3}{2}x} + \ldots + e^{-2\pi i\frac{N-1}{2}x}\big]^2\\&= \big[e^{-\pi i(N-1)x}\frac{e^{2\pi i N x}-1}{e^{2\pi i x}-1}\big]^2\\&= \big[\frac{\sin(\pi N x)}{\sin(\pi x)}\big]^2,\end{aligned}$$

and so $K_N(x) = \frac{1}{N}\big[\frac{\sin(\pi N x)}{\sin(\pi x)}\big]^2$. It is the positivity which renders the Fejer kernel so much more useful than the Dirichlet kernel. In fact, the family $\{K_N\}_{N\geq 1}$ forms an *approximate identity*:

Definition 5.1. *A family of functions $\{\Phi_N(x)\}_{N\geq 1} \subset L^\infty(S^1)$ is called an approximate identity, provided*

- $\int_0^1 \Phi_N(x)\,dx = 1$.
- $\sup_N \int_0^1 |\Phi_N(x)|\,dx < \infty$.
- *For any* $\delta > 0$*, we have*

$$\lim_{N\to\infty} \int_{|x|>\delta} |\Phi_N(x)|\,dx = 0.$$

Observe that since $K_N(x) := \frac{1}{N}\sum_{|n|=0}^{N-1}(N-|n|)e^{2\pi i n x}$ we have $\int_0^1 K_N(x)\,dx = 1$, and since $K_N(x) \geq 0$ the second condition is immediate from the first. Finally, since

$$|K_N(x) \leq CN^{-1}|x|^{-2},$$

the last condition is also fulfilled. Finally, we state

Proposition 5.2. *Let* $\{\Phi_N(x)\}_{N\geq 1}$ *be an approximate identity on* S^1*. Then for any* $f \in C^0(S^1)$*, we have*

$$\lim_{N\to\infty} (\Phi_N * f)(x) = f(x).$$

In particular, we can infer

Corollary 5.3. *For any* $f \in C^0(S^1)$*, we have*

$$\lim_{N\to\infty} (K_N * f)(x) = f(x)\,\forall x \in S^1.$$

Proof. (Proposition) Put $\sup_N \int_0^1 |\Phi_N(x)|\,dx = M < \infty$. Pick $x \in S^1$. Also, fix $\varepsilon > 0$. By continuity of f, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{2M}$ if $|x - y| < \delta$. Then write

$$\begin{aligned}
(\Phi_N * f)(x) - f(x) &= \int_0^1 \Phi_N(x-y)f(y)\,dy - f(x) \\
&= \int_0^1 \Phi_N(x-y)[f(y) - f(x)]\,dy \\
&= \int_{|x-y|<\delta} \Phi_N(x-y)[f(y) - f(x)]\,dy \\
&+ \int_{|x-y|\geq\delta} \Phi_N(x-y)[f(y) - f(x)]\,dy \\
&=: A + B.
\end{aligned}$$

Then we get

$$|A| \leq \frac{\varepsilon}{2M} \sup_N \int_0^1 |\Phi_N(x)|\,dx < \frac{\varepsilon}{2}$$

uniformly in N. Moreover, for N large enough, we have

$$|B| \leq 2\|f\|_{L^\infty} \int_{|x|>\delta} |\Phi_N(x)|\,dx < \frac{\varepsilon}{2}.$$

It follows that for N large enough, we have

$$|(\Phi_N * f)(x) - f(x)| < \varepsilon.$$

□

Note that the preceding proof shows that in $\Phi_N * f$ converges in fact uniformly toward f as $N \to \infty$.

6. A property of $\mathcal{A}(S^1)$.

Here we check the identity (2.1) and hence that $\mathcal{A}(S^1)$ is indeed an algebra. First, assume that f is a trigonometric polynomial, thus

$$f(x) = \sum_{|k|\leq n} a_k e^{2\pi ikx},\ a_k \in \mathbf{C}.$$

Then if $g \in \mathcal{A}(S^1)$ we get

$$\widehat{fg}(m) = \int_0^1 \big(\sum_{|k|\leq n} a_k e^{2\pi ikx}\big) g(x) e^{-2\pi imx}\,dx = \sum_{|k|\leq n} a_k \int_0^1 g(x)) e^{-2\pi i(m-k)x}\,dx = \sum_k \widehat{f}(k)\widehat{g}(m-k)$$

Next, for $f, g \in \mathcal{A}(S^1)$ arbitrary, we have that $K_N * f$ is a trigonometric polynomial, and so (writing $\mathcal{F}$ for the Fourier transform and using Lemma 2.2 as well as the definition of K_N)

$$\mathcal{F}\big([K_N * f]g\big)(n) = \sum_k (1 - \frac{|k|}{N})_+ \widehat{f}(k)\widehat{g}(n-k)$$

Letting $N \to \infty$ and using the dominated convergence theorem, we get

$$\widehat{fg}(n) = \sum_k \widehat{f}(k)\widehat{g}(n-k)$$

Moreover, since

$$\sum_n |\widehat{fg}(n)| \leq |\sum_k \widehat{f}(k)\widehat{g}(n-k)| \leq (\sum_k |\widehat{f}(k)|)(\sum_n |\widehat{g}(n)|) < \infty,$$

we see $fg \in \mathcal{A}(S^1)$.